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A NOTE ON ANNIHILATOR IDEALS OF COMPLEX BORDISM CLASSES

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Recent studies of the complex bordism homology theory $\Omega_*^U(\ )$ have shown that for a finite complex X the integer hom-dim $_{\Omega_*^U}\Omega_*^U(X)$ provides a useful numerical invariant measuring certain types of complexity in X. Associated to an element $\alpha \in \Omega_*^U(X)$ one has the annihilator ideal $A(\alpha) \subset \Omega_*^U$. Numerous relations between $A(\alpha)$ and hom-dim $_{\Omega_*^U}\Omega_*^U(X)$ are known. In attempting to deal with these invariants it is of course useful to study special cases, and families of special cases. In this note we study the annihilator ideal of the canonical element $\sigma \in {}_{2N}^{U}\Omega(X)$ where X is a complex of the form

$$S^{2N} \cup_p e^{2N+1} \cup e^{2N+2n_1-1} \cup \cdots \cup e^{2N+2n_k-1}$$

and $N >> n_1, \cdots n_k > 1$, and p an odd prime. We show that $A(\sigma) \not\ni [V^{2p^2-2}], \cdots, [V^{2p^s-2}], \cdots$, where $[V^{2p^s-2}] \in \Omega_{2p^s-2}^U$ is a Milnor manifold for the prime p. This provides another piece of evidence that for such a complex X, hom-dim $_{\Omega_*^U}\Omega_*^U(X)$ is 1 or 2.

In [9], [11] and [12] the study of the annihilator ideal of the canonical class $\alpha \in \widetilde{\Omega}_0^U(X)$ in a stable complex X of the form

$$X = S^0 \cup_p e^1 \cup_f e^{2n-1}$$

played a crucial role in the applications of [9] and [11] to the stable homotopy of spheres. In the closing remarks of [4] it was suggested that more generally for a stable complex of the form (where p is an *odd* prime)

$$Y = S^0 \cup_p e^1 \cup_{f_1} e^{2n_1-1} \cdots \cup_{f_k} e^{2n_k-1}$$

the annihilator ideal of the canonical class $\alpha \in \widetilde{\Omega}_0^U(Y)$ had the form $(p, [CP(p\text{-}1)]^t)$ for some integer t, and that hom-dim $_{\Omega_*^U}\Omega_*^U(Y) \leq 2$. Our objective in this note is to make the following elementary contribution to these matters.

THEOREM. *Let Y be a stable complex of the form*

$$S^0 \cup_p e^1 \cup_{f_1} e^{2n_1-1} \cup_{f_2} e^{2n_2-1} \cdots \cup_{f_k} e^{2n_k-1}$$

where p is odd prime. Let $\sigma \in \widetilde{\Omega}_0^U(Y)$ denote the canonical class. Then

$$[V^{2p^s-2}] \notin A(\sigma)$$

for any $s > 1$, *where* $A(\sigma) \subset \Omega^U_*$ *denotes the annihilator ideal of* σ, *and* $[V^{2p^s-2}] \in \Omega^U_{2p^s-2}$ *is a Milnor manifold for the prime* p.

REMARK. An easy computation using Landweber-Novikov operations [1] [2] [5] [7] shows, for any stably spherical bordism element $\sigma \in \widetilde{\Omega}^U_*(X)$ of additive order p, on a complex X, that

$$A(\sigma) \subset (p, [V^{2p-2}], (V^{2p^2-2}], \cdots) .$$

It is therefore not unreasonable to ask if some particular Milnor manifold $[V^{2p^s-2}]$ belongs to $A(\sigma)$.

The proof of this result follows closely the developments of [3] [4] [10]. The fundamental fact that we shall use is implicit in both [3] and [10] and may be stated as follows:

PROPOSITION. *Let* X *be a finite complex and suppose that* $\sigma \in \Omega^U_*(X)$ *is a stably spherical class and satisfies the following conditions*

(1) $p\sigma = 0$, p *a prime*

(2) $\mu_p(\sigma) \neq 0 \in \widetilde{H}_*(X; Z_p)$ *where* μ_p *is the composite* $\mu_p: \widetilde{\Omega}^U_*(X) \longrightarrow \widetilde{H}_*(X; Z) \xrightarrow{\rho} \widetilde{H}_*(X; Z_p)$ *of the Thom homomorphism* μ *and reduction mod* p ρ.

If $[V^{2p^s-2}] \in A(\sigma)$ *then*

$$S_i\beta \; u \neq 0 \in \widetilde{H}^{n+2p^s-1}(X; Z_p)$$

where

$n = deg\ \sigma$

$u \in \widetilde{H}^n(X; Z_p)$ *is dual to* $\mu_p(\sigma)$

$\beta \in \mathscr{A}(p)$ *the Bockstein* $\left\{ \begin{array}{l} \mathscr{A}(p) \textit{ the mod} \\ p \textit{ Steenrod} \\ \textit{algebra.} \end{array} \right.$

$S_i \in \mathscr{A}(p)$ *the primitive element of degree* $2p^i-2$.

The proof of the preceding proposition may be deduced directly from [10; 1.2] as in § 2 of [10] or [3; 2.4].

Proof of Theorem. Suppose to the contrary that $[V^{2p^s-2}] \in A(\sigma)$ for some $s > 1$.

Let us denote by s_α, $\alpha = (a_1, \cdots)$ the Landweber-Novikov operation corresponding to the sequence α. (See [1] [2] [5] [7] or [11] for information about Landweber-Novikov operations). The following formula for the action of the Landweber-Novikov operations on Ω^U_* may be found in [11; 2.1] (and the succeeding remark):

$$s_{p\Delta_{p^{s-1}-p^s-2}} [V^{2p^s-2}] \equiv [V^{2p^{s-1}-2}] \bmod p .$$

Applying the Cartan formula to the fact that σ is a spherical class we obtain:

$$\begin{aligned} [V^{2p^{s-1}-2}]\,\sigma &= (s_{p\Delta_{p^{s-1}}-p^{s-2}}[V^{2p^s-2}])\,\sigma && \text{(Cartan formula)} \\ &= s_{p\Delta_{p^{s-1}}-p^{s-2}}([V^{2p^s-2}]\,\sigma) \\ &= s_{p\Delta_{p^{s-1}}-p^{s-2}}(0) = 0\,. \end{aligned}$$

Hence $[V^{2p^{s-1}-2}] \in A(\sigma)$. In this way we see that (recall we may choose $[V^{2p-2}] = [\mathbf{C}P(p-1)]$).

$$p,\ [V^{2p-2}],\ \cdots,\ [V^{2p^s-2}] \in A(\sigma)\,.$$

Note next that

$$\tilde{H}^i(Y; Z_p) \cong \begin{cases} Z: & i = 0, 1, 2n_1-1, \cdots, 2n_k-1 \\ 0: & \text{otherwise.} \end{cases}$$

For $i = 0, 1, 2n_1-1, \cdots, 2n_k-1$, let $e_i \in \tilde{H}^i(Y; Z_p)$ denote a nonzero class. Note that we may choose $e_1 = \beta e_0$.

Applying the preceding proposition to the fact that

$$[\mathbf{C}P(p-1)] \in A(\sigma)$$

we conclude that (recall $S_1 = P^1$)

$$P^1\ \beta e_0 \neq 0 \in \tilde{H}^{2p-1}(Y; Z_p)\,.$$

Hence we may assume that $n_i = p$ and

$$e_{2p-1} = e_{2n_i-1} = P^1\beta e_0\,.$$

Next apply the preceding proposition to the fact that $[V^{2p^2-2}] \in A(\sigma)$ to conclude that

$$S_2\beta e_0 \neq 0 \in H^{2p^2-1}(Y; Z_p)\,.$$

Recall that

$$S_2 = P^pP^1 - P^1P^p\,.$$

Thus

$$P^pP^1\beta e_0 - P^1P^p\beta e_0 \neq 0 \in \tilde{H}^{2p^2-1}(Y; Z_p)\,.$$

Recall now the following Adem relations:

$$\begin{aligned} P^1P^p &= -P^{p+1} \\ P^1\beta P^p &= P^{p+1}\beta = P^1P^p\beta\,. \end{aligned}$$

Therefore

$$P^1P^p\beta e_0 = P^1\beta P^p e_0 .$$

But

$$P^p e_0 \in \tilde{H}^{2p(p-1)}(Y; Z_p) = 0$$

and hence

$$P^1P^p\beta e_0 = 0 .$$

Therefore we conclude that

$$P^pP^1\beta e_0 \neq 0 \in \tilde{H}^{2p^2-1}(Y; Z_p) .$$

Next we propose to apply the factorization theorem of [6], [8] to deduce a contradiction. Consider therefore

$$P^1P^1\beta e_0 .$$

Recall the Adem relations

$$P^1P^1 = 2P^2$$
$$P^2\beta - \beta P^2 = \beta P^1P^1 - P^1\beta P^1 .$$

We have therefore

$$P^1P^1\beta e_0 = 2P^2\beta e_0 = 2(\beta P^2 e_0 + \beta P^1P^1 e_0 - P^1\beta P^1 e_0) .$$

Now note that

$$P^2 e_0 = 0$$
$$P^1 e_0 = 0$$

for dimensional reasons. Therefore

$$P^1P^1\beta e_0 = 2(0) = 0 .$$

The class $P^1\beta e_0 \in \tilde{H}^{2p-1}(Y; \underset{\sim}{z}_p)$ therefore satisfies

$$P^1(P^1\beta e_0) = 0$$
$$\beta(P^1\beta e_0) = 0$$

(the latter from dimensional considerations). We may therefore apply the formula of [6] [8] to conclude that

$$P^pP^1\beta e_0 = \beta\Lambda(P^1\beta e_0) + 2P^{p-2}\mathscr{R}(P^1\beta e_0)$$

modulo a suitable indeterminacy, where Λ and $\mathscr{R}$ are secondary operations of degree $2p(p-1)-1$ and $4(p-1)$ respectively. Note that

$$\Lambda P^1\beta e_0 \in \tilde{H}^{2p^2-2}(Y; Z_p) = 0$$

and hence

$$P^pP^1\beta e_0 = 2P^{p-2}\mathscr{R}P^2\beta e_0$$

modulo a suitable indeterminacy. And so it remains for us to eliminate the possibility that

$$P^{p-2}\mathscr{R}P^2\beta e_0 \neq 0 .$$

This will require a careful analysis of the cell structure of Y thru dimensions $6p-6$. Recall that

$$Y = S^0 \bigcup_p e^1 \cdots .$$

Let $M = S^0 \bigcup_p e^1$. It follows from [14] that (N. B. it is necessary to divide into two cases, according to $p > 3$ or $p = 3$)

$$\pi_i^s(M) = \begin{cases} Z_p\colon i = 0, 2p-3, 2p-2, 4p-5, 4p-4, 6p-7 \text{ or } 6p-6 \\ 0\colon \text{otherwise for } 0 \leqq i \leqq 6p-6 . \end{cases}$$

A stable map

$$\alpha\colon S^{2p-2} \longrightarrow M$$

represents a nonzero element of $\pi_{2p-2}^s(M)$ iff

$$P^1\beta\colon \tilde{H}^0(M \bigcup_\alpha e^{2p-1}\colon Z_p) \longrightarrow \tilde{H}^{2p-1}(M \bigcup_\alpha e^{2p-1}; Z_p)$$

is an isomorphism. Thus from the preceding analysis of the fact that $[CP(p-1)] = [V^{2p-2}] \in A(\sigma)$ we find

$$Y = S^0 \bigcup_p e^1 \bigcup_\alpha e^{2p-1} \cdots$$

where $[\alpha] \in \pi_{2p-2}^s(S^0 \bigcup_p e^1)$ is a generator. Next we form the cofibration

$$S^{2p-2} \xrightarrow{\alpha} M \xrightarrow{q} N$$

defining N as $S^\circ \bigcup_p e^1 \bigcup_\alpha e^{2p-1}$. Note that

$$\pi_{2p-2}^s(N) = 0$$

and therefore the $2p-2$ skeleton of Y has the form

$$N \vee (S^{2p-1} \vee \cdots \vee S^{2p-1}) .$$

We propose now to calculate the homotopy of N in low dimensions. From the definition of N we obtain an exact sequence

$$\cdots \pi_i^s(S^{2p-2}) \xrightarrow{\alpha_*} \pi_i^s(M) \xrightarrow{q_*} \pi_i^s(N) \xrightarrow{\partial_*} \pi_{i-2}^s(S^{2p-2}) \cdots .$$

Consulting [14] and the little table of the homotopy of M above we see that only a few values of i lead to p-primary information. The first such section is:

$$\longrightarrow \pi_{4p-4}^s(S^{2p-2}) \xrightarrow{\alpha_*} \pi_{4p-4}^s(M) \xrightarrow{q_*} \pi_{4p-4}^s(N) \xrightarrow{\partial_*} \pi_{4p-5}^s(S^{2p-2})$$
$$\xrightarrow{\alpha_*} \pi_{4p-5}^s(M) \longrightarrow \pi_{4p-5}^s(N) \longrightarrow \cdots .$$

Now note that

$$\alpha_*:\ \pi^s_{4p-5}(S^{2p-2}) \longrightarrow \pi^s_{4p-5}(M)$$

is an isomorphism on the p-components which are Z_p. Thus our sequence divides into

$$0 \longrightarrow \pi^s_{4p-4}(M) \xrightarrow{q_*} \pi^s_{4p-4}(N) \longrightarrow 0$$
$$0 \longrightarrow \pi^s_{4p-5}(S^{2p-2}) \longrightarrow \pi^s_{4p-5}(M) \longrightarrow 0$$
$$0 \longrightarrow \pi^s_{4p-5}(N) \longrightarrow \pi^s_{4p-6}(S^{2p-2}) = 0\ .$$

We thus find that

$$\pi^s_i(N) \cong \begin{cases} Z_p\colon\ i = 4p-4 \\ \text{a group of order prime to } p\colon\ 2p-1 < i < 4p-4. \end{cases}$$

It will be important to remember that the generator of $\pi^s_{4p-4}(N)$ comes from $\pi_{4p-4}(M)$. Denote this generator by β. In a similar manner we find that

$$\pi^s_i(N) \cong \begin{cases} Z_p\colon\ i = 6p-6 \\ \text{a group of order prime to } p\colon\ 4p-4 \leqq i \leqq 6p-6, \end{cases}$$

and the generator of $\pi^s_{6p-6}(N)$ comes from $\pi^s_{6p-6}(M)$. Denote this generator by γ. Observe that since β and γ pull off the top cell of N that

$$P^1\colon\ H^{2p-1}(N \bigcup\nolimits_\beta e^{4p-3}; Z_p) \longrightarrow H^{4p-3}(N \bigcup\nolimits_\beta e^{4p-3}; Z_p)$$

and

$$\mathscr{R}\colon\ H^{2p-1}(N \bigcup\nolimits_\gamma e^{6p-5}; Z_p) \longrightarrow H^{6p-5}(N \bigcup\nolimits_\gamma e^{4p-3}; Z_p)$$

are zero. The first of these shows that

$$\mathscr{R}\colon\ H^{2p-1}(e^{6p-5} \bigcup\nolimits_\gamma N \bigcup\nolimits_\beta e^{4p-3}; Z_p) \longrightarrow H^{6p-5}\,(e^{2p-5} \bigcup\nolimits_\gamma N \bigcup\nolimits_\beta e^{4p-3}; Z_p)$$

is again defined but is still zero, modulo zero. (N. B. The indeterminacy of the operation $\mathscr{R}$ in the sum of the images of the operations $2P^1\beta-\beta P^1$ and P^2, which vanish for $e^{6p-5} \bigcup_\gamma N \bigcup_\beta e^{4p-3}$ for dimensional reasons).

Let us denote by

$$r\colon\ N \vee (S^{2p-1} \vee \cdots \vee S^{2p-1}) \longrightarrow M$$

the natural map. The analysis preceding shows that thru the $6p-6$ skeleton Y^{6p-6}, of Y, there is a map

$$p\colon\ Y^{6p-6} \longrightarrow \begin{cases} N \\ N \cup_\beta e^{4p-3} \\ N \cup_\gamma e^{6p-5} \\ e^{6p-5} \cup_\gamma N \cup_\beta e^{4p-3} \end{cases}$$

inducing an isomorphism of Z_p cohomology thru dimensions 0 and 1 and an injections thru dimensions $6p$-6. The particular space on the right depending on the cells of Y occuring.

Therefore $\mathscr{R}\ P^1\beta\, e_0 \neq 0 \in H^{6p-5}(Y; Z_p)$ iff the corresponding result is true for one of the spaces N, $N \cup_\beta e^{4p-3}$, $N \cup_\gamma e^{6p-5}$, $e^{6p-5} \cup_\gamma N \cup_\beta e^{4p-3}$. However, in constructing these spaces we carefully checked that $\mathscr{R}\ P^1\beta e_0 = 0$ modulo 0. Therefore

$$\mathscr{R}\, P^1\beta e_0 = 0 \in H^{6p-5}(Y; Z_p)$$

modulo 0 and hence

$$P^p P^1 \beta e_0 = P^{p-2}\, \mathscr{R}\, P^1 \beta e_0 = 0 \in H^{2p^2-1}(Y; Z_p)$$

modulo 0 from which we conclude $[V^{2p^22-2}] \notin A(\sigma)$. But as we saw at the beginning of the proof this is contrary to the assumption that $[V^{2p^2-2}] \in A(\sigma)$ tor some $s > 1$ and the the theorem is established. $**$

Question. Does there exist a simple Adem type relation that would show $\mathscr{R}\, P^1\beta e_0 = 0$ modulo 0 in $H^{6p-5}(Y; Z^p)$ without going through the cell by cell analysis above?

COROLLARY. *Let Y be a stable complex of the form*

$$Y = S^\circ \cup_p e^1 \cup_{f_1} e^{2n_1-1} \cdots \cup_{f_k} e^{2n_k-1}$$

where p is an odd prime. Let $\sigma \in \Omega_0^U(Y)$ denote the canonical class. Then for each integer $s > 1$ the class $[V^{2p^s-2}]\,\sigma$ is not stably spherical, that is, for $s > 1$

$$[V^{2p^2-2}]\sigma \notin \operatorname{Im}\{\widetilde{\Omega_*^{fr}}(Y) \longrightarrow \widetilde{\Omega_*^U}(Y)\}\ .\ **$$

REFERENCES

1. J. F. Adams, *On Novikov's Work on Complex Cobordism Operations*, University of Chicago Lecture Notes (1866)
2. ———, *Quillen's Work on Formal Group Laws and Complex Cobordism*, Univ. of Chicago Lecture Notes (1970).
3. P. E. Conner and L. Smith, *On the complex bordism of finite complexes II*, J. Differential Geometry, (to appear).
4. ———, *On the complex bordism of complexes with few cells*. University of Virginia, Preprint 1970. (To appear in Kyoto U. Math. J.)
5. P. S. Landweber, *Cobordism operations and Hopf algebras*, TAMS **129** (1967), 94-110.

6. A. Liulevicius, *Factorization of cyclic reduced power by secondary cohomology operations*, Memoirs of Amer. Math. Soc., **42** (1962).
7. S. P. Novikov. *The method of algebraic topology from the viewpoint of cobordism theory*, Izv. Ak. Nauk. SSSR, **31** (1967), 855-951.
8. N. Shimada and T. Yamanoshita, *On the Triviality of the mod p Hopf Invariant*, Japan J. Math., **31** (1961), 1-25.
9. L. Smith, *On realizing complex bordism modules I*, Amer. J. Math., **92** (1970), 793-856.
10. ———, *On Ideals in* Ω_*^U, Pacific J. Math., **37** (1971), 527-537.
11. ———, *On Realizing Complex Bordism Modules II*, Amer. J. Math., **93** (1971), 226-263.
12. ———, *Annihilator Ideals of Complex Bordism Classes and Toda's α-Sequence in the Stable Homotopy of Spheres*, University Virginia, Preprint 1970. Indiana Math. J. **20** (1971), 1119-1123.
13. R. E. Stong, *Notes on Cobordism Theory*, Princeton Uuiversity Press, 1968.
14. H. Toda, *p-Primary components of homotopy groups IV*, Mem. Coll. Sci. Kyoto, Series A, **32** (1959), 297-332.

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