A FORM OF THE MOMENT PROBLEM
FOR LIE GROUPS

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A form of the Hamburger moment problem for the real
line is generalized and solved for an arbitrary Lie group.
The solution relates the unitary representations of the Lie
group to certain symmetric representations of the associated
universal enveloping algebra.

The celebrated Hamburger moment problem for the real line asks:
Which linear functionals on the algebra of polynomials can be realized
by integrating these polynomials against a fixed positive measure?
Hamburger's classic solution shows that a linear functional has this
property if and only if it assigns positive values to positive polyno-
mials [6, 10].

By taking Fourier transforms, we may rephrase the problem as
follows: Which linear functionals on the algebra of distributions
supported at the origin can be realized by evaluating these distribu-
tions at a fixed function of positive type? The solution shows that
a linear functional has this property if and only if it assigns positive
values to distributions of positive type.

In this latter form we may extend the problem to an arbitrary
Lie group, and search for a solution in the same form. Here we
confirm that this extended problem does indeed admit such a solution
(Theorem 3). We then reformulate the result in terms of the space
of unitary representations of the group to give a less satisfactory
extension of the problem in its original form (Theorem 4). In the
process we find that we can characterize the symmetric representa-
tions of the associated Lie algebra which can be "lifted" to unitary
representations of the group (Theorem 5). As in the classical case
[1], the heart of the matter lies in showing that positive linear func-
tionals on the moment algebra are necessarily continuous in an ap-
propriate sense (Corollary 2).

The algebra of distributions of compact support. In order to
formulate the problem on a given Lie group, we shall need an
algebraic structure containing both the Lie group and its Lie algebra.
For this purpose we choose the algebra of all distributions with
compact support defined on the group. The properties of this algebra
which we shall use are all obtained by direct extensions of arguments
developed by Laurent Schwartz in his investigations of distribution
defined on R^n [9, also 2, 3, 4] and are only summarized here.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Denote by $E^{(k)}$ the space of all k-times continuously-differentiable complex-valued functions defined on G, topologized by uniform convergence of all derivatives on all compacta; and by E the projective limit of all infinitely differentiable functions, with the projective limit topology. Denote by $E^{(k)'}$ the dual space of all complex-valued distributions of order of most k with compact support on G, topologized with the strong dual topology; and by E' the inductive limit of all distributions with compact support on G, topologized with the dual inductive limit topology. Then E and E' are both Montel spaces, and each is the strong dual of the other. In particular, every linear functional on E' which is bounded on the bounded subsets of E' is necessarily continuous on E' and so must lie in E. The bounded subsets of E' are relatively compact; moreover, every bounded subset of E' is contained and bounded in $E^{(k)'}$ for some k [cf. 11, Theorem 34.4].

Denote further by E_K, where K is a compact subset of G, the quotient space of E module the subspace of functions vanishing, together with all of their derivatives, on K; and by E'_K the dual subspace of E' consisting of distributions whose supports be in K. Then as K runs through the family of all compact subsets of G, directed by inclusion, E is obtained as the strict projective limit, and E' as the strict inductive limit, of E_K and E'_K, respectively. In particular, it follows that every bounded set in E' is contained in K'_K for some K[cf. 11, Theorem 34.4].

The space E' admits an associative multiplication, defined by convolution over G and denoted here simply by juxtaposition, such that the mapping $\{s, t\} \rightarrow st$ is continuous jointly in s and t. Under the multiplication, E' becomes a Montel algebra with unit e (the point measure at the origin of G) which contains:

    (a) an isomorphic image of the Lie group G as the group of point measures under convolution,

    (b) an isomorphic image of the Lie algebra $\mathfrak{g}$ as the algebra of tangent vectors at the origin of G under the Lie product.

Moreover, both of these isomorphisms are topological. We denote by A the associative subalgebra of E' generated algebraically by G, and by U the associative subalgebra generated algebraically by $\mathfrak{g}$ and e. Then A may be identified with the algebraic group algebra of G and U with the universal enveloping algebra of $\mathfrak{g}$. A is dense in $E^{(0)'}$, and hence in E', while U coincides with $E'_{\{e\}}$.

In terms of the multiplication in E' we can describe the distributions of order k as k-fold derivatives of distributions of order 0, or Radon measures, on G. More precisely for every bounded set B in $E^{(k)'}$, and hence in E', there is a bounded set C in $E^{(0)'}$ and a finite

set D (consisting in fact of monomials in the generators of $\mathfrak{g}$) in $U^{(k)} = U \cap E^{(k)\prime}$, respectively, such that every $t \in B$ can be expressed in the form $t = \sum d_j m_i$ with $m_i \in C$ and $d_i \in D$ [11, Theorem 34.5]. Thus $E^{(k)\prime}$ is spanned by $U^{(k)} E^{(0)\prime}$.

Order properties. The space E admits an involution $*$: $\varphi \to \varphi^*$ defined by the formula $\varphi^*(g) = \overline{\varphi(g^{-1})}$, $g \in G$. This involution is conjugate-linear and bicontinuous in the topology of E. We denote by H the set of symmetric functions φ, for which $\varphi^* = \varphi$, and by P the set of functions φ of positive type, for which $\sum_{ij} \lambda_i \bar{\lambda}_j \varphi(g_i g_i^{-1}) \geqq 0$, $\lambda_i \in C$, $g_i \in G$. Then H is a closed real-linear subspace of E, and P is a closed convex cone in H. Moreover, we know that $P \cap (-P) = \{0\}$ (since if $\varphi \in P$ then $|\varphi(g)| \leqq \varphi(e)$, $g \in G$) and that $P + (-P)$ is dense in H [cf. 7, p. 142].

The space E also admits an involution $*$: $t \to t^*$, induced by the involution in E according to the formula $t^*(\varphi) = \overline{t(\varphi^*)}$. This involution is a conjugate-linear anti-automorphism, bicontinuous in the topology of E'. Under this involution, E' becomes a Montel $*$-algebra which contains both A and U as $*$-subalgebras.

We denote by H' the set of symmetric elements $t \in E'$ for which $t^* = t$, and by P' the set of elements t of positive type, for which $t(\varphi) \geqq 0$ for all $\varphi \in P$. All elements of the form tt^* lie in P' and convex combinations of such elements are dense in P'. Every element t of P' has the form $t = t_r + it_i$, where $t_r = \frac{1}{2}(t + t^*)$ and $t_i = \frac{1}{2}i(t - t^*)$ lie in H', and every element t of H' has the form $t = t_p - t_n$, where $t_p = \frac{1}{4}(t + e)^2$ and $t_n = \frac{1}{4}(t - e)^2$ lie in P'. Hence, $P' + (-P') = H'$, and $H' + iH' = E$. Since $P + (-P)$ is dense in H, it follows that $P' \cap (-P') = \{0\}$. Finally, since multiplication is jointly continuous in E', it follows that P' is invariant under all mappings of the form $t \to sts^*$, where $s \in E'$.

The cones P and P' are dual cones, in the sense that $\varphi \in P$ if and only if $t(\varphi) \geqq 0$ for all $t \in P'$, and $t \in P'$ if and only if $t(\varphi) \geqq 0$ for all $\varphi \in P$. In terms of these cones we can define dual partial orderings in H and H' by putting $\theta \leqq \varphi$ if and only if $\varphi - \theta \in P$, and $s \leqq t$ if and only if $t - s \in P'$. Then these orderings are compatible with the algebra and topology of H and H' and exhibit all the usual ordering properties [9]. In particular, if s, $t \in E'$, then $(s + t)(s + t)^* \leqq (s + t)(s + t)^* + (s - t)(s - t)^* = 2(ss^* + tt^*)$. By iterating this argument, we find that if $t_1, \cdots, t_n \in E'$, then $(\Sigma t_i)(\Sigma t_i)^* \leqq 2^n (\Sigma t_i t_i^*)$.

In terms of the ordering in H, we can now identify the bounded sets of E' as follows:

LEMMA 1. *A subset B of E' is bounded if and only if there exist*

(a) *a compact subset K of G, and* (b), *a single element d in $U \cap P'$, such that $B \subset E_K'$, and for all $t \in B$ we have*

$$- d \leqq t_r \leqq + d$$
$$- d \leqq t_i \leqq + d$$

Proof. Suppose first that B is bounded. Then we already know that $B \subset E_K' \cap E^{(k)'}$ for some compact subset K of G and integer k. Now for any $t \in E'$ we have $2t_r = t + t^* \leqq tt^* + e$, and similarly for $-2t_r$ and $\pm 2t_i$. Hence it suffices to show that if $t \in B$, then there exists a single element $d \in U$ such that $tt^* \leqq d$.

Suppose first that $B \subset E^{(0)'}$. Then every $m \in B$ is a finite measure supported in K. If $\varphi \in P$, then $mm^*(\varphi) = \iint \varphi(gh^{-1}) \, dm(g) \, d\bar{m}(h) \leqq \varphi(0) \left(\int_G d \, |m| \, (g) \right)^2 = \|m\|^2 \, e(\varphi)$, where $\|m\|$ denotes the total variation of m, and $e(\varphi) = \varphi(0)$. Since φ is arbitrary in P, it follows that $mm^* \leqq \|m\|^2 \, e$. Since B is bounded is $E^{(0)'}$, there exists a constant $|B|$ such that $m \in B$ implies $\|m\| \leqq |B|$. Hence $mm^* \leqq |B|^2 e$.

Suppose now that $B \subset E^{(k)'}$ where $k > 0$. Then we know that there are a bounded subset C of $E^{(0)'}$ and a finite subset D of $U^{(k)}$, respectively, such that $t \in B$ implies $t = \Sigma \, d_i m_i$, with $m_i \in C$, $d_i \in D$. Hence $tt^* = (\Sigma \, d_i m_i) \, (\Sigma \, d_i m_i)^* \leqq 2^n \, \Sigma (d_i m_i) \, (d_i m_i)^*$. But we also know that $m_i m_i^* \leqq \|m_i\|^2 e$, and hence that $(d_i m_i) \, (d_i m_i)^* = d_i m_i m_i^* d_i^* \leqq \|m\|^2 \, d_i d_i^*$. It follows that $tt^* \leqq 2^n \Sigma \, \|m_i\|^2 \, d_i d_i^*$. Since C is bounded in $E^{(0)'}$, there is a constant $|C|$ such that $m \in C$ implies $\|m\| \leqq |C|$, and since D is finite, we have $tt^* \leqq N 2^N \, |C|^2 \sum_{d \in D} dd^*$, where N is the number of elements in D. Hence if $t \in B$, then tt^* is bounded above by an element in $U^{(2k)} \cap P'$ which is independent of t, as required.

The converse, which we shall not need here, can be obtained by observing that every subset B with the properties of the theorem is necessarily bounded on bounded subsets of P, and hence on E.

It follows from this result that every positive linear functional on E' is bounded on the bounded subsets of E'. Hence we have.

COROLLARY 2. *Every positive linear functional defined on E' is necessarily continuous, i.e., is defined by an element of E.*

The moment problem. We are now in a position to take up the moment problem. First we recall that if φ is a C^∞ function of positive type on G, then φ defines a positive linear functional on E'. In particular, φ defines a linear functional on U which is positive on $U \cap P'$.

Suppose now that we have a linear functional on U. Under what conditions is it defined by a C^∞ function of positive type on G? A necessary condition is that the functional be positive on $U \cap P'$.

Our solution to the moment problem shows that this necessary condition is also sufficient. The proof is patter ned after refernce [1].

THEOREM 3. *Every linear functional on the algebra U which is positive on $U \cap P'$ admits an extension to a linear functional on the algebra E' which is positive on P', and hence is defined by a C^∞ function of positive type on G.*

Proof. Let θ be any linear functional on U which is positive on $U \cap P'$. Define the sublinear functional p on H' according to the formula $p(t) = \inf. \{\theta(d): d \in U \cap H'$ and $t \leq d\}$. Then from Lemma 1 we see that $p(t)$ is defined for all t in H', and p is obviously sublinear. Since $\theta(d) = p(d)$ for all d in $U \cap H'$, we know from the Hahn-Banach theorem that θ admits an extension to a linear functional θ^-, defined on all of H', such that $\theta^-(t) \leq p(t)$. If $-t \in P'$, then $p(t) \leq 0$, so $\theta^-(t) \leq 0$, and θ^- is positive on P'. By complex linearity θ^- now admits a unique extension to all of E'. According to Corollary 2, θ^- is automatically continuous on E', and hence is defined by an element φ of P: $\theta^-(t) = t(\varphi)$.

Unitary representations. To obtain a solution of the Hamburger moment problem in its original form when G is in the real line, we have only to pass to the dual group via the (inverse) Fourier transform. In fact, we can carry out this procedure whenever G is Abelian. In the general case, however, the dual group must be replaced with the dual space of all irreducible unitary representations of G.

Every continuous representation ρ of G as a group of unitary operators on a Hilbert space H defines a continuous *-representation ρ of E' as a *-algebra of unbounded operators on a certain invariant dense subspace D of H [8]. If an element t of E' is of positive type, then $(\rho(t)v,v) \geq 0$ for all $v \in D$, so that $\rho(t)$ is a positive operator on D. Hence if v is any vector in D, then the linear functional θ determined by v according to the formula $\theta(t) = (\rho(t)v,v)$ is positive on E', and so is defined by a function φ of positive type in E. Conversely, if φ is such a function, then we know that there exists a unitary representation ρ of G on a Hilbert space H, and a vector v in H such that the positive functional θ defined by φ on E' is determined by v [7]. This leads us to the following restatement of Theorem 3:

THEOREM 4. *Every linear functional on the algebra U which is positive on all elements whose representatives are positive in each irreducible unitary representation of G is determined by a vector in*

some unitary representation of G.

Not all *-representations of U are determined by unitary representations of G in the manner described above [8]. Those which are must evidently be completely reducible, and must represent elements of positive type by positive operators. Theorem 3 enables us to conclude that these necessary conditions are also sufficient:

THEOREM 5. *Every completely reducible *-representation of U in which elements of positive type are represented by positive operators is defined by a unitary represesentation of G in the manner described above.*

REMARKS. It is remarkable that the hypothesis of Theorem 3 apparently involves only the local structure of the group G, while the conclusion evidently involves the global structure. In fact, however, the global structure of G is reflected in the order structure of the associated enveloping algebra U. Thus, distinct locally isomorphic Lie groups, having the same Lie algebra, will define different positive cones in the enveloping algebra. Hence the notion of positive type depends upon the group G as well as upon the enveloping algebra U. This curious fact can be verified directly for the locally isomorphic Abelian groups R and $T = R/Z$.

It would be desirable for applications to obtain an intrinsic characterization of the elements of positive type in the enveloping algebra. We know that every element of the form tt^* lies in $U \cap P'$. If $G = R$, elements of this form exhaust $U \cap P'$. If $G = R^2$, however, convex combinations of elements of this form are not even dense in $U \cap P'$ [5, p. 232], and no reasonable characterization of $U \cap P'$ is known.

In general, the function of positive type obtained in Theorem 3 is not unique, even when $G = R$. In order to ensure uniqueness, we must require in addition that the linear functional on U be not too badly unbounded. The following criterion, familiar in the classical problem, can be derived from the work of Nelson [8, section 9].

Let $d_i \cdots, d_n$ denote here a complete set of generators for the Lie algebra $\mathfrak{g}$, and put $\varDelta = \varSigma d_i^2$. Then $\varDelta$ lies in $U \cap P'$, and if θ is any positive linear functional on U then the moments $\theta(\varDelta_n)$ of $\varDelta$ are all $\geqq 0$. If in addition these moments are all such that the series $\varSigma \lambda^n \theta(\varDelta^n)/n!$ converges for some real λ, then we say the functional θ is *analytic* on U. Every analytic positive functional on U admits a *unique* extension to an analytic positive functional on E' which is defined by an analytic function of positive type in E. [8].

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