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# SOME TRIPLE INTEGRAL EQUATIONS

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**In this paper we solve the triple integral equations**

$$(1)\qquad \mathfrak{M}^{-1}\left\{\frac{\Gamma(\xi+s/\delta)}{\Gamma(\xi+\beta+s/\delta)}\Phi(s);\ x\right\}=0,\ 0\leqq x<a,\ b<x<\infty\ ,$$

$$(2)\qquad \mathfrak{M}^{-1}\left\{\frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\Phi(s);\ x\right\}=f_2(x),\ a<x<b\ ,$$

**where** $\alpha, \beta, \xi, \eta, \delta>0, \sigma>0$, **are real parameters,** $f_2(x)$ **is a known function,** $\Phi(s)$ **is to be determined and**

$$(3)\qquad \mathfrak{M}\{h(x);\ s\}=H(s),\ \mathfrak{M}^{-1}\{H(s);\ x\}=h(x)\ ,$$

**denote the Mellin transform of** $h(x)$ **and its inversion formula respectively.**

The above equations are an extension of the dual integral equations solved in a recent paper by Erdélyi [2] by means of a systematic application of the Erdélyi-Kober operators of fractional integration [4].

Using the properties of some slightly extended forms of the Erdélyi-Kober operators we show, in a purely formal manner, that the solution of the triple integral equations can be expressed in terms of the solution of a Fredholm integral equation of the second kind. Srivastav and Parihar [5] have solved a very special case of the equations by a completely different method from that used in this paper. The method of solution employed here will be seen to follow closely that used by Cooke [1] to obtain the solution to some triple integral equations involving Bessel functions; indeed Cooke's equations may be regarded as a special case of equations (1) and (2) and it is shown that a solution of his equations can be readily obtained from that presented in this paper.

**2. The integral operators.** We shall use the integral operators defined by

$$(4)\qquad I_{\eta,\alpha}(a,\ x\colon\sigma)f(x)=\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x(x^\sigma-t^\sigma)^{\alpha-1}t^{\sigma(\eta+1)-1}f(t)dt\ ,\quad \alpha>0\ ,$$

$$(5)\qquad =\frac{x^{1-\sigma(\alpha+\eta+1)}}{\Gamma(1+\alpha)}\frac{d}{dx}\int_a^x(x^\sigma-t^\sigma)^{\alpha}t^{\sigma(\eta+1)-1}f(t)dt\ ,\quad -1<\alpha<0\ ,$$

$$(6)\qquad K_{\eta,\alpha}(x,b\colon\sigma)f(x)=\frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)}\int_x^b(t^\sigma-x^\sigma)^{\alpha-1}t^{\sigma(1-\alpha-\eta)-1}f(t)dt\ ,\quad \alpha>0\ ,$$

$$= -\frac{x^{\sigma(\eta-1)+1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_x^b (t^\sigma - x^\sigma)^\alpha t^{\sigma(1-\alpha-\eta)-1} f(t)dt\ , \quad -1 < \alpha < 0\ , \tag{7}$$

where $a < x < b$, $\sigma > 0$.

When $a = 0$, $b = \infty$, these become the extended form of the Erdélyi-Kober operators used in [2] and when $\sigma = 2$ they are the same as the operators defined by Cooke [1].

From the theory of Abel integral equations it follows that the inverse operators are given by

$$I^{-1}_{\eta,\alpha}(a, x: \sigma)f(x) = I_{\eta+\alpha,-\alpha}(a, x: \sigma)f(x)\ , \tag{8}$$

$$K^{-1}_{\eta,\alpha}(x, b: \sigma)f(x) = K_{\eta+\alpha,-\alpha}(x, b: \sigma)f(x)\ . \tag{9}$$

We shall also find it convenient to have expressions for integral operators of the type

$$L_{\eta,\alpha}(0, x: \sigma)f(x) = I^{-1}_{\eta,\alpha}(a, x: \sigma)\, I_{\eta,\alpha}(0, a: \sigma)f(x)\ , \qquad 0 < a < x\ , \tag{10}$$

$$M_{\eta,\alpha}(x, b: \sigma)f(x) = K^{-1}_{\eta,\alpha}(x, a: \sigma)\, K_{\eta,\alpha}(a, b: \sigma)f(x)\ , \qquad x < a < b\ . \tag{11}$$

When $0 < \alpha < 1$, we see on using the results (4), (5) and (8) that

$$L_{\eta,\alpha}(0, x: \sigma)f(x) = \frac{\sigma x^{1-\sigma(\eta+1)}}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x^\sigma - t^\sigma)^{-\alpha} t^{\sigma-1} dt \int_0^a (t^\sigma - u^\sigma)^{\alpha-1} u^{\sigma(\eta+1)-1} f(u)du\ .$$

Inverting the order of integration and using the result

$$\frac{d}{dx} \int_a^x \frac{t^{\sigma-1}dt}{(x^\sigma - t^\sigma)^\alpha (t^\sigma - u^\sigma)^{1-\alpha}} = \frac{x^{\sigma-1}(a^\sigma - u^\sigma)^\alpha}{(x^\sigma - u^\sigma)(x^\sigma - a^\sigma)^\alpha}\ , \quad u < a < x,\ 0 < \alpha < 1\ ,$$

we find

$$L_{\eta,\alpha}(0, x: \sigma)f(x) = \frac{\sigma \sin(\alpha\pi)}{\pi} \frac{x^{-\sigma\eta}}{(x^\sigma - a^\sigma)^\alpha} \int_0^a \frac{u^{\sigma(\eta+1)-1}(a^\sigma - u^\sigma)^\alpha}{x^\sigma - u^\sigma} f(u)du\ . \tag{12}$$

Similarly we can show that

$$M_{\eta,\alpha}(x, b: \sigma)f(x) = \frac{\sigma \sin(\alpha\pi)}{\pi} \frac{x^{\sigma(\alpha+\eta)}}{(a^\sigma - x^\sigma)^\alpha} \int_a^b \frac{u^{\sigma(1-\alpha-\eta)-1}(u^\sigma - a^\sigma)^\alpha}{u^\sigma - x^\sigma} f(u)du\ , \tag{13}$$

where $0 < \alpha < 1$.

When $-1 < \alpha < 0$, the formulae for $L_{\eta,\alpha}$ and $M_{\eta,\alpha}$ are exactly the same as those given by the above equations.

We also have the expressions

$$\begin{aligned}(14)\qquad & I_{\eta+\alpha,-\alpha}(0, a: \sigma) I_{\eta,\alpha}(0, x: \sigma)f(x) \\ &= [I^{-1}_{\eta,\alpha}(0, x: \sigma) - I^{-1}_{\eta,\alpha}(a, x: \sigma)] I_{\eta,\alpha}(0, x: \sigma)f(x) \\ &= f(x) - I^{-1}_{\eta,\alpha}(a, x: \sigma)[I_{\eta,\alpha}(0, a: \sigma) + I_{\eta,\alpha}(a, x: \sigma)]f(x) \\ &= -I^{-1}_{\eta,\alpha}(a, x: \sigma) I_{\eta,\alpha}(0, a: \sigma)f(x) = -L_{\eta,\alpha}(0, x: \sigma)f(x) ,\end{aligned}$$

$$(15)\qquad K_{\eta+\alpha,-\alpha}(a, b: \sigma) K_{\eta,\alpha}(x, b: \sigma)f(x) = -M_{\eta,\alpha}(x, b: \sigma)f(x) .$$

Two well known results [2] which play an important part in our solution are

$$(16)\qquad \mathfrak{M}(I_{\eta,\alpha}(0, x: \sigma)f(x); s\} = \frac{\Gamma(1 + \eta - s/\sigma)}{\Gamma(1 + \eta + \alpha - s/\sigma)} \mathfrak{M}\{f(x); s\} ,$$

$$(17)\qquad \mathfrak{M}\{K_{\eta,\alpha}(x, \infty: \sigma)f(x); s\} = \frac{\Gamma(\eta + s/\sigma)}{\Gamma(\eta + \alpha + s/\sigma)} \mathfrak{M}\{f(x); s\} .$$

In what follows we are concerned with three ranges of the variable $x$, namely

$$(18)\qquad I_1 = \{x: 0 \leqq x < a\}, I_2 = \{x: a < x < b\}, I_3 = \{x: b < x < \infty\} ,$$

and we shall write any function $f(x)$, $x \geqq 0$, in the form

$$(19)\qquad f(x) = \sum_{i=1}^{3} f_i(x) ,$$

where

$$(20)\qquad f_i(x) = \begin{cases} f(x), x \in I_i , \\ 0, \text{ otherwise} , \end{cases} \qquad i = 1, 2, 3 .$$

With these definitions it is easily seen that if we evaluate the equations

$$(21)\qquad g(x) = I_{\eta,\alpha}(0, x: \sigma)f(x), h(x) = K_{\eta,\alpha}(x, \infty: \sigma)f(x) ,$$

on the intervals $I_1$, $I_2$ and $I_3$ respectively, we get

$$\begin{aligned}(22)\qquad & g_1(x) = I_{\eta,\alpha}(0, x: \sigma)f_1(x) , \\ & h_1(x) = K_{\eta,\alpha}(x, a: \sigma)f_1(x) + K_{\eta,\alpha}(a, b: \sigma)f_2(x) + K_{\eta,\alpha}(b, \infty: \sigma)f_3(x) ,\end{aligned}$$

$$\begin{aligned}(23)\qquad & g_2(x) = I_{\eta,\alpha}(0, a: \sigma)f_1(x) + I_{\eta,\alpha}(a, x: \sigma)f_2(x) , \\ & h_2(x) = K_{\eta,\alpha}(x, b: \sigma)f_2(x) + K_{\eta,\alpha}(b, \infty: \sigma)f_3(x) ,\end{aligned}$$

$$\begin{aligned}(24)\qquad & g_3(x) = I_{\eta,\alpha}(0, a: \sigma)f_1(x) + I_{\eta,\alpha}(a, b: \sigma)f_2(x) + I_{\eta,\alpha}(b, x: \sigma)f_3(x) , \\ & h_3(x) = K_{\eta,\alpha}(x, \infty: \sigma)f_3(x) .\end{aligned}$$

**3. Solution of the integral equations.** Using the notation of equations (19) and (20) we can write the triple integral equations (1) and (2) as

$$\mathfrak{M}^{-1}\left\{\frac{\Gamma(\xi + s/\delta)}{\Gamma(\xi + \beta + s/\delta)}\Phi(s);\, x\right\} = g(x)\ , \tag{25}$$

$$\mathfrak{M}^{-1}\left\{\frac{\Gamma(1 + \eta - s/\sigma)}{\Gamma(1 + \eta + \alpha - s/\sigma)}\Phi(s);\, x\right\} = f(x)\ , \tag{26}$$

where $g_1 = g_3 = 0$, $f_2$ is given and $g_2, f_1$ and $f_3$ are unknown functions.

If we write

$$\Phi(s) = \mathfrak{M}\{\phi(x);\, s\}\ , \tag{27}$$

and use the formulae (16) and (17) we find that equations (25) and (26) assume the operational form

$$I_{\eta,\alpha}(0,\, x\colon \sigma)\,\phi(x) = f(x)\ , \tag{28}$$

$$K_{\xi,\beta}(x,\, \infty\colon \delta)\,\phi(x) = g(x)\ . \tag{29}$$

Using the formulae (8) and (9) and solving the above equations for $\phi(x)$ we obtain

$$\phi(x) = I_{\eta+\alpha,-\alpha}(0,\, x\colon \sigma)f(x) \tag{30}$$

$$= K_{\xi+\beta,-\beta}(x,\, \infty\colon \delta)\,g(x)\ . \tag{31}$$

Now remembering that $g_1 = g_3 = 0$, and using the relations (22), (23) and (24) to evaluate equation (28) on the interval $I_1$, equation (30) on $I_2$, equation (31) on $I_3$, equation (29) on $I_2$ and equation (31) on $I_1$, we arrive at the following results

$$f_1(x) = I_{\eta,\alpha}(0,\, x\colon \sigma)\,\phi_1(x)\ , \tag{32}$$

$$\phi_2(x) = I_{\eta+\alpha,-\alpha}(0,\, a\colon \sigma)f_1(x) + I^{-1}_{\eta,\alpha}(a,\, x\colon \sigma)f_2(x)\ , \tag{33}$$

$$\phi_3(x) = K^{-1}_{\xi,\beta}(x,\, \infty\colon \delta)\,g_3(x) = 0\ , \tag{34}$$

$$g_2(x) = K_{\xi,\beta}(x,\, b\colon \delta)\,\phi_2(x)\ , \tag{35}$$

$$\phi_1(x) = K_{\xi+\beta,-\beta}(a,\, b\colon \delta)\,g_2(x)\ . \tag{36}$$

After eliminating $f_1(x)$ between equations (32) and (33), and eliminating $g_2(x)$ between equations (35) and (36), we find that the functions $\phi_1(x)$ and $\phi_2(x)$ satisfy the pair of simultaneous integral equations

$$\phi_2(x) = -L_{\eta,\alpha}(0,\, x\colon \sigma)\,\phi_1(x) + I^{-1}_{\eta,\alpha}(a,\, x\colon \sigma)f_2(x)\ , \tag{37}$$

$$\phi_1(x) = -M_{\xi,\beta}(x,\, b\colon\delta)\,\phi_2(x)\ , \tag{38}$$

where we have used the formulae (14) and (15).

From these results it is easily seen that $\phi_2(x)$ can be determined from the Fredholm integral equation of the second kind

$$\phi_2(x) = L_{\eta,\alpha}(0, x\colon \sigma) M_{\xi,\beta}(x, b\colon \delta)\phi_2(x) + I^{-1}_{\eta,\alpha}(a, x\colon \sigma) f_2(x) \ . \tag{39}$$

The solution to the triple integral equations can then be obtained from equations (27), (34), (38) and (39).

As an example we consider the case when $0 < \alpha < 1$, and $-1 < \beta < 0$, or $0 < \beta < 1$; in this instance equation (39) when written out in detail is

$$\begin{aligned} &\phi_2(x) - \int_a^b \phi_2(u) S(x, u) du \\ &\quad = \frac{x^{1-\sigma(\eta+1)}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{t^{\sigma(\alpha+\eta+1)-1}}{(x^\sigma - t^\sigma)^\alpha} f_2(t) dt \ , \end{aligned} \tag{40}$$

where

$$\begin{aligned} S(x, u) = \frac{\sigma\delta}{\pi^2} \sin(\alpha\pi) \sin(\beta\pi) & \frac{x^{-\sigma\eta} u^{\delta(1-\beta-\xi)-1}}{(x^\sigma - a^\sigma)^\alpha (u^\delta - a^\delta)^{-\beta}} \\ & \int_0^a \frac{t^{\sigma(\eta+1)+\delta(\beta+\xi)-1} (a^\sigma - t^\sigma)^\alpha}{(x^\sigma - t^\sigma)(u^\delta - t^\delta)(a^\delta - t^\delta)^\beta} dt \ . \end{aligned} \tag{41}$$

**4. An application.** Certain mixed boundary value problems [4] may be reduced to the solution of triple integral equations of the type

$$\int_0^\infty \psi(u) J_{2p}(ux) du = 0 \ , \qquad 0 \leqq x < a \ , \quad b < x < \infty \ , \tag{42}$$

$$\int_0^\infty u^{-2n} \psi(u) J_{2q}(ux) du = F(x) \ , \qquad a < x < b \ , \tag{43}$$

where $J_{2p}(ux)$ is the Bessel function of the first kind of order $2p$, $F(x)$ is a prescribed function and $\psi(u)$ is to be determined. When $p = q$ these are the equations investigated by Cooke [1]. We now show, in a fairly straightforward manner, that the above equations can be transformed into equations of the type (1) and (2).

Denoting the Mellin transform of $\psi(u)$ by

$$\mathfrak{M}\{\psi(u); s\} = \Psi(s) \ , \tag{44}$$

and using the result [3]

$$\mathfrak{M}\{\xi^{-2n} J_{2q}(\xi); s\} = 2^{s-1-2n} \frac{\Gamma(q - n + s/2)}{\Gamma(1 + n + q - s/2)} \ , \tag{45}$$

we have, on applying the Faltung theorem for Mellin transforms [3], that the integral equations (42) and (43) can be written in the form

$$\mathfrak{M}^{-1}\left\{\frac{\Gamma(p+s/2)}{\Gamma(q-n+s/2)}\,\Phi(s);\,x\right\}=0\,,\quad 0\leqq x<a\,,\quad b<x<\infty\,, \tag{46}$$

$$\mathfrak{M}^{-1}\left\{\frac{\Gamma(1+p-s/2)}{\Gamma(1+n+q-s/2)}\,\Phi(s);\,x\right\}=2^{1+2n}x^{-2n}F(x)\,,\quad a<x<b\,, \tag{47}$$

where

$$\Phi(s)=2^{s}\frac{\Gamma(q-n+s/2)}{\Gamma(1+p-s/2)}\,\Psi(1-s)\,. \tag{48}$$

These are the same as equations (1) and (2) with

$$\begin{aligned}&\sigma=\delta=2,\,\xi=\eta=p,\,\alpha=q-p+n,\,\beta=q-p-n\,,\\&f_2(x)=2^{1+2n}x^{-2n}F(x)\,.\end{aligned} \tag{49}$$

Using the results of the previous section we have therefore that the solution of equations (46) and (47) can be found in terms of a function $\phi(x)$ by

$$\Phi(s)=\mathfrak{M}\{\phi(x);\,s\}\,, \tag{50}$$

where $\phi_3(x)=0$ and the functions $\phi_1(x)$ and $\phi_2(x)$ are obtained from equations (38) and (39) with the parameters $\xi,\eta$, etc. given by equations (49).

Finally, in order to find the solution of the integral equations (42) and (43) in terms of $\phi(x)$, we proceed in the following way.

From equation (44) we have that the solution is

$$\begin{aligned}\psi(u)&=\mathfrak{M}^{-1}\{\Psi(s);\,u\}\\&=\mathfrak{M}^{-1}\left\{2^{s-1}\frac{\Gamma(1/2+p+s/2)}{\Gamma(1/2+q-n-s/2)}\,\mathfrak{M}\{\phi(x);\,1-s\};\,u\right\}\,,\end{aligned} \tag{51}$$

on using equations (48) and (50). Inverting the order of integration in the last equation we get

$$\begin{aligned}\psi(u)&=\int_0^\infty\phi(x)\mathfrak{M}^{-1}\left\{2^{s-1}\frac{\Gamma(1/2+p+s/2)}{\Gamma(1/2+q-n-s/2)}\,;\,ux\right\}dx\\&=\int_0^\infty\left(\frac{ux}{2}\right)^{1+n+p-q}\phi(x)\,J_{p+q-n}(ux)dx\,,\end{aligned} \tag{52}$$

after applying the result (45). When $p=q$ this solution is exactly the same as that found by Cooke [1, pp. 61–62].

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