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BIHOLOMORPHIC MAPS IN HILBERT SPACE HAVE A FIXED POINT

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The results in this paper reveal a dichotomy in regard to the existence of fixed points for smooth real maps and biholomorphic maps in Hilbert space. Kakutani has shown that there exists a homeomorphism of the closed unit sphere of Hilbert space onto itself which has no fixed point. A slight modification of his example shows that there is a diffeomorphism having the same property. Our results show that in the complex case every biholomorphic map of the unit ball onto itself in Hilbert space has a fixed point.

A function h defined on an open subset D of a Banach space into a Banach space is called holomorphic in D if h has a Fréchet derivative at each point of D. Standard results about holomorphic maps may be found in [4]. By biholomorphic we mean a holomorphic map with a holomorphic inverse. It is a known result that in C^n an injective holomorphic map is biholomorphic. The corresponding result does not seem to be known in infinite dimensions even assuming the range is an open set.

Our proofs make use of some results obtained recently for holomorphic mappings in Banach spaces. One knows that in the plane every bijective holomorphic map of the unit disk which takes zero to zero is given by a rotation. In Hilbert Space the analogous result is that every biholomorphic map of the unit ball onto the unit ball which leaves zero fixed is given by a unitary operator. This result follows easily from the work of R. S. Phillips [6]. Also L. Harris [3] has obtained more general results in this direction. Our result is the following theorem for a complex Hilbert space H.

Theorem: Suppose $B = \{z \in H: ||z|| < 1\}$ and h is a biholomorphic map of B onto B. Then h is biholomorphic in a larger region, maps $\bar{B}$ onto $\bar{B}$, and has a fixed point in $\bar{B}$.

In § 2 we give a proof of the above result and in § 3 we show that the fixed points are either isolated points or the fixed point sets are affine subspaces.

2. *Proof*. Let H be a complex Hilbert space and

$$B = \{z \in H: ||z|| < 1\} .$$

The first lemma can be obtained by an argument similar to that used by Phillips [6] or from the work of Harris [3].

LEMMA. (Phillips, Harris) *If T maps B biholomorphically onto B and $T(0) = 0$, then T is unitary.*

We now obtain an explicit representation of all biholomorphic maps of B onto B. Let $\{e_\alpha\}_{\alpha \in I}$ be an orthonormal basis in H and let $\alpha_0 \in I$ be fixed. Define: $f: B \to H$ by

$$f\Big(ze_{\alpha_0} + \sum_{\alpha \neq \alpha_0} z_\alpha e_\alpha\Big) = \frac{z - \beta}{1 - \bar{\beta} z} e_{\alpha_0} + \frac{\sqrt{1 - |\beta|^2}}{1 - \bar{\beta} z} \sum_{\alpha \neq \alpha_0} z_\alpha e_\alpha \tag{1}$$

where β is a fixed complex number, $|\beta| < 1$.

THEOREM 1. *Suppose $h: B \to B$ is a biholomorphic map of B onto B, with $h(x^0) = 0$. Then $h = T \circ f \circ S$ where T and S are unitary operators and f is defined by (1) with $|\beta| = \|x^0\|$. Therefore h is biholomorphic in $\{x \in H: \|x\| < 1/\|x^\circ\|\}$ and h has a fixed point in B.*

Proof. Let us first show that f is $1 - 1$ and onto B. Suppose that $\|x\| = r < 1$ and $x = ze_{\alpha_0} + \sum_{\alpha \neq \alpha_0} z_\alpha e_\alpha$. Then

$$\begin{aligned}\|f(x)\|^2 &= [|z - \beta|^2 + (1 - |\beta|^2)(\sum_{\alpha \neq \alpha_0} |z_\alpha|^2)]/|1 - \bar{\beta} z|^2 \\ &= 1 - (1 - |\beta|^2)(1 - r^2)(|1 - \bar{\beta} z|^2)^{-1} < 1 .\end{aligned}$$

Hence $f(B) \subset B$.

Let

$$g(x) = \frac{z + \beta}{1 + \bar{\beta} z} e_{\alpha_0} + \frac{\sqrt{1 - |\beta|^2}}{1 + \bar{\beta} z} \sum_{\alpha \neq \alpha_0} z_\alpha e_\alpha .$$

Then $g \circ f = f \circ g = I$ (the identity map). Also $g(B) \subset B$ by repeating the above argument for f with β replaced by $-\beta$. Hence f is $1 - 1$ and onto. A rather tedious but straight forward argument shows that the Frechét derivative $Df(x, \circ)$ of f at x is given by:

$$\begin{aligned}Df(x; y) &= \frac{1 - |\beta|^2}{(1 - \bar{\beta} z)^2} y_{\alpha_0} e_{\alpha_0} \\ &\quad + \sum_{\alpha \neq \alpha_0} \left[\frac{\bar{\beta}\sqrt{1 - |\beta|^2}}{(1 - \bar{\beta} z)^2} z_\alpha y_{\alpha_0} + \frac{\sqrt{1 - |\beta|^2}}{1 - \bar{\beta} z} y_\alpha \right] e_\alpha\end{aligned}$$

where $y = y_{\alpha_0} e_{\alpha_0} + \sum_{\alpha \neq \alpha_0} y_\alpha e_\alpha$, for all $\|x\| < 1/|\beta|$. Similarly one may show that f^{-1} is holomorphic. We now show that $h = T \circ f \circ S$. Let $\beta = \|h(0)\|$ and let T be a unitary operator such that $T(-\beta e_{\alpha_0}) = h(0)$. Since $f^{-1}(-\beta e_{\alpha_0}) = 0$, then the map $f^{-1} \circ T^{-1} \circ h$ is a biholomorphic map of B onto B with fixed origin. Hence by the Lemma, $f^{-1} \circ T^{-1} \circ h$ is a unitary operator and we have the desired representation.

We now show that h has a fixed point in $\bar{B}$.

Our original proof made use of the following interesting result of Earle and Hamilton [2].

THEOREM. *If D is a bounded connected open subset of a complex Banach space then any holomorphic map g from D strictly inside D*(i.e., $\exists\varepsilon > 0$ *such that* $\|g(x) - y\| > \varepsilon$ *for all* $x \in D, y \notin D$) *has a fixed point.*

However J. W. Helton noted that our mapping was weakly continuous and the following direct proof is due to him.

Suppose $x_k \xrightarrow{w} x_0$, where $x_k = z^{(k)}e_{\alpha_0} + \sum_{\alpha\neq\alpha_0} z_\alpha^{(k)} e_\alpha$ is a net in B, and $x_0 = z^{(0)}e_{\alpha_0} + \sum_{\alpha\neq\alpha_0} z_\alpha^{(0)} e_\alpha$. We wish to show that the net $f(x_k) \xrightarrow{w} f(x_0)$. Let $y \in H$, $y = y_0 e_{\alpha_0} + \sum_{\alpha\neq\alpha_0} y_\alpha e_\alpha$, then

$$< y, f(x_k) > = y_0\left(\frac{\bar{z}^{(k)} - \bar{\beta}}{1 - \beta\bar{z}^{(k)}}\right) + \frac{\sqrt{1 - |\beta|^2}}{1 - \beta\bar{z}^{(k)}} \sum_{\alpha\neq\alpha_0} y_\alpha \bar{z}_\alpha^{(k)} .$$

Since $x_k \xrightarrow{w} x_0$ we have that

$$\frac{\bar{z}^{(k)} - \bar{\beta}}{1 - \beta\bar{z}^{(k)}} \longrightarrow \frac{\bar{z}^0 - \bar{\beta}}{1 - \beta\bar{z}^0} \text{ and } \sum_{\alpha\neq\alpha_0} y_\alpha \bar{z}_\alpha^{(k)} \longrightarrow \sum_{\alpha\neq\alpha_0} y_\alpha \bar{z}_\alpha^0$$

hence f is weakly continuous. Since S and T are weakly continuous, h is weakly continuous in a region containing $\bar{B}$. Thus by an application of the Schauder-Tychnoff Theorem h has a fixed point in $\bar{B}$ since $\bar{B}$ is weakly compact.

3. Description of the fixed point sets. We thank the referee for pointing out that our original results in this section could be extended to infinite dimensions in the following simple way.

Suppose as before that B is the open unit ball in Hilbert space and that $h: B \to B$ is a biholomorphic map of B onto B. An affine subspace of B or its closure means the intersection of B or its closure with a closed complex affine subspace of H. The following result then follows easily from our representation of h.

THEOREM 2. *Every biholomorphic map of B preserves affine subspaces.*

Now suppose h fixes a point y in B. Let g be a biholomorphic map which sends y to 0. Then ghg^{-1} fixes 0, so it is linear and its fixed point set is the affine subspace C. But the fixed point set of h is $g^{-1}(C)$ which is also affine by Theorem 2. Hence

THEOREM 3. *If $h: B \to B$ is biholomorphic and has a fixed point in B, its fixed point set is an affine subspace.*

Finally, suppose h has no fixed point in B. It must have at least one fixed point on the boundary. If there are two fixed points, h must leave invariant the one-dimensional affine subspace C which contains them. Choose a biholomorphic map g so that $g(C)$ contains the origin. The map $h_0 = ghg^{-1}$ then maps the one-dimensional subspace $g(C)$ onto itself, fixing two boundary points. Denote one of these fixed points by x_0. Then $h_0 = Uf$, where U is unitary and

$$f(x) = \frac{(< x, x_0 > - \beta)x_0 + (1 - |\beta|^2)^{1/2}(x - < x, x_0 > x_0)}{1 - \bar{\beta} < x, x_0 >}\,.$$

It is easy to see that h_0 has only two fixed points, proving

THEOREM 4. *If $h: B \to B$ is biholomorphic and has no fixed point in B, then its fixed point set in B closure consists of either one point or two points.*

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