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THE TRANSLATION GROUPS OF n-UNIFORM TRANSLATION HJELMSLEV PLANES

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The purpose of this paper is twofold: first, to determine the full translation groups for all n-uniform translation affine Hjelmslev planes for all positive integers n; and second, to prove that all such groups occur as the full translation groups of Pappian Hjelmslev Planes.

1. Introduction. For brevity's sake, we introduce the following three conventions: Hjelmslev plane will be abbreviated to H-plane; we will always mean affine (rather than projective) when we write translation H-plane; and throughout the paper, translation group will denote the group of *all* translations. H. Lüneburg has previously defined translation H-planes [7] and has determined the translation groups of all uniform translation H-planes [7, Satz 8.3]. The author has defined a class of finite H-planes called n-uniform H-planes in such a way that the finite uniform H-planes are just the n-uniform H-planes with $n = 1$ and 2 [3]. In § 2, we prove (see Theorem 2.6.) that only certain groups can occur as translation groups of n-uniform translation H-planes; and in § 3, we establish the converse. As algebraic corollaries to the geometric theorem of § 2, we obtain results on the additive structures of the finite Desarguesian H-rings. (See Corollary 3.1 and Remark 3.3.). This is possible, because every Desarguesian H-ring coordinatizes a Desarguesian affine H-plane, because every Desarguesian affine H-plane is a translation H-plane, and because every finite Desarguesian H-plane is n-uniform for some n.

In § 3, we quote a result of W. E. Clark and the author on the additive structure of finite commutative Desarguesian H-rings; we use this result to show that all groups permitted by Theorem 2.6 do in fact occur as translation groups of Pappian affine H-planes. Then every translation group of an n-uniform translation H-plane A is isomorphic to the translation group of a Pappian affine H-plane B. One may always take B to have the same invariants as A. One may also always choose B so that its associated ordinary affine plane has prime order.

2. The translation groups of finite translation H-planes. The reader is referred to P. Dembowski [2] or to the papers in the bibliography for definitions of affine and projective H-planes. We will write $P \sim Q$, $g \not\sim h$, etc., to mean the point P is neighbor to Q, the line g is not neighbor to h, etc. Associated with every finite affine or projective H-plane are two invariants denoted by s and t. We may take t to be the number of lines through a point P which are neighbor to the line g where (P, g) is an arbitrary flag of the H-plane; then $s + t$ will denote the total number of lines incident with P. It is well known that s/t is the order of the ordinary affine or projective plane associated with the H-plane. (See [4] and [7].)

DEFINITION 2.1. Let P be a point of an H-plane π. We define $\bar{P}$ to be the following incidence structure. The points of $\bar{P}$ are the points Q of π such that $Q \sim P$. The lines of $\bar{P}$ are the nonempty point sets $l^P = l \cap \bar{P}$, l being a line of π. Incidence is given by inclusion.

DEFINITION 2.2. We define a 1-*uniform* affine (projective) H-plane to be a finite ordinary affine (projective) plane. We call a finite affine or projective H-plane n-*uniform* ($n \geqq 2$) provided that

(a) $\bar{P}$ is an $(n-1)$-uniform affine H-plane for each point P in π.

(b) For each $\bar{P}$, every line l^P is the restriction of the same number of lines from π.

The following result is part of [3, Proposition 2.2]. The reader should thoroughly acquaint himself with the content of this proposition as it will be used frequently in the rest of the paper.

PROPOSITION 2.1. *Let π be an n-uniform projective or affine H-plane. Then π satisfies the following properties:*

(1) *If $r = s/t$, then $s = r^n$ and $t = r^{n-1}$.*

(2) *Distinct intersecting neighbor lines of π meet in r^i points for some integer i such that $1 \leqq i \leqq n - 1$.*

(3) *The dual of* (2) *holds in π.*

(4) *If $P \in h$, the number of lines through P which intersect h in r^i or more points is r^{n-i} for $i = 1, 2, \cdots, n$.*

(5) *The dual of* (4) *holds in π.*

We write "$P(\cong i)Q$" and read "P is i-equivalent to Q" to mean P is joined to Q by exactly r^i lines; we write "$P(\sim i)Q$" and read "P is at least i-equivalent to Q" to mean P is joined to Q by r^i or more lines.

(6) *$(\sim i)$ is an equivalence relation on points for $i = 0, 1, \cdots, n$.*

(7) *The following conditions imply $|l \cap k| > 1$: $R, Q \in l$; $R, S \in k$; $R(\cong i)Q$; $Q(\sim i + 1)S$; i is a nonnegative integer $< n$.*

(8) *If P is any point of π, the number of points Q of π such that $Q(\sim i)P$ is $r^{2(n-i)}$ for $i = 1, 2, \cdots, n$.*

In light of Proposition 2.1 (1), an n-uniform H-plane may be thought of as having three invariants r, s, and t. However, s and t are determined by r and n; and thus, we shall write *the* invariant of an n-uniform H-plane π to refer to r. Since $r = s/t$, the invariant of π is the order of the ordinary affine or projective plane associated with π. Next we prove

LEMMA 2.2. *Let P, Q, R be points of an n-uniform H-plane which satisfy $P, Q \in g$; $P, R \in h$; $Q, R \in k$. Further suppose $P(\cong i)Q(\cong i)R(\cong i)P$, $i < n$, and $g \not\sim h$. Then $h \not\sim k \not\sim g$.*

Proof. Proposition 2.1 (5) implies the number of points X such that $X \in g$ and $X(\sim i + 1)P$ is r^{n-i-1}. By (7), any line joining R to such an X is neighbor to h, hence not neighbor to g. Then no line joins R to two such X. By (6), the number of lines joining R to each such X is r^i. Then the number of lines joining R to all such X is $r^{n-1} = t$. Thus all lines through R which are neighbor to h meet g in points X which satisfy $X(\sim i + 1)P$. Then $k \not\sim h$, and by symmetry $k \not\sim g$.

To state the next several lemmas, we need some notation and a definition. We will write ${}^iP$ to denote $\{Q : Q(\sim n - i)P\}$. Thus ${}^0P = \{P\}$ and ${}^nP$ is the set of all points of the H-plane.

DEFINITION 2.3. A mapping σ defined on the point set of an affine H-plane is called a *dilatation* if the following condition is satisfied: $P, Q \in g$; $(P)\sigma \in h$; $g \| h$ imply $(Q)\sigma \in h$.

LEMMA 2.3. *Let σ be a dilatation of an n-uniform affine H-plane. Let P, Q, R, T be points such that $Q(\cong j)P(\cong j)R$, $T(\sim j + 1)P$, and $(P)\sigma(\cong i + j)(Q)\sigma$. Then*

(a) *$(R)\sigma(\cong i + j)(P)\sigma$, and*

(b) *$(T)\sigma(\sim i + j + 1)(P)\sigma$ if $i + j < n$.*

Proof. Let g be any line through P and Q, h be any line through P which is not neighbor to g. We first prove the lemma for all $R, T \in h$ such that $R(\cong j)P$, $T(\sim j+1)P$. We have $R, T(\sim j)Q$. Since $h \not\sim g$, Proposition 2.1 (7) implies $R(\cong j)Q$. Let k be any line through R ane Q, m be any line through T and Q. By Lemma 2.2, $|h \cap k| = 1$. Let g', h' be the lines through $(P)\sigma$ parallel respectively to g, h; let k', m' be the lines through $(Q)\sigma$ parallel respectively to k, m. In one form of the definition of affine H-planes (See [7] or [3], not [2].), the following condition is assumed: $|h \cap k| = 1$ and $k \| k'$ imply $|h \cap k'| = 1$. Then also $|h' \cap k'| = 1$. Similarly, $|h' \cap g'| = 1$; and since $|m \cap g| > 1$, $|m' \cap g'| > 1$. Since $(P)\sigma(\cong i + j)(Q)\sigma$ and $h' \not\sim k'$, Proposition 2.1 (7) implies $(R)\sigma(\sim i+j)(P)\sigma$. Since $m' \sim g' \not\sim h'$, the same argument implies $(T)\sigma(\sim i+j)(P)\sigma$. We have $(R)\sigma(\cong i+j)(P)\sigma$, for otherwise the above argument would yield $(Q)\sigma(\sim i+j+1)(P)\sigma$. Next, suppose $(T)\sigma(\cong i+j)(P)\sigma$. Then since $h' \not\sim g'$, $(T)\sigma(\cong i+j)(Q)\sigma$. If $i+j<n$, Lemma 2.2 implies $m' \not\sim g'$. By the contradiction, we conclude that $(T)\sigma(\sim i+j+1)(P)\sigma$. To see that the conclusions of the lemma hold for points R and T on a line h through P such that $h \sim g$, we apply the above results, replacing g and Q by g^* and Q^* where $Q^* \in g^* \not\sim g$ and $Q^*(\cong j)P$. (The existence of such a point $Q^* \in g^*$ is assured by Proposition 2.1 (5).)

LEMMA 2.4. *Let σ be a dilatation of an n-uniform affine H-plane. Let $(P)\sigma(\cong i)(Q)\sigma$ for nonneighbor points P, Q. Then if $k \leqq n-i$, $({}^{n-k}P)\sigma = {}^{n-k-i}((P)\sigma)$.*

Proof. Taking $j=0$ in Lemma 2.3 yields $({}^{n}P)\sigma \subset {}^{n-i}((P)\sigma)$ and $({}^{n-1}P)\sigma \subset {}^{n-i-1}((P)\sigma)$. It follows from Proposition 2.1 (8) that for each $k=0,1,\cdots,n-1$, there exists a point R_k such that $R_k(\cong k)P$. Using induction and Lemma 2.3, we get $({}^{n-k}P)\sigma \subset {}^{n-i-k}((P)\sigma)$ for all $k \leqq n-i$. If we can prove that the last containment is equality when $k=0$, then another induction proof using Lemma 2.3 will yield the full conclusion of Lemma 2.4. Thus it suffices to prove that ${}^{n-i}((P)\sigma) \subset \text{Image}\,(\sigma)$.

We let g denote the line joining P and Q; g', the line through $(P)\sigma$ which is parallel to g. Let h' be any line through $(P)\sigma$ not neighbor to g', and let R' be any point of h' satisfying $R'(\sim i)(P)\sigma$. Let k' be any line joining R' to $(Q)\sigma$. Since $(Q)\sigma(\cong i)(P)\sigma$ and $R'(\sim i)(P)\sigma$, $R'(\sim i)(Q)\sigma$. Since $h' \not\sim g'$, $R'(\cong i)(Q)\sigma$. If $R'(\sim i+1)(P)\sigma$, then $k' \sim g'$; hence $k' \not\sim h'$. If $R'(\cong i)(P)\sigma$, Lemma 2.2 implies $k' \not\sim h'$. Then in all cases $|k' \cap h'| = 1$. Let h be the line through P which is parallel to h', k be the line through Q which is parallel to k'. Then $|k \cap h| = 1$. If $\{R\}$ is $k \cap h$, then $(R)\sigma = R'$. To see that Image (σ) contains points R' on lines $h' \sim g'$, repeat the above argument using (in place of g and Q) a line g^* through P such that $g^* \not\sim g$ and a point Q^* on g^* such that $Q^* \not\sim P$. Lemma 2.3 implies $(P)\sigma(\cong i)(Q^*)\sigma$. Since $g^* \not\sim g$, a previous argument implies $(g^*)' \not\sim g'$, hence $(g^*)' \not\sim h'$. (Here $(g^*)'$ denotes the line through $(P)\sigma$ which is parallel to g^*.) Then ${}^{n-i}((P)\sigma) \subset \text{Image}\,(\sigma)$, and the proof of the lemma is complete.

Lemmas 2.3 and 2.4 combine to yield

PROPOSITION 2.5. *Let σ be a dilatation of an n-uniform affine H-plane. Let $P(\cong j)Q$ and $(P)\sigma(\cong i+j)(Q)\sigma$ for some $j<n-i$.*

Then for all $k \leqq n - i$, $({}^{n-k}P)\sigma = {}^{n-k-i}((P)\sigma)$.

The reader is referred to [7] or [2] for the definition of translation H-planes and for the results on translation H-planes which we quote and use below. If π is a set of subgroups (called *components*) of the group T, $J(T, \pi)$ denotes the incidence structure with parallel relation defined as follows: the points are the elements of T; the lines are the right cosets of the components; incidence is given by inclusion; and lines are parallel if and only if they are cosets of the same component of π. If A is any translation H-plane and if T^* is the translation group of A, then T^* is abelian and there exist T, π such that $A \cong J(T, \pi)$ and $T^* \cong T$. Every element t^* of T^* may be defined on T by $(x)t^* = x + t$ for all $x \in T$, some fixed $t \in T$. If $J(T, \pi)$ is an affine H-plane A and if T is abelian, then A is a translation H-plane with translation group isomorphic to T. Finally, we note that the invariant of A must be a prime power, since the ordinary affine plane associated with A is a translation plane.

THEOREM 2.6. *Let A be an n-uniform translation H-plane with invariant $r = p^x$ and translation group T^*. Then there exist nonnegative integers k_1, k_2, j such that T^* is the direct sum of $2xk_1$ cyclic subgroups of order p^j and of $2xk_2$ cyclic subgroups of order p^{j+1}.*

Proof. We represent A by $J(T, \pi)$ where $T \cong T^*$. Let ${}^iT$ denote the set of all elements of T in ${}^i0$. Let $\tau \in {}^iT$, τ^* denote the translation which adds τ to each element of T. Then all lines connecting 0 and τ are "traces" of τ^*, i.e., are fixed by τ^*. Then if $\beta \in T$, all lines through β parallel to these traces are also traces of τ^*, hence contain $(\beta)\tau^*$. Then $\beta(\sim n - i)(\beta)\tau^*$; and if $\beta \in {}^iT$, $\tau + \beta = (\beta)\tau^* \in {}^iT$. Then ${}^iT$ is a subgroup of T. Let ${}^i\pi$ denote the set of all intersections of ${}^iT$ with components of π. Then ${}^i0$ is isomorphic to $J({}^iT, {}^i\pi)$. Since A is n-uniform, ${}^i0$ is an i-uniform affine H-plane; since ${}^iT$ is an abelian group, ${}^i0$ is a translation H-plane.

We prove the theorem by induction on n. The 1-uniform translation H-planes are just the finite translation planes, and it is well known that such planes have elementary abelian translation groups. Since the order of the translation group of such a plane equals r^2, the number of points in the plane, the theorem is satisfied with $j = 1 = k_1$ and $k_2 = 0$. Now let A be an n-uniform translation H-plane with $n > 1$. By the induction hypothesis, ${}^{n-1}T$ is the direct sum of $2xk_1$ cyclic subgroups of order p^j and of $2xk_2$ cyclic subgroups of order p^{j+1} for suitable k_1, k_2, j. We may assume $j > 0, k_1 > 0$. Let σ be the dilatation of A defined by $(\beta)\sigma = p\beta$ for all $\beta \in T$. By Lemma 2.4, Image $(\sigma) = {}^i0$ for some $i < n$. If $i = 0$, T is elementary abelian. The theorem is then satisfied with $j=1$, $k_1=n$, $k_2=0$, since the number of points in A is r^{2n}.

Henceforth, we assume $i>0$. Since $i<n$, we may apply the induction assumption to ${}^iT$ and conclude that T is a p-group even for $i \neq 0$. If $i \neq 0$, Lemma 2.4 implies that $p({}^{n-1}T) = {}^{i-1}T$. Then ${}^{i-1}T$ is the direct sum of $2xk_1$ cyclic subgroups of order p^{j-1} and $2xk_2$ cyclic subgroups of order p^j.

Now $o({}^iT) = p^{2x} \cdot o({}^{i-1}T)$, and $o(T) = p^{2x} \cdot o({}^{n-1}T)$. Thus, letting σ^* denote the restriction of σ to ${}^{n-1}T$, we see that Ker (σ^*) and Ker (σ) have the same order. Then T and ${}^{n-1}T$ both have the same number $k = 2x(k_1+k_2)$ of summands. By counting elements of order p, we see that, in general, no p-group may have fewer summands than any of its subgroups. ${}^iT$ and ${}^{i-1}T$ also have k summands unless $j=1$. Assume $j=1$ so that ${}^{i-1}T$ is the direct sum of $2xk_2$ cyclic subgroups of order p. Applying the induction assumption to ${}^iT$ and observing that $o({}^iT) = p^{2x} \cdot o({}^{i-1}T)$, we see that either

(2.1) ${}^iT$ is the direct sum of $2x(k_2+1)$ cyclic subgroups of order p,

or

(2.2) ${}^iT$ is the direct sum of $2x(k_2-1)$ cyclic subgroups of order p and of $2x$ cyclic subgroups of order p^2.

Assume that (2.2) is satisfied, and apply the induction assumption to ${}^{i+1}T, {}^{i+2}T, \cdots, {}^{n-2}T$. Since ${}^{n-1}T$ has more summands than ${}^iT$, there is an integer l such that $0 \leqq l < n-1$ and ${}^{i+l+1}T$ is isomorphic to the direct sum of ${}^{i+l}T$ and of $2x$ cyclic subgroups of order p. Then $({}^{i+l+1}T)\sigma = ({}^{i+l}T)\sigma \neq 0$ which contradicts Lemma 2.4. We conclude that (2.1) is the only possibility for ${}^iT$ when $j=1$.

If $j>1$, applying the induction assumption to ${}^iT$, we see that

(2.3) ${}^iT$ must be the direct sum of $2x(k_1-1)$ cyclic subgroups of order p^{j-1} and of $2x(k_2+1)$ cyclic subgroups of order p^j.

Since (2.1) is just a degenerate case of (2.3), we see that (2.3) must be satisfied whenever T is not elemetary abelian. Then if T is not elementary abelian, T must contain a subgroup S which is the direct sum of $2x(k_1-1)$ cyclic subgroups of order p^j and $2x(k_2+1)$ cyclic subgroups of order p^{j+1}. Since $o(T)=o(S)$, $T=S$, and the proof is complete.

We have also proved

LEMMA 2.7. *For all $m \leqq n$, either ${}^mT$ is elementary abelian, or else there exist nonnegative integers j, k_1, k_2, x such that*

(a) *${}^{m-1}T$ is the direct sum of $2xk_1$ cyclic subgroups of order p^j and of $2xk_2$ cyclic subgroups of order p^{j+1};*

(b) *${}^{m}T$ is the direct sum of $2x(k_1 - 1)$ cyclic subgroups of order p^j and of $2x\ (k_2 + 1)$ cyclic subgroups of order p^{j+1}.*

We now use Lemma 2.7 to obtain the following improvement of Theorem 2.6.

THEOREM 2.6A. *Let $A = J(T, \pi)$ be an n-uniform translation H-plane with invariant $r = p^x$ and translation group isomorphic to T. Then there exist integers l, k with $0 \leqq l < k$ and subgroups C_i of T which satisfy the following conditions:*

(a) $T = C_1 \oplus \cdots \oplus C_k$;

(b) *for $i \leqq l$, C_i is the direct sum of $2x$ cyclic subgroups of order p^{j+1};*

(c) *for $i > l$, C_i is the direct sum of $2x$ cyclic subgroups of order p^j;*

(d) *for $i \leqq n = kj + l$,*

$$ {}^{i}0 = p^{q+1}\cdot(C_1 \oplus \cdots \oplus C_e) \oplus p^{q}\cdot(C_{e+1} \oplus \cdots \oplus C_k) $$

where q, e are given by $n - i = kq + e$, $0 \leqq e < k$.

Proof. By Theorem 2.6, we have that T is the direct sum of $2xk_1$ cyclic subgroups of order p^j and $2xk_2$ cyclic subgroups of order p^{j+1}. Set $k = k_1 + k_2$. Using Lemma 2.7 and Proposition 2.1 (8), it is easy to see that for $m \leqq k$,

$$ {}^{m}T = D_1 \oplus \cdots \oplus D_m $$

where each D_i is the direct sum of $2x$ cyclic subgroups of order p. By Lemma 2.4, there exists an integer c such that $p^bT = {}^{b-c}T$ for all $b \geqq c$. Clearly, $c = k$. Assume that for some m with $0 \leqq m < \min(k, n - k)$, there exist subgroups E_i of T satisfying

(2.4) ${}^{m+k}T = E_1 \oplus \cdots \oplus E_m \oplus D_{m+1} \oplus \cdots \oplus D_k$;

(2.5) $pE_i = D_i$ for $1 \leqq i \leqq m$;

(2.6) each E_i is the direct sum of $2x$ cyclic groups of order p^2.

Certainly the above requirements are satisfied for $m=0$. Let $\{d_i\colon 1 \leqq i \leqq 2x\}$ be a basis for D_{m+1}. Since $p({}^{m+k+1}T) = {}^{m+1}T \supset D_{m+1}$, there exist $e_i \in {}^{m+k+1}T$ satisfying $pe_i = d_i$. Let E_{m+1} be the group generated by $\{e_i\}$. Suppose e is an element of

$$ E_{m+1} \cap (E_1 \oplus \cdots \oplus E_m \oplus D_{m+2} \oplus \cdots \oplus D_k) . $$

Then by (2.5),

$$ pe \in D_{m+1} \cap (D_1 \oplus \cdots \oplus D_m) . $$

Then $pe = 0$; hence

$$e \in D_{m+1} \cap (D_1 \oplus \cdots \oplus D_m \oplus D_{m+2} \oplus \cdots \oplus D_k) .$$

Then $e = 0$. We have proved that the sum

$$E = E_1 + \cdots + E_{m+1} + D_{m+2} + \cdots + D_k$$

is direct. We know $E \subset {}^{m+k+1}T$, and Proposition 2.1 (8) implies $E = {}^{m+k+1}T$. It is now easy to see that (2.4) – (2.6) are all satisfied if we substitute $m + 1$ for m.

Proceeding in this manner, we eventually obtain

$$T = F_1 \oplus \cdots \oplus F_k$$

where F_i is the direct sum of $2x$ cyclic groups of order p^{j+1} if $i \leqq k_2$ and of $2x$ cyclic groups of order p^j when $i > k_2$. The result now follows from setting $l = k_2$, $C_i = F_{l+1-i}$ for $1 \leqq i \leqq l$ and $C_i = F_{k+l+1-i}$ for $l < i \leqq k$. We may assure $l < k$ by changing the value of j if necessary.

3. The translation groups of finite Desarguesian affine *H*-planes.

The reader is referred to Klingenberg [5], [6] or Dembowski [2] for the definition of Desarguesian and Pappian affine H-planes as well as for all the results on such planes stated below. We do repeat the following definition.

DEFINITION 3.1. A Desarguesian H-ring (henceforth abbreviated to H-ring) is an associative ring with identity which satisfies the following three conditions:

(a) Every divisor of zero is a two-sided divisor of zero, and the set N of divisors of zero is an ideal.

(b) Every nondivisor of zero has an inverse.

(c) If $n, m \in N$, then there is an $h \in H$ such that $nh = m$ or $n = mh$; and there is a $k \in H$ such that $kn = m$ or $n = km$.

If H denotes an H-ring, then Klingenberg defined [6] an incidence structure $\sum_p (H)$ as follows: points are left "homogeneous triples" of elements of H; lines are right "homogeneous triples"; a point and line are incident if and only if the inner products of their corresponding triples are zero. Klingenberg proved [6, S 28, S 29, proof of S 29] that $\sum_p (H)$ is a projective H-plane whose affine H-planes are all isomorphic Desarguesian affine H-planes with translation groups isomorphic to $H^+ \oplus H^+$. The affine H-planes belonging to $\sum_p (H)$ are themselves coordinatizable (in an affine manner) by the ring H and Klingenberg denotes such an affine H-plane by $\sum_a (H)$. Call a projective H-plane P Desarguesian if and only if P is isomorphic to $\sum_p (H)$ for some H-ring H.

By definition, all affine Desarguesian H-planes are translation H-planes. The author has proved [3, Theorem 5.4] that all finite $\sum_p (H)$ and hence also all finite $\sum_a (H)$ are n-uniform for various n. Let $\sum_a (H)$ or $\sum_p (H)$ be n-uniform with invariants r, s, t. Then $o(H) = s = r^n$ and $o(N) = t = r^{n-1}$ (See [3, Lemma 5.1].) It is clear from [3, Theorem 5.3 and Lemma 5.2 (1)] that $o(N^i) = r^{n-i}$ for $1 \leqq i \leqq n$. In particular, N is nilpotent of degree n. We are now in a position to state and prove the following algebraic corollary to Theorem 2.6.

COROLLARY 3.1. *Let H be a finite H-ring with radical N. Let r^* denote $o(H/N)$. Then r^* is a prime power p^x. H^+ is the direct sum of xk_1 cyclic subgroups of order p^j and of xk_2 cyclic subgroups of order p^{j+1} for some nonnegative integers k_1, k_2, j.*

Proof. Since H is a finite H-ring, $\sum_a (H)$ is an n-uniform translation H-plane. Since $r^* = o(H/N)$, r^* is the invariant of $\sum_a (H)$; hence r^* is a prime power. The result now follows from Theorem 2.6 and the previous observation that the translation group of $\sum_a(H)$ is isomorphic to $H^+ \oplus H^+$.

We remark that W. E. Clark and the author [1] have given an algebraic proof of Corollary 3.1. Nevertheless, it is interesting that the corollary should be an immediate consequence of a geometric theorem.

LEMMA 3.2. *Let H be a finite H-ring with radical N, $\sum_a (H)$ be n-uniform. Then for each point (c, d) of $\sum_a (H)$, one has ${}^i(c, d) = \{(c + a, d + b): a, b \in N^{n-i}\}$, $0 \leqq i < n$.*

Proof. Let $a \in N^{n-i} - N^{n-i+1}$, $b \in N^{n-j} - N^{n-j+1}$. We assume $i \geqq j$. Let $[x, y]$ denote the line whose incident points are $\{(tx, ty): t \in H\}$. The lines through $(0, 0)$ are the lines of the form $[x, y]$. (See [6, S 23]. Note that $[x, y]$ is a line if and only if not both $x, y \in N$.) Let $[x, y]$ be a line through (a, b). Then there exists $t_0 \in H$ such that $a = t_0x$, $b = t_0y$; hence $x \in H - N$ and $t_0 \in N^{n-i}$. Let $u \in H - N$, $w \in N^i$, $v = ux^{-1}y + w$. Then $[u, v]$ contains (a, b). There are $(s-t)r^{n-i}$ satisfactory pairs u, v; and, since $[u, v] = [u', v']$ if and only if $u' = zu$, $v' = zv$ for a unit z, these must give rise to at least r^{n-i} distinct lines. Then $(a, b) \in {}^i(0, 0)$. Similarly, if $j \geqq i$, $(a, b) \in {}^j(0, 0)$. Let $X = \{(a, b): a, b \in N^{n-i}\}$. Then $X \subset {}^i(0, 0)$. Since $|X| = r^{2i} = |{}^i(0, 0)|$, ${}^i(0, 0) = X$. This yields the result when $(c, d) = (0, 0)$. To obtain the full result, one merely considers the translation $\tau(c, d)$ which maps each point (x, y) to $(c + x, d + y)$.

REMARK 3.3. Let $H, N, r^* = p^x, k_1, k_2$ be as in Corollary 3.1. Set $k = k_1 + k_2$. Let i be any nonnegative integer less than n where n satisfies $N^{n-1} \neq N^n = 0$. Let q, r be the nonnegative integers which satisfy $i = kq + r$ and $r < k$. Then $(N^{n-i})^+$ is the direct sum of $x(k - r)$ cyclic subgroups of order p^q and of xr cyclic subgroups of order p^{q+1}.

Proof. Let $T = H^+ \oplus H^+$. Let π be the set of lines $[x, y]$. It is clear from [6, S 23] that $\sum_a (H) \cong J(T, \pi)$. By Lemma 3.2, $N^{n-i} \times N^{n-i} = {}^i(0, 0) = {}^iT$. Then the conclusion follows from Lemma 2.7 and Corollary 3.1.

In [1], W. E. Clark and the author prove the following result:

PROPOSITION 3.4. *Let there be given a prime integer p and nonnegative integers x, k_1, k_2, j, such that $x > 0$ and $k_1 j + k_2(j + 1) > 0$. Then there exists a commutative H-ring H with radical N such that $o(H/N) = p^x$ and so that H^+ is tha direct sum of xk_1 cyclic subgroups of order p^j and of xk_2 cyclic subgroups of order p^{j+1}.*

Klingenberg proves (See [5] or [2].) that if H is a commutative H-ring, then $\sum_a (H)$ is Pappian. We then obtain the following strong converse to Theorem 2.6 as an immediate corollary to Proposition 3.4.

COROLLARY 3.5. *Let $r = p^x, k_1, k_2, j$ be given with $k_1 j + k_2(j+1) > 0$, $x > 0$. Then there exists a Pappian affine H-plane with invariant r whose translation group is the direct product of $2xk_1$ cyclic subgroups of order p^j and $2xk_2$ cyclic subgroups of order p^{j+1}.*

Corollary 3.5 says that all translation groups of n-uniform translation H-planes can be obtained as translation groups of Pappian H-planes. Actually it says somewhat more: namely, if T is the translation group of an n-uniform translation H-plane A whose invariant is p^x, then T can be obtained as the translation group of a Pappian affine H-plane B whose invariant is p^y where y is any positive integer such that $y \mid xk_1$ and $y \mid xk_2$. In particular, one can always take $y = x$ so that A and B will have the same invariant. Also all translation groups can be obtained as the translation groups of Pappian affine H-planes whose associated affine planes are of prime order.

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