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COMMUTATIVE ASSOCIATIVE RINGS AND ANTI-FLEXIBLE RINGS

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Let R be a simple anti-flexible ring of characteristic distinct from 2 and 3. Anderson and Outcalt have proved that R^+ is a commutative associative ring. The same authors have also shown that a commutative associative ring P of characteristic not 2 gives rise to a simple anti-flexible ring provided P has a suitably defined symmetric belinear form on it. The purpose of this paper is to give an explicit construction of such a symmetric bilinear form and determine the suitable commutative associative rings.

It is proved that for any commutative associative ring R, which is either free of zero divisors or a zero ring, there is a class of simple anti-flexible rings associated with R. It is also shown that a subclass of commutative associative rings may be used to obtain a more general class of anti-flexible rings, namely prime ones, which are not necessarily simple even if they have both of the chain conditions. Finally two important examples on certain prime anti-flexible rings are given.

The results mentioned above of Anderson and Outcalt appear in [1]. In [3], Slater has shown that in semi-prime alternative rings the Nucleus and the center of an ideal of R are contained in the Nucleus the center of R respectively, which turns out to be very valuable in the structure theory of such rings. One of the examples shows that such results do not hold in anti-flexible rings. The other example will be of use in a later paper [2].

All algebraic structures will be of characteristic not 2. Unless mentioned otherwise the term "ring" means an anti-flexible ring which is defined by the identity

$$(x, y, z) = (z, y, x)$$

where $(x, y, z) = (xy)z - x(yz)$ is the associator. R^+ is the ring obtained from additive group of R together with the multiplication "$\cdot$" defined by $x \cdot y = \frac{1}{2}(xy + yx)$ for all $x, y \in R$, where xy, yx are multiplications of x and y in R.

$$N(R) = \{n \in R\colon (n, x, y) = 0 = (x, n, y) \text{ for all } x, y \in R\}$$
$$Z(R) = \{z \in N(R)\colon [z, x] = 0 \text{ for all } x \in R\}$$

are defined to be the Nucleus and the center of R respectively, where $[z, x] = zx - xz$ is the commutator.

2. Simple rings.

DEFINITION 2.1. Let R be a commutative associative ring, and let Ω be nonempty set such that $\Omega \cap R = \emptyset$. Define the free Ω-extension of R to be the commutative associative ring R^* generated by $R \cup \Omega$ with the multiplication $pqr \cdots st$ for the finitely many elements $p, q, r, \cdots, s, t \in R \cup \Omega$, such that the restriction of this multiplication to R is the multiplication of R and the identity of R, if it has any, is the identity of R^*. We say that R^* is of D-index n if

$$d_1 d_2 \cdots d_n = 0$$

for all $d_i \in D$, $i = i, \cdots, n$, where D is a subset of Ω and n is a positive integer.

We should mention here that the existence of such extension of R is guaranteed by the rings of polynomials over R and their quotient rings for suitable ideals.

THEOREM 2.2. (i) *Let R be a commutative associative ring without zero divisors, or let R be a zero ring. Then there exists a commutative associative ring R^* containing R and a bilinear mapping $\langle , \rangle$ of $R^* \times R^*$ into R^* such that the ring $\mathscr{R} = (R^*, \otimes)$ is a simple anti-flexible ring, where for $x, y \in R^*$, $x \otimes y$ is defined as $xy + \langle x, y \rangle$, xy being the multiplication in R^*.*

(ii) *Let R be a simple anti-flexible ring of characteristic not 3. Then for any commutative multiplication "o" defined on the set R such that $x^{o2} = x^2$ for all $k \in R$, the ring (R, o)* is *commutative and associative and there is a bilinear form on (R, o) which defines R.*

Proof. (i) (a) Assume that R has no zero divisors. Suppose that Ω is a set containing a totally ordered subset Ω_1 of at least two distinct elements. Let R^* be the free Ω-extension of R of Ω_1-index 2. Without loss of generality, assume that R has an identity element e, therefore R^* has an identity element e. In R^*, defined a bilinear form $\langle , \rangle$ as follows:

(a_1) $\langle r, s \rangle = 0$ if either r or s belongs to the set

$$\mathscr{S} = R \cup P \cup RP$$

where,

$$P = \text{the set } \Omega \backslash \Omega_1$$

and the set of all finite products of elements of $\Omega\backslash\Omega_1$.

(a_2) $\langle rx, sy\rangle = e = -\langle sy, rx\rangle$ if $x, y \in \Omega_1$ such that $x < y$ and $r, s \in \mathscr{S}\backslash\{0\}$.

(a_3) $\langle rx, sx\rangle = 0$ for all $r, s \in \mathscr{S}$ and all $x \in \Omega_1$. In R^* define a new multiplication "$\otimes$" by

$$r \otimes t = rt + \langle r, t\rangle ,$$

and let $\mathscr{R} = (R^*, \otimes)$ be the ring obtained by the additive group of R^* together with the multiplication "$\otimes$". In order to prove that $\mathscr{R}$ is a simple anti-flexible ring, by Theorem 3.11 of [1] it suffices to show that the bilinear form $\langle\, ,\rangle$ satifies the following conditions:

(1) $\langle x, x\rangle = 0$,

(2) $\langle x^2, x\rangle = 0$, for all $x \in R^*$,

(3) $\langle\langle R^*, R^*\rangle, R^*\rangle = 0$,

(4) $\langle R^*, R^*\rangle \neq (0)$,

(5) $\langle I, R^*\rangle \subsetneqq I$ for any proper ideal I of R^*.

It follows from (a_1) and (a_3) that (1) holds. To see (2), consider an arbitrary element w of R^*. Since R has an identity element, w has the following form:

$$w = \alpha_0 s_0 + \alpha_1 s_1 x_1 + \alpha_2 s_2 x_2 + \cdots + \alpha_n s_n x_n$$

where α_i are integers, $s_i \in \mathscr{S}$, $x_i \in \Omega_1 (i=1, 2, \cdots, n)$ and $x_1 < x_2 < \cdots < x_n$. Then,

$$w^2 = \alpha_0^2 s_0^2 + 2\sum_{i=1}^{n} \alpha_0\alpha_i s_0 s_i x_i$$

So,

$$\langle w^2, w\rangle = 2\,\alpha_0 \sum_{\substack{i=1\\ j=1}}^{n} \alpha_i\alpha_j \langle s_0 s_i x_i,\, s_j x_j\rangle$$

$$= 2\,\alpha_0 \sum_{\substack{i=1\\ j=1}}^{n} g_{ij} \;.$$

By (a_3), g_{ii} are all zero and by (a_2)

$$g_{ij} = -g_{ji} \text{ for } i \neq j \;.$$

Therefore

$$\langle w^2, w\rangle = 0 \;,$$

(3) and (4) are immediate.

(5) follows from the following argument. For each proper ideal I of R^*, there exists at least one element $\alpha s x$ in I such that α is an integer, $s \in \mathscr{S}$ and $x \in \Omega_1$. Since Ω_1 contains at least two distinct

elements, the set $\langle I, R^* \rangle$ contains the identity element e, Therefore $\langle I, R \rangle \subsetneqq I$. Thus, $\mathscr{R}$ is a simple anti-flexible ring.

(b) Assume that R is a zero ring. By the Zermelós well ordering axiom, the generating set R_1 of R can be imbedded in a totally ordered set Ω_1. Then consider Ω to be a set containing Ω_1. Thus, starting with the ring (0), we obtain R^* to which an identity element e may be adjoined. To define the bilinear form $\langle \, , \rangle$ on R^*, set the defining conditions as

$$(b_1) = (a_1), \; (b_2) = (a_2), \; (b_3) = (a_3)$$

with,

$$\mathscr{S} = P \cup \{0\}, \text{ where } P \text{ is as in } (a_1) \, .$$

Then an analogous proof to that of (a) shows that $\mathscr{R} = (R^*, \otimes)$ is a simple anti-flexible ring.

(ii) The proof of this part follows from the following argument: Let R be a ring and suppose that there is defined a commutative multiplication "o" on R such that $x^2 = x^{o2}$ for all $x \in R$. Then

$$(R, o) = R^+ \, .$$

For if, $x, y \in R$, then

$$(x + y)^2 = (x + y)^{o2}$$
$$x^2 + xy + yx + y^2 = x^{o2} + 2xoy + y^{o2}$$

or

$$xoy = \tfrac{1}{2}(xy + yx) \, .$$

Therefore $(R, o) = R^+$ and is a commutative associative ring which gives rise to R by the bilinear form $\langle x, y \rangle = xy - xoy$.

REMARKS. (i) The class of rings without zero divisors includes fields, integral domains, polynomial rings over such rings, group algebras of abelian groups, radical-quotient rings of commutative associative rings in which $x \neq y$ and xy is nilpotent imply either x is nilpotent or y is nilpotent, etc.

(ii) In (a), if R contains a zero divisor, then the condition (2) fails: suppose that $q \in R$, such that $qt = 0$ for some $t \in R$. Then consider

$$w = \alpha q + \beta t x_1 + \gamma x_2$$

with α, β, γ nonzero integers; $x_1, x_2 \in \Omega_1$ with $x_1 < x_2$. Then

$$w^2 = \alpha^2 q^2 + 2\alpha\gamma q x_2$$

and,

$$\langle w^2, w\rangle = -2\alpha\beta\gamma e \neq 0 .$$

The following corollary gives simple anti-flexible algebras of arbitrary dimension.

COROLLARY 2.3. *Let* $R = F$ *be a field in Theorem* (2.2) *and suppose that* $\Omega = \Omega_1$ *is a totally ordered set. Then* $\mathscr{R}$ *is a simple anti-flexible algebra over* F, *and dimension of* $\mathscr{R}$ *is* $|\Omega_1|$. $\mathscr{R}$ *is associative if and only if* $|\Omega_1| = 1$.

3. **Prime rings.** The purpose of this section is to show that there exist various types of prime anti-flexible rings which are not simple. R is prime if for any two ideals A, B of R, $AB=(0)$ implies $A = (0)$ or $B = (0)$.

PROPOSITION 3.1. *Let* R *be a commutative associative ring generated by a totally ordered set* R_1 *which contains at least two distinct elements. Suppose that* $xy = yx = 0$ *for all distinct* $x, y \in R_1$, *and* $x^2 = 0$ *for all* $x \in R_1$, *except for a fixed* $z \in R_1$, *in which case the* z^n*'s are all distinct for* $n \geqq 1$. *Then, there exists a prime anti-flexible, not simple ring* $\mathscr{R}$ *based on* R.

Proof. Let Ω be a nonempty set such that $\Omega \cap R = \varnothing$. Let R^* be the free Ω-extension of R. Consider the set

$$\mathscr{S} = P \cup \{z^n\}_{n\geqq 2} \cup P\{z^n\}_{n\geqq 2}$$

and a fixed element $a \in \Omega$, where P is the set Ω and the set of finite products of elements of Ω. Define a bilinear form in R^* by

(a) $\langle r, s\rangle = 0$ if r or $s \in \mathscr{S}$

(b) For any $x, y \in R_1$, if $x < y$, then

$$\langle x, y\rangle = \langle gx, y\rangle = \langle x, hy\rangle = \langle gx, hy\rangle = a$$
$$\langle y, x\rangle = \langle y, gx\rangle = \langle hy, x\rangle = \langle hy, gx\rangle = -a$$

for all $g, h \in P$.

(c) $\langle gx, hx\rangle = 0 = \langle x, x\rangle = \langle gx, x\rangle = \langle x, hx\rangle$ for all $g, h \in P$ and all $x \in R_1$.

Then for $r, s \in R^*$, define

$$r \otimes s = rs + \langle r, s\rangle .$$

It is not difficult to verify that the bilinear form has the properties (1)-(4) mentioned in the proof of Theorem (2.2). Therefore $\mathscr{R} =$

$(R^*, \otimes)$ is an anti-flexible ring. $\mathscr{R}$ is not simple because $a \in \Omega$ generates a proper ideal of $\mathscr{R}$. To see this we observe that $a \neq 0$ and any $x \in R_1$ does not belong to this ideal. Similarly, each z^n for $n \geqq 2$ generates a proper ideal of $\mathscr{R}$. In any case, each ideal contains a finite sum of elements of the form $\alpha_i p_i z^{n_i}$ for $n_i \geqq 1$, α_i are integers and $p_i \in P$. It is clear that the product of any two elements in $\mathscr{R}$ of this type is not zero whenever both of them are not zero. Thus $\mathscr{R}$ is a prime ring.

COROLLARY 3.2. *In Proposition* (3.1), *let R be a zero algebra generated by a finite set R_1 of at least two distinct elements, over a field F. Suppose that $\Omega = \{a\}$. Then $\mathscr{R}$ is a prime, anti-flexible, not simple algebra over F. Moreover, $\mathscr{R}$ has both of the chain conditions on ideals.*

Proof. Suppose that $R_1 = \{x_1, x_2, \cdots, x_n\}$ with the natural ordering $x_1 < x_2 < \cdots x_n$. If we define the bilinear form $\langle , \rangle$ as in the Proposition (3.1), then, $\mathscr{R} = (R^*, \otimes)$ is an anti-flexible algebra based on R. $\mathscr{R}$ is prime because any ideal of $\mathscr{R}$ contains the element a, and $a \otimes a = a^2 \neq 0$. $\mathscr{R}$ is not simple since a generates a proper ideal of $\mathscr{R}$. $\mathscr{R}$ has both of the chain conditions on ideals, because the only proper ideals of $\mathscr{R}$ are the ideals generated by the proper subsets of

$$\{x_1, x_2, \cdots, x_n; a\} .$$

COROLLARY 3.3. *There exist finite dimensional anti-flexible algebras which are prime but not simple.*

Proof. Suppose that R is as in Corollary (3.2), and $\Omega = \{a = w_1, w_2, \cdots, w_m\}$. It is possible to construct R^* in such a way that for each $i = 1, \cdots, m$, there exists a positive integer $n_i \geqq 2$ such that $w_i^{n_i} = w_i$. Then, defining the bilinear form $\langle , \rangle$ as in Proposition (3.1), $\mathscr{R}$ becomes a prime anti-flexible but not simple algebra over F. The fact that $\mathscr{R}$ is finite dimensional is an easy consequence of the conditions imposed on elements of Ω and the finiteness of both R_1 and Ω.

REMARK. The type of commutative associative rings which are used in Proposition (3.1) can easily be found as follows:

Let Q be a zero ring generated by a totally ordered set Q_1. Consider $Q[z]$, the ring of polynomials in z. Let $Q[z]_2$ be the ring of 2×2 matrices on $Q[z]$. Set

$$R_1 = \left\{ \bar{z} = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } \bar{s} = \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} : \in Q_1 \right\} .$$

Let R be the subring of $Q[z]_2$ generated by the set R_1. Then R has the required properties.

4. Two examples.

PROPOSITION 4.1. *There exists an anti-flexible ring R such that both R and R^+ are prime.*

Proof. Let R be the free commutative associative ring generated by a totally ordered set S of at least three elements. Let I be the ideal of R generated by monomials of degree two or more in S. On R define a bilinear form $\langle\, , \rangle$ as follows:

(a) $\langle r, s\rangle = 0$ if r or s belong to the set $\{a, I\}$ where a is a fixed element of S.

(b) $\langle x, y\rangle = a = -\langle y, x\rangle$ if $x, y \in S\backslash\{a\}$ and $x < y$.

(c) $\langle x, x\rangle = 0$ for all $x \in S$.

Then the conditions (a) = (c) satisfy the properties (I) – (IV) of the proof of Theorem (2.2), with $R^* = R$, and hence $\mathscr{R} = (R, \otimes)$ with $r \otimes s = rs + \langle r, s\rangle$ becomes an anti-flexible ring. It follows from (a) – (c) that any ideal of $\mathscr{R}$ must contain elements of the form $a + p$ with $p \in I$. Since for $p, q \in I$

$$(a + p) \otimes (a + q) = a^2 + aq + pa + pq \neq 0$$

$\mathscr{R}$ is prime. To see that $\mathscr{R}^+$ is also prime, we observe that $\mathscr{R}^+$ has no nonzero divisors of zero, because for any $r, s \in \mathscr{R}$,

$$\begin{aligned} (r, s)_\otimes &= \tfrac{1}{2}(r \otimes s + s \otimes r) \\ &= rs = 0 \end{aligned}$$

if and only if one of r, s is 0.

4.2. *Nucleus and the Center of Ideals.*

Given R and a proper ideal A of R, the following inclusions are hoped to hold:

$$\begin{aligned} N(A) &\subseteqq N(R) \\ Z(A) &\subseteqq Z(R) . \end{aligned}$$

In semi-prime alternative rings these inclusions hold [3] and are very useful in the related structure theory [4], [5]. It is unfortunate that the same results do not hold for the class of anti-flexible rings.

EXAMPLE. Let $\mathscr{R}$ be the ring obtained by Proposition (3.1), and let I be the ideal generated by z^n, for some $n \geqq 2$. I is a proper ideal of $\mathscr{R}$. Since $z^n \in \mathscr{S}$, $z^n \otimes r = z^n r$ for every $r \in \mathscr{R}$. Therefore I is a commutative associative ring and hence

$$N(I) = I \text{ and } Z(I) = I .$$

On the other hand $N(R) = (0) = Z(R)$. To see this consider any $x, y \in R$, with $x < y$ and $b \in \mathscr{S}$. By the construction of R^*, if $s_1, s_2 \in \mathscr{S}$ then $s_1 s_2$ is distinct from both s_1 and s_2. Following this argument and calculating the associator $(x, y, b)_\otimes$ we get

$$\begin{aligned}(x, y, b)_\otimes &= (x \otimes y) \otimes b - x \otimes (y \otimes b)\\ &= \langle x, y\rangle b + \langle b, y\rangle x - \langle xb, y\rangle\\ &= ab - a \neq 0 .\end{aligned}$$

This implies that neither x, y of R_1 nor b of $\mathscr{S}$ can be in the nucleus of $\mathscr{R}$. Therefore,

$$N(R) = (0) .$$

Thus

$$N(I) \nsubseteqq N(R)$$

and

$$Z(I) \nsubseteqq Z(R) .$$

REMARK. In this paper the term "simple" is relaxed up to the ideals which are integer multiples of R.

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