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RESTRICTION OF THE PRINCIPAL SERIES OF $SL(n, C)$ TO SOME REDUCTIVE SUBGROUPS

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Let $n = n_1 + \cdots + n_r$, where $r \geqq 2$ and the n_i's are positive integers. Then every element of $G = SL(n, C)$ can be written as a block matrix $(g_{ij})_{1 \leqq i,\, j \leqq r}$, where each block g_{ij} is a $n_i \times n_j$ matrix. Let $G_{n_1,\cdots,n_r}$ denote the subgroup of all diagonal block matrices, i.e., g_{ij} is the 0-matrix for $i \neq j$. Let T^χ be any element of the non-degenerate principal series of G. The main purpose of this paper is to decompose the restriction of T^χ to $G_{n_1,\cdots,n_r}$ into irreducible representations.

As we shall see by an induction argument, it is sufficient to consider the restriction of T^χ to $G_{n-1,1}$. Now by the Frobenius reciprocity theorem this restriction problem is equivalent to the decomposition of the induced representations to G of some irreducible representations of $G_{n-1,1}$. Note that

$$G_{n-1,1} \subset G_0 = \{(g_{ij})_{1 \leqq i,j \leqq n} \in G \mid g_{in} = 0,\ 1 \leqq i \leqq n-1\}\ ,$$

and hence those induced representations may be obtained by inducing some representations W of G_0. The W's are in turn equivalent to the restrictions of the elements of the non-degenerate principal series to G_0. Therefore they are all irreducible according to Gelfand and Naimark [3], and in fact are divided into n distinct classes of irreducible representations of G_0 [4]. The problem is now completed by applying again the Frobenius reciprocity theorem. It turns out that this restriction problem is equivalent to the problem of decomposing the tensor product of an element of the nondegenerate and an element of the degenerate principal series of G. In fact Theorem 4.2 gives the decompositions of such tensor products in terms of the non-degenerate principal series only. The results contained in this paper were parts of the author's thesis at the University of California, Los Angeles. The author would like to express his gratitude to Professor Donald G. Babbitt for guiding the preparation of the thesis. The author would also like to thank the referee for many helpful suggestions.

1. Some results on induced representations and the Frobenius reciprocity theorem. In this section we shall recall some results on induced representations due to Mackey ([5], [6]) and then prove some corollaries of the Frobenius reciprocity theorem ([6]) which are useful for later application.

Every locally compact group considered will be separable and every representation is understood to be unitary.

Let us recall quickly the definition of induced representations.[1] Let H be a closed subgroup of G. Let L be a representation of H in the Hilbert space $\mathfrak{H}(L)$. Let μ be any quasi invariant measure in the homogeneous space $\mathfrak{M} = H\backslash G$ of right H cosets. By definition of quasi invariance, the right translate of μ by an element y of G is equivalent to μ. Let $\lambda(\cdot, y)$ be the corresponding Radon-Nikodym derivative. Consider the space ${}^{\mu}\mathfrak{H}^{L}$ of all functions f from G to $\mathfrak{H}(L)$ such that

(a) $(f(x), v)^2$ is a Borel function of x for all $v \in \mathfrak{H}(L)$.

(b) $f(\xi x) = L_{\xi}(f(x))$ for all $\xi \in H$ and $x \in G$.

(c) By (b) $(f(x), f(x))^2$ is in fact a function on $\mathfrak{M}$. We assume $\int (f(x), f(x))\, d\mu(\dot{x}) < \infty$ where $\dot{x}$ is the right coset containing x. If functions equal almost everywhere are identified then ${}^{\mu}\mathfrak{H}^{L}$ becomes a Hilbert space. For each $y \in G$, let T_y map $f \in {}^{\mu}\mathfrak{H}^{L}$ into g where $g(x) = \lambda(\dot{x}, y)^{1/2} f(xy)$. Then it can be proved that T is a representation of G which is determined within unitary equivalence by the measure class of μ. This representation is called the representation of G induced from L and is denoted by $\mathrm{ind}_{H\uparrow G} L$ or ${}_{G}U^{L}$ or simply U^{L} if there is no ambiguity.

On the other hand let V be any representation of G. then the restriction of V to the subgroup H is denoted by $V|_H$ or simply V_H.

The following theorems were proved by Mackey.

THEOREM 1.1. (Theorem 4.1 of [5]). *Let $H \subset K$ be closed subgroups of G. Let L be a representation of H and let $M = \mathrm{ind}_{H\uparrow K} L$. Then $\mathrm{ind}_{H\uparrow G} L$ and $\mathrm{ind}_{K\uparrow G} M$ are equivalent representations.*

THEOREM 1.2 (Theorem 5.2 of [5]). *Let L and M be representations of the closed subgroups H_1 and H_2 of the groups G_1 and G_2 respectively. Then the outer Kronecker product $\mathrm{ind}_{H_1\uparrow G_1} L \times \mathrm{ind}_{H_2\uparrow G_2} M$ is equivalent to $\mathrm{ind}_{H_1\times H_2\uparrow G_1\times G_2}(L \times M)$ where $L \times M$ is the outer Kronecker product of L and M.*

Let H_1 and H_2 be closed subgroups of G. We shall say that H_1 and H_2 are discretely related if there exists a subset of G whose complement has Haar measure zero and which is itself the union of

[1] See, e.g. § 2 of [5].

[2] $(\cdot,\cdot)$ denotes the inner product in $\mathfrak{H}(L)$.

countably many $H_1: H_2$ double cosets.

THEOREM 1.3 (Theorem 7.1 of [5]). *Let H_1 and H_2 be two discretely related closed subgroups of G. Let L be a representation of H_1. For each $x \in G$ consider the subgroup $H_2 \cap (x^{-1}H_1x)$ of H_2 and let ${}_xV$ denote the representation of H_2 induced by the representation $\eta \mapsto L_{x\eta x^{-1}}$ of this subgroup. Then ${}_xV$ is determined within unitary equivalence by the double coset $H_1 \times H_2 = D(x)$ and we may write ${}_DV = {}_xV$ where $D = D(x)$. Finally $\mathrm{ind}_{H_1 \uparrow G} L$ restricted to H_2 is the direct sum of the ${}_DV$ over those double cosets D which are not of measure zero.*

THEOREM 1.4 (Theorem 7.2 of [5]). *Let H_1 and H_2 be as in Theorem 1.3 and let L and M be representations of H_1 and H_2 respectively. For each $(x, y) \in G \times G$ consider the representations*

$$s \longmapsto L_{xsx^{-1}} \quad \textit{and} \quad s \longmapsto M_{ysy^{-1}}$$

of the subgroup $(x^{-1}H_1x) \cap (y^{-1}H_2y)$. Let us denote their tensor product (or Kronecker product in the terminology of [5]) by $N^{x,y}$. Then the induced representation of $N^{x,y}$ to G is determined within unitary equivalence by the double coset $H_1xy^{-1}H_2$ and the direct sum of these induced representations over those double cosets which are not of measure zero is equivalent to the tensor product $\mathrm{ind}_{H_1 \uparrow G} L \otimes \mathrm{ind}_{H_2 \uparrow G} M$.

THEOREM 1.5 (Theorem 10.1 of [5]). *Let H be a closed subgroup of G and let M be a representation of H which is a direct integral over a Borel measure space (Y, μ) of representations ${}^yL$; $M = \int {}^yL \, d\mu(y)$. Then $\int \mathrm{ind}_{H \uparrow G} {}^yL \, d\mu(y)$ is equivalent to $\mathrm{ind}_{H \uparrow G} M$.*

Let $\mathfrak{M}$ be a separable locally compact space and let μ be a finite measure on $\mathfrak{M}$. Let r be an equivalence relation on $\mathfrak{M}$. Let r also denote the natural mapping of $\mathfrak{M}$ onto the quotient space Y. Assume r regular in the sense of § 11 of [5]. Then μ induces a natural measure $\tilde{\mu}$ on Y.

LEMMA 1.6 (Lemma 11.1 of [5]). *Let μ, $\tilde{\mu}$ be as above. Then for each $y \in Y$ there exists a finite Borel measure μ_y in $\mathfrak{M}$ such that $\mu_y(\mathfrak{M}\text{-}r^{-1}\{y\}) = 0$ and $\int f(y) \int g(x) d\mu_y(x) d\tilde{\mu}(y) = \int f(r(x)) g(x) d\mu(x)$ whenever $f \in \mathscr{L}_1(Y, \tilde{\mu})$ and g is bounded and measurable on $\mathfrak{M}$. μ_y is called the quotient measure obtained from μ by way of the equivalence relation r.*

LEMMA. 1.7 (Lemma 11.4 of [5]). *Let* μ, r, Y, $\mathfrak{M}$ *be as above and let* k *be a nonnegative function on* $\mathfrak{M}$ *which is* μ*-summable. Let* υ *be the measure whose Radon-Nikodym derivative with respect to* μ *is* k. *Then* $\tilde{\upsilon}$ *is absolutely continuous with respect to* $\tilde{\mu}$, *the Radon-Nikodym derivative being* λ *say. Moreover in the decomposition of* υ, υ_y *may be taken to be that measure absolutely continuous with respect to* μ_y, *whose Radon-Nikodym derivativative is zero or* $x \mapsto k(x)/\lambda(y)$ *depending upon whether or not* $\lambda(y)$ *is zero.*

THEOREM 1.8 (Theorem 5.1 of [6]). *Let* H *be a closed subgroup of* G. *Let the regular representations of* H *and* G *be of type I and let their canonical decomposition into factor representations be* $\int_X F^x d\zeta(x)$ *and* $\int_Y N^y d\eta(y)$ *respectively where* F^x (resp. N^y) *is a multiple of the irreducible representation* L^x (res. M^y) *of* H (resp. G) *and* ζ *and* η *are finite measures such that* $\zeta(X) = \eta(Y)$. *Then there exists a Borel measure* α *on* $X \times Y$ *and an* α*-measurable function from* $X \times Y$ *to the countable cardinals,* $(x, y) \mapsto n(x, y)$, *such that for all Borel subsets* E *and* E' *of* X *and* Y *respectively we have*

$$\alpha(E \times Y) = \zeta(E);\ \alpha(X \times E') = \eta(E')$$

and such that for ζ *almost all* x *in* X

(i) $\operatorname{ind}_{H\uparrow G} L^x \cong \int_Y n(x, y) M^y \, d\beta_x(y)$ *and for* η *almost all* y *in* Y

(ii) $M^y \mid H \cong \int_X n(x, y) L^x \, d\gamma_y(x)$ *where the* βx (resp. γy) *are the quotient measures obtained from* α *by way of the equivalence relation* $r(x, y) = x$ (resp. $r(x, y) = y$).

The Theorem 1.8 is often called the Frobenius reciprocity theorem. Let us derive some corollaries of Theorem 1.8 which are easier for application in some special cases. In fact it is hard to compute α in general. However what we expect is the following: suppose by some other way we know that one of the statements (i) or (ii) is valid, then what can be said about the other?

The answer of this question is contained in the following corollaries.

COROLLARY 1.9. *Let* G *and* H *be as in Theorem* 1.8. *Assume also that they are of type I. Then the following are equivalent.*

(i) *for* ζ *almost all* x, $\operatorname{ind}_{H\uparrow G} L^x$ *is quasi-equivalent to a subrepresentation of the regular representation of* G.

(ii) *for* η *almost all* y, $M^y|_H$ *is quasi-equivalent to a subrepresentation of the regular representation of* H.

Proof. By (i) of Theorem 1.8 and the uniqueness of direct integral decompositions into irreducible reperesentations for type I groups (see e.g., [2]), (i) is equivalent to:

(iii) for ζ almost all x, β_x is absolutely continuous with respect to η.

Suppose (iii) is true. Then the Fubini's theorem and Lemma 1.6 show that:

(iii)′ α is absolutely continuous with respect to $\zeta \times \eta$. Conversely suppose α is absolutely continuous with respect to $\zeta \times \eta$. Let us apply Lemma 1.7 for the equivalence relation $r(x, y) = x$, $\mu = \zeta \times \eta$, $\upsilon = \alpha$. Since it is clear that $\tilde{\mu} = \eta(Y)\zeta$ and $\mu_x = \eta(Y)^{-1}\eta$ for every $x \in X$, (iii) follows immediately. The equivalence between (ii) and (iii)′ is proved in a similar manner.

To have a more precise statement we must include the multiplicity function.

COROLLARY 1.10. *Let G and H be as in Corollary* 1.9. *Let $\omega(x, y)$ and $n'(x, y)$ be $\zeta \times \eta$-measurable functions where $n'(x, y)$ is a countable cardinal for every x, y. Then the following are equivalent.*

(i) *for ζ almost all x,* $\mathrm{ind}_{H\uparrow G} L^x \cong \int_Y n'(x, y) M^y d\beta'_x(y)$, *where* $d\beta'_x(y) = \omega(x, y) d\eta(y)$.

(ii) *for η almost all y,* $M^y|_H \cong \int_X n'(x, y) L^x d\gamma'_y(x)$, *where* $d\gamma'_y(x) = \omega(x, y) d\zeta(x)$.

Proof. Let α, β_x, γ_y be as in Theorem 1.8. As in the proof of Corollary 1.9, (i) or (ii) imply that α is absolutely continuous with respect to $\zeta \times \eta$. Let $f(x, y)$ be the corresponding Radon-Nikodym derivative. Apply again Lemma 1.7 for the relation $r(x, y) = x$, $\mu = \zeta \times \eta$, $\upsilon = \alpha$. As noted in the proof of Corollary 1.9 $\tilde{\mu} = \eta(Y)\zeta$, and $\mu_x = \eta(X)^{-1}\eta$ for every x in X. On the other hand it is also obvious that $\tilde{\upsilon} = \zeta$ (see e.g., the connection between α and ζ in Theorem 1.8). Therefore the function λ in Lemma 1.7 satisfies $\lambda(x) = d\tilde{\upsilon}/d\tilde{\mu}(x) = \eta(Y)^{-1}$, and the Radon-Nikodym derivative of the corresponding quotient measures is given by $d\beta_x/d\mu_x(y) = f(x, y)/\eta(Y)^{-1}$. Therefore

$$d\beta_x(y) = f(x, y) d\eta(y) \,. \tag{1}$$

Using again the uniqueness of direct-integral decomposition into irreducible representations for type I groups and taking (1) into account we see that (i) is equivalent to

$$\text{(iii)} \quad \begin{cases} \omega(x, y) d\zeta(x) d\eta(y) \sim \alpha \; (= f(x, y) d\zeta(x) d\eta(y)) \\ n'(x, y) = n(x, y), \; \zeta \times \eta - \text{a.e.} \end{cases}$$

Similarly (ii) is equivalent to (iii).

2. **Description of some representations of $G_{n_1}, \cdots, {}_{n_r}$.** Although the representations of $G_{n_1}, \cdots, n_r$ can be described by using the known results on reductive Lie groups, we prefer to use another method which is interesting in its own right and is used to simplify our computations later on.

Let H and K be two subgroups of a group G. Then G is said to be the "*generalized direct product*" of H and K if: (i) $HK = G$; (ii) $hk = kh$ for $h \in H$ and $k \in K$. In the case $H \cap K = \{id_G\}$. G is simply the direct product of H and K.

Let G_1 and G_2 be two groups. Let Z_1 (resp. Z_2) be a subgroup of the center of G_1 (resp. G_2). Suppose that there exists an isomorphism t from Z_1 onto Z_2. It is clear that $Z = \{(z, t(z)) \mid z \in Z_1\}$ is a normal subgroup of $G_1 \times G_2$. Let υ be the canonical homomorphism of $G_1 \times G_2$ onto $G = G_1 \times G_2/Z$. Put $H_i = \upsilon(G_i)$ $(i = 1, 2)$. Then it is easy to see that G is the generalized direct product of H_1 and H_2. Moreover H_1 and H_2 are isomorphic to G_1 and G_2 respectively. Under these isomorphisms, t becomes the automorphism $h \mapsto h^{-1}$ of $H_1 \cap H_2$. Suppose now G_1 and G_2 are topological groups, Z_1 and Z_2 are closed subgroups of G_1 and G_2 respectively, and t is also a homeomorphism. Then G, equipped with the quotient topology, is a topological group containing H_1 and H_2 as closed subgroups. If this is the case we say that G is the *topological generalized direct product* of G_1 and G_2 via t. Assume that G is a separable locally comact group. If G is the (algebraic) generalized direct product of two closed subgroups H and K, then it can be shown that G is (topologically and algebraically equivalent to) the topological generalized direct product of H and K via the automorphism $z \mapsto z^{-1}$ of $H \cap K$.[3]

We turn now to the representation theory of generalized direct products. Note that while this notion is a generalization of that of direct products, it is also contained, in part, in the theory of group extensions.

PROPOSITION 2.1. *Let G be the generalized direct product of two closed subgroups H and K. Let H' be closed subgroup of H containing $H \cap K$. Then $G' = H'K$ is a closed subgroup of G. Let V be a representation of G' in the Hilbert space $\mathfrak{H}$. Put $W_1 = \mathrm{ind}_{H' \uparrow H} (V|_{H'})$. Then $\mathrm{ind}_{G' \uparrow G} V$ is equivalent to the representation of G defined by*

$$(2) \qquad g = hk \longmapsto W_1(h) W_2(k) \quad (h \in H,\ k \in K)\ ,$$

where W_2 is a repesentation of K equivalent to some multiple of $V|_K$.

[3] See, e.g., [1], Chapter 7, § 2, no. 9.

Proof. We first remark that the map $H'h \mapsto G'h (h \in H)$ is a homeomorphism of $H'\backslash H$ onto $G'\backslash G$ which intertwines the actions of H by right translations. Moreover it transforms a quasi invariant measure $\tilde{\mu}$ of $H'\backslash H$ into a quasi invariant measure μ of $G'\backslash G$.[4] For every function f from G into $\mathfrak{H}$, put $\hat{f} = f|_H$. Then $f \mapsto \hat{f}$ is an isometry of the Hilbert spaces ${}^{\mu}\mathfrak{H}^V$ and ${}^{\tilde{\mu}}\mathfrak{H}^{V_{H'}}$ in § 1. In fact it sets up an equivalence between $\mathrm{ind}_{G' \uparrow G} V$ and the representation (2). The fact that W_2 is equivalent to a multiple of $V|_K$ can be checked directly or by using Theorem 12.1 of [5].

The following corollary is useful for later application

COROLLARY 2.2. *Let G, H, K, H', G' be as in Proposition 2.1. Let V be a one-dimensional representation of G'. Then $\mathrm{ind}_{G' \uparrow G} V$ is equivalent to the representation defined by*

$$g = hk \longmapsto V(k)W(h) \,, \quad h \in H \,, \quad k \in K \,, \tag{3}$$

where $W = \mathrm{ind}_{H' \uparrow H}(V|_{H'})$.

Let us consider the important particular case in which H is abelian.

LEMMA 2.3. *Let G be the generalized direct product of a closed subgroup K and an abelian closed subgroup H. Let U be any irreducible representation of G. Then $U|_H$ is a multiple of some character χ of H and $V = U|_K$ is an irreducible representation of K such that*

$$V|_{H \cap K} = \text{mult of } \chi|_{H \cap K} \,. \tag{4}$$

Conversely let χ be any character of H and V be any irreducible representation of K satisfying (4). Then $g = hk \mapsto \chi(h)V(k)$ is a well-defined irreducible representation of G.

Proof. Let U be an irreducible representation of G in the Hilbert space $\mathfrak{H}$. Since H is abelian it is contained in the center of G. Therefore by Schur's Lemma $U(h) = \chi(h)I$ where $\chi(h)$ is a complex number and I is the unit operator of $\mathfrak{H}$. It is clear that χ is a character of H. Let $\mathfrak{H}'$ be a nonzero closed subspace of $\mathfrak{H}$ which is invariant under $U(k)$, $k \in K$. Let W be the component of K on $\mathfrak{H}'$ then $W(k) = \chi(k)I'$, $k \in H \cap K$ where I' is the unit operator of $\mathfrak{H}'$. Hence $g = hk \mapsto \chi(h)W(k)$ is a well-defined subrepresentation of U. Thus $\mathfrak{H}' = \mathfrak{H}$. This shows that $U|_K$ is irreducible. The converse is clear.

[4] This can be seen by a direct computation. See however [5] for the correspondence between quasi-invariant measures and λ-functions.

COROLLARY 2.4. *G is of type I if and only if K is.*

We shall apply these results to the subgroup $G_{n_1,\cdots,n_r}$. First we shall recall some facts on the representation theory of $SL(n, C)$. Let K_n(resp. D_n) be the subgroup of $SL(n, C)$ consisting of all upper-triangular (resp. diagonal) matrices. Let χ be a character of D_n, then χ extends uniquely to a one-dimensional representation of K_n which induces to an irreducible representation of $SL(n, C)$ (see [3] and [4]). This representation is called the element of the nondegenerate principal series of $SL(n, C)$ corresponding to χ and denoted by T^{χ}. Since the dual $\hat{D}_n$ of D_n is parametrized by $\mathbf{Z}^{n-1} \times \mathbf{R}^{n-1}$ ([3], see also [4] for another parametrization of $\hat{D}_n$) we also use the notation $T^{(m_2,\cdots,m_n;\rho_2,\cdots,\rho_n)}$ for the element of the nondegenerate principal series corresponding to $(m_2, \cdots, m_n; \rho_2, \cdots, \rho_n) \in \mathbf{Z}^{n-1} \times \mathbf{R}^{n-1}$. A fundamental domain of $\hat{D}_n$ is a maximal subset $\hat{D}_n^0$ of $\hat{D}_n$ with respect to the following property: let χ_1, χ_2 be two different elements of $\hat{D}_n^0$, then the corresponding elements T^{χ_1} and T^{χ_2} of the nondegenerate principal series are not equivalent.

Let $\hat{D}_n^0$ be any fundamental domain of $\hat{D}_n$. Then the regular representation of $SL(n, C)$ can be decomposed into $\int_{\hat{D}_n^0} \infty\, T^{\chi} d\chi$, where $d\chi$ is the restriction of the Haar measure of $\hat{D}_n$ to $\hat{D}_n^0$.[5]

We now return to the group $G_{n_1,\cdots,n_r}$. Let $K_{n_1,\cdots,n_r}$ (resp. $D_{n_1,\cdots,n_r}$) be the subgroup of $G_{n_1,\cdots,n_r}$ consisting of all diagonal block matrices (g_{ij}) such that each block g_{ii} is an upper triangular (resp. scalar) matrix. It is clear that $H = SL(n_1, C) \times \cdots \times SL(n_r, C)$ can be embedded in $G_{n_1,\cdots,n_r}$ and $G_{n_1,\cdots,n_r}$ becomes the generalized direct product of H and $D_{n_1,\cdots,n_r}$. Moreover $H \cap D_{n_1,\cdots,n_r} = C_1 \times \cdots \times C_r$, where C_i is the center of $SL(n_i, C)$. Thus by Lemma 2.3 every irreducible representation of $G_{n_1,\cdots,n_r}$ is of the form

$$U(g) = \alpha(d) T_1(g_r) \times \cdots \times T_r(g_r) ,$$

for

$$g = dg_1 \cdots g_r,\ d \in D_{n_1,\cdots,n_r} ,\ g_i \in SL(n_i, C)\ (1 \leqq i \leqq r) .$$

Recall that α is a character of $D_{n_1,\cdots,n_r}$ and each T_i is an irreducible representation of $SL(n_i, C)$ whose restriction to C_i is a multiple of $\alpha|_{C_i}$. In the case T_i is the element T^{χ_i} of the nondegenerate principal series of $SL(n_i, C)$, U may be obtained by inducing a one-dimensional representation ρ of $K_{n_1,\cdots,n_r}$ according to Corollary 2.2 and Theorem 1.2; ρ is uniquely determined by the conditions

[5] See [3] and [7] for a description of $\hat{D}_n^0$ and the decomposition of the regular representation.

$$\rho\,|_{D_{n_1,\cdots,n_r}} = \alpha,\ \rho\,|_{D_{n_i}} = \chi_i\ ,$$

where D_{n_i} is the diagonal subgroup of $SL(n_i, C)$.

DEFINITION 2.5. The irreducible representation of $G_{n_1,\cdots,n_r}$ defined as above is called the element of the principal series of $G_{n_1,\cdots,n_r}$ corresponding to ρ and is denoted by U^ρ.

REMARK. Again the one-dimensional representations of $K_{n_1,\cdots,n_r}$ are determined uniquely by their restrictions to D_n. Therefore the elements of the principal series of $G_{n_1,\cdots,n_r}$ are parametrized by $\boldsymbol{Z}^{n-1} \times \boldsymbol{R}^{n-1}$. For each i, $1 \leqq i \leqq r$, choose a fundamental domain $\hat{D}^0_{n_i}$ of $\hat{D}_{n_i}$. Let $\hat{D}^+_n$ be the subset of $\hat{D}_n$ consisting of those characters whose restrictions to D_{n_i} belong to $\hat{D}_{n_i}$. Then it is easy to verify that $\hat{D}^+_n$ is a fundamental domain of $\hat{D}_n$ corresponding to the group $G_{n_1,\cdots,n_r}$ in the sense that it is a maximal subset of $\hat{D}_n$ with respect to the property: let ρ_1, ρ_2 be two different elements of $\hat{D}^+_n$, then the corresponding elements U^{ρ_1} and U^{ρ_2} of the principal series of $G_{n_1,\cdots,n_r}$ are not equivalent. Suppose such a set is chosen, we have.

PROPOSITION 2.6. *The regular representation of $G_{n_1,\cdots,n_r}$ can be decomposed as follows:* $\int_{\hat{D}^+_n}^{\oplus} U^\rho\, d\rho$, *where $d\rho$ is the restriction of the Haar measure of $\hat{D}_n$.*

Proof. Using the decomposition of the regular representation of $SL(n_i, C)$ recalled earlier and Theorem 1.2, we see that the regular representation of $H = SL(n_1, C) \times \cdots \times SL(n_r, C)$ can be decomposed as follows: $\int_{\hat{D}^0_{n_1}} \cdots \int_{\hat{D}^0_{n_r}}^{\oplus} T^{\chi_1} \times \cdots \times T^{\chi_r} d\chi_1 \cdots\cdot d\chi_r$. Therefore the regular representation of $G_{n_1,\cdots,n_i}$ is equivalent to

$$\int_{\hat{D}^0_{n_1}} \cdots \int_{\hat{D}^0_{n_r}}^{\oplus} \operatorname*{ind}_{H \uparrow G_{n_1,\cdots,n_r}} (T^{\chi_1} \times \cdots \times T^{\chi_r})\, d\chi_1 \cdots d\chi_r\ .$$

Note that we have used the Theorems 1.1 and 1.5. Now by Theorems 1.1 and 1.2 we have

$$\operatorname*{ind}_{H \uparrow G_{n_1,\cdots,n_r}} (T^{\chi_1} \times \cdots \times T^{\chi_r}) \cong \operatorname*{ind}_{K_{n_1} \times \cdots \times K_{n_r} \uparrow G_{n,\cdots,n_r}} (\chi_1 \times \cdots \times \chi_r)\ .$$

Put $H' = K_{n_1} \times \cdots \times K_{n_r}$. Then it is clear that $K_{n_1,\cdots,n_r}$ is the generalized direct porduct of H' and $D_{n_1,\cdots,n_r}$ such that

$$D_{n_1,\cdots,n_r} \cap H' = C_1 \times \cdots \times C_r\ .$$

Put $\chi = \chi_1 \times \cdots \times \chi_r$. Then we have by Theorem 1.1

$$\operatorname*{ind}_{H'\uparrow G_{n_1,\cdots,n_r}}\chi \cong \operatorname*{ind}_{K_{n_1,\cdots,n_r}\uparrow G_{n_1,\cdots,n_r}} (\operatorname*{ind}_{H'\uparrow K_{n_1,\cdots,n_r}} \chi)\,.$$

By Corollary 2.2

$$\operatorname*{ind}_{H'\uparrow K_{n_1,\cdots,n_r}}\chi \cong \chi\,|_{H'}\cdot \operatorname*{ind}_{C_1\times\cdots\times C_r\uparrow D_{n_1,\cdots,n_r}} \chi_{C_1\times\cdots\times C_r}\,.$$

Since $C_1 \times \cdots \times C_r$ is a compact (in fact finite) subgroup of the abelian group $D_{n_1,\cdots,n_r}$, we can write[6] $\operatorname*{ind}_{H'\uparrow K_{n_1,\cdots,n_r}}\chi \cong \chi\,|_{H'}\cdot \int' \lambda d\lambda$, where $\int'$ is taken over the set of all character λ of $D_{n_1,\cdots,n_r}$ whose restriction to $C_1 \times \cdots \times C_r$ is $\chi\,|_{C_1\times\cdots\times C_r}$. Thus

$$\begin{aligned}\operatorname*{ind}_{H'\uparrow K_{n_1,\cdots,n_r}}\chi &\cong \int' \chi\,|_{H'}\cdot \lambda d\lambda \\ &\cong \int' \rho d\rho\end{aligned}$$

where $\int'$ is taken over the set of all one-dimensional representations of $K_{n_1,\cdots,n_r}$ extending χ.

3. Restriction of the nondegenerate principal series to $G_{n_1,\cdots,n_r}$. Before treating the general case, we consider a special case which is itself the main step for solving the general porblem, namely the restriction of the nondegenerate principal series to $G_{n-1,\,1}$.

THEOREM 3.1. *Let $T^{(m_2,\cdots,m_n;\,\rho_2,\cdots,\rho_n)}$ be any element of the non-degenerate principal series of $SL(n, C)$. Then its restriction to $G_{n-1,\,1}$ is equivalent to $\sum_{k_2,\cdots,k_n}\int\cdots\int U^{(k_2,\cdots,k_n;\,\sigma_2,\cdots,\sigma_n)}\,d\sigma_2\cdots d\sigma_n$, where $U^{(k_2,\cdots,k_n;\,\sigma_2,\cdots,\sigma_n)}$ is an element of the princippl series of $G_{n-1,\,1}$ and $\sum\int\cdots\int$ is the summation-integral over the set of all*

$$(k_2, \cdots, k_n;\ \sigma_2, \cdots, \sigma_n) \in \mathbf{Z}^{n-1}\times \mathbf{R}^{n-1}$$

such that $(k_2, \cdots, k_{n-1};\ \sigma_2, \cdots, \sigma_{n-1}) \in \hat{D}^0_{n-1}$ and $\sum_2^n k_i \equiv \sum_2^n m_i\ (\mathrm{mod}\ n)$.

Proof. Let $G_0 = \{(g_{ij})_{1\leqq i,\,j\leqq n}\,|\,g_{in}=0,\ 1\leqq i \leqq n-1\}$.

By Theorem 3 of [4], $T^{(m_2,\cdots,\rho_n)}\,|_{G_0}$ is equivalent to some fixed representation W_i of G_0 if $\sum_2^n m_j \equiv i\ (\mathrm{mod}\ n)$. In fact, $W_0, \cdots, W_{n-1}$ are all irreducible as indicated in [3]. Recall that $T^{(m_2,\cdots,\rho_n)}$ is obtained by inducing the one-dimensional representation $(m_2, \cdots, \rho_n)$ of K_n. Since the complement of G_0K_n in G has Haar measure zero ([3];

[6] This can be one by using the Fourier analysis on abelian groups or by Corollary 1.10.

[4]) and $G_0 \cap K_n = K_{n-1,1}$, we have by applying Theorems 1.3 and 1.1:

$$
(5) \qquad \begin{aligned} T^{(m_2,\cdots,\rho_n)} |_{G_0} &\cong \operatorname*{ind}_{K_{n-1,1} \uparrow G_0} (m, \cdots, \rho_n) |_{K_{n-1,1}} \\ &\cong \operatorname*{ind}_{G_{n-1} \uparrow G_0} U^{(m_2,\cdots,\rho_n)} . \end{aligned}
$$

Therefore,

$$
(6) \qquad \operatorname*{ind}_{G_{n-1,1} \uparrow G_0} U^{(m_2,\cdots,\rho_n)} \cong W_i \text{ iff } \sum_2^n m_j \equiv i \pmod n .
$$

Note that a direct integral whose components are all equivalent to the same fixed representation is in fact equivalent to a multiple of that representation. Hence (6), Theorem 1.1 and Proposition 2.6 imply that the regular representation of G_0 is decomposed as $\infty W_0 \oplus \cdots \oplus \infty W_{n-1}$. Therefore we can apply Corollary 1.10 and get:

$$
W_i |_{G_{n-1,1}} \cong \sum_{k_2,\cdots,k_n} \int \cdots \int U^{(k_2,\cdots,k_n;\, \sigma_2,\cdots,\sigma_n)}\, d\sigma_2 \cdots d\sigma_n
$$

where $\sum \int \cdots \int$ in the summation integral over the set of all $(k_2, \cdots, \sigma_n)$ such that

$$
(k_2, \cdots, k_{n-1}; \sigma_2, \cdots, \sigma_{n-1}) \in \hat{D}^0_{n-1} \text{ and } \sum_2^n k_j \equiv i \equiv \sum_2^n m_j \pmod n .
$$

COROLLARY 3.2.

$$
(7) \qquad \operatorname*{ind}_{G_{n-1,1} \uparrow G} U^{(k_2,\cdots,\sigma_n)} \cong \sum_{m_2,\cdots,m_n} \int \cdots \int T^{(m_2,\cdots,\rho_n)}\, d\rho_2 \cdots d\rho_n
$$

where $\sum \int \cdots \int$ *is the summation integral over the set of all* $(m_2, \cdots, \rho_n) \in \hat{D}^0_n$ *such that* $\sum_2^n m_i \equiv \sum_2^n k_i \pmod n$.

Proof. Corollary 1.10 also gives the decomposition of $\operatorname{ind}_{G_0 \uparrow G} W_i$ (the notation as in Theorem 3.1). This together with (5) give the desired decomposition.

REMARK 1. Since $G_{1,n-1}$ and $G_{n-1,1}$ are conjugate in G we also get the decomposition of $\operatorname{ind}_{G_{1,n-1} \uparrow G} U^{(k_2,\cdots,\sigma_n)}$. It turns out to be the same as that of $\operatorname{ind}_{G_{n-1,1} \uparrow G} U^{(k_2,\cdots,\sigma_n)}$, hence the two representations are equivalent.

2. A fundamental domain as defined earlier is also a fundamental domain of $\hat{D}_n$ with respect to the action of the permutation group (the Weyl group) on $\hat{D}_n$. Since every permutation preserves $\sum_2^n k_i$

(mod n), dropping the restriction to the fundamental domain in the right hand side of (7) amounts to repeat every representation occurring there $n!$ times. Therefore the left-hand side of (7) must be replaced by $n!$ times of itself. This technique will be used often and we shall not mention it explicitly again.

LEMMA 3.3. *Let* $n = n_1 + n_2$, $n_1 > 1$. *Let* $\chi = (m_2, \cdots, m_n; \rho_2, \cdots, \rho_n)$ *be any one-dimensional representation of* $K_{1,\cdots,1,n_2}$. *Then*

$$\operatorname*{ind}_{K_{1,\cdots,1,n_2}\uparrow G} \chi \cong \sum \int \cdots \int_{\infty} T^{(k_2,\cdots,\sigma_n)} d\sigma_2 \cdots d\sigma_n \tag{8}$$

where the summation extends over the set of all $(k_2, \cdots, k_n)$ *such that* $\sum_2^n k_i \equiv \sum_2^n m_i \pmod n$.

Sketch of the Proof. $K_{1,\cdots,1,n_2} = D_{1,\cdots,1,n_2+1} K_{1,n_2}$ ($K_{1,n_2} \subset SL(n_2+1, C)$ are embedded in $G_{1,\cdots,1\,n_2+1}$ as usual) and $K_{1,n_2} \supset D_{1,\cdots,1,n_2+1} \cap SL(n_2+1, C)$, hence Corollary 2.2 shows that

$$\operatorname*{ind}_{K_{1,\cdots,1,n_2}\uparrow G_{1,\cdots,n_2+1}} \chi \cong \chi|_{D_{1,\cdots,1,n_2+1}} \operatorname*{ind}_{K_{1,n_2}\uparrow SL(n_2+1,C)} \chi|_{K_{1,n_2}}. \tag{9}$$

Theorem 1.1 shows that the left hand side of (9) is equivalent to

$$\operatorname*{ind}_{G_{1,\cdots,1,n_2}\uparrow G_{1,\cdots,1,n_2+1}} U^{\chi},$$

where U^{χ} is the element of the principal series of $G_{1,\cdots,1,n_2}$ determined by χ. On the other hand Corollary 3.2 (and its remark) and Theorem 1.1 give the decomposition of the right hand side of (9) into a direct integral of some elements of the principal series of $G_{1,\cdots,1,n_2+1}$. Therefore the Lemma can be done by using an induction on n_1. The detailed computation based on some change of variables similar to that in Theorem 3.5 and will be omitted here.

Let us consider another special case where $r = 2$, i.e., $n = n_1 + n_2$. Since G_{n_1,n_2} and G_{n_2,n_1} are conjugate in G ([3]), we can assume $n_1 \geqq n_2$. The case $n_2 = 1$ is contained in Theorem 3.1, hence we can suppose $n_2 \geqq 2$. Put

$$s = \begin{pmatrix} I_{n_1} & 0 \\ s_1 & I_{n_2} \end{pmatrix},$$

where I_{n_1}, I_{n_2} are unit matrices and

$$s_1 = (\underbrace{s_0}_{n_2} \ \underbrace{0}_{n_1-n_2}) \} n_2,$$

$$s_0 = \begin{pmatrix} 0 & & \cdot 1 \\ & 1 \cdot^{\cdot} & \\ \pm 1 & & 0 \end{pmatrix},$$

the $+$ or $-$ sign is chosen so that $s_0 \in SL(n_2, C)$.

LEMMA 3.4. *The complement of the double coset $K_n s G_{n_1, n_2}$ in G has Haar measure zero.*

Proof. It is known (see [3] and [4]) that every element of G except those in a finite number of manifolds of lower dimension can be written as kz, where $k \in K_n$, $z \in Z_n$ (the unipotent lower triangular matrices). Put

$$z = \begin{pmatrix} z_1 & 0 \\ z' & z_2 \end{pmatrix},$$

where $z_i \in Z_{n_i}$ $(i = 1, 2)$. Consider

$$g = \begin{pmatrix} c^{n_2} k_1 z_1 & 0 \\ 0 & c^{-n_1} k_2 z_2 \end{pmatrix} \in G_{n_1 \cdot n_2},$$

where $k_i \in K_{n_i}$ and c is a nonzero complex number. Then

$$\begin{pmatrix} c^{-n_2} k_2^{-1} & 0 \\ 0 & c^{n_1} k_2^{-1} \end{pmatrix} sg = \begin{pmatrix} z_1 & 0 \\ c^{(n_1+n_2)} k_2^{-1} s_1 k_1 z_1 & z_2 \end{pmatrix}.$$

For fixed z_1, z_2, we want to find k_1, k_2 such that

$$\begin{pmatrix} c^{-n_2} k_1^{-1} & 0 \\ 0 & c^{n_1} k_2^{-1} \end{pmatrix} sg = \begin{pmatrix} z_1 & 0 \\ z' & z_2 \end{pmatrix}$$

i.e., $c^{(n_1+n_2)} k_2^{-1} s_1 k_1 z_1 = z'$, i.e.

(10) $$c^{(n_1+n_2)} k_2^{-1} s_1 k_1 = z' z_1^{-1}.$$

Let us write

$$k_1 = \begin{pmatrix} k_1' & k' \\ 0 & k_1'' \end{pmatrix},$$

where k_1' and k_1'' are upper triangular of orders n_2 and $(n_1 - n_2)$ respectively. (10) is equivalent to

(11) $$c^{(n_1+n_2)} (k_2^{-1} s_0 k_1' \quad k_2^{-1} s_0 k') = z' z_1^{-1}.$$

It is easy to see that complement of

$$\{c^{(n_1+n_2)} (k_2^{-1} s_0 k_1' \quad k_2^{-1} s_0 k') \mid c \in C^*;\ k_1',\ k_2 \in K_{n_2},\ k' \text{ is a } n_2 \times (n_1 - n_2)\text{-matrix}\}$$

in the set of all $n_2 \times n_1$ matrices has Haar measure zero.[7] In other words, for fixed z_1 and for almost all z', the equation (11) and hence (10) has a solution. Thus for almost all z, there exists $g \in G_{n_1,n_2}$ and $k \in K_{n_1,n_2} \subset K_n$ such that $ksg = z$.

We can now apply Theorem 1.3 and get $T^\chi|_{G_{n_1,n_2}} \cong \operatorname{ind}_{H' \uparrow G_{n_1,n_2}} \tilde{\chi}$. It is easy to see that $H' = G_{n_1,n_2} \cap s^{-1} K_n s$ is the subgroup of all matrices of the from

$$\begin{pmatrix} \delta_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \delta_{n_2} & & & & \\ & & & k & & & \\ & & & & \delta_{n_2} & & \\ & & & & & \ddots & \\ 0 & & & & & & \delta_1 \end{pmatrix},$$

where $\delta_i \in C^*$, and k is an upper triangular matrix of order $n_1 - n_2$ such that $\delta_1^2 \cdots \delta_n^2 \det k = 1$. Since $shs^{-1} = h$ for every $h \in H'$, $\tilde{\chi}$ is simply the restriction of χ to H'. Put

$$K' = K_{\underbrace{1,\cdots,1}_{n_2},\, n_1-n_2,\, \underbrace{1,\cdots,1}_{n_2}},$$

and let L' be the subgroup of all matrices of the form

$$\begin{pmatrix} \delta_1 & & & & & & 1 \\ & \ddots & & & & & \\ & & \delta_{n_2} & & & & \\ & & & I_{n_1-n_2} & & & \\ & & & & \delta_{n_2}^{-1} & & \\ & & & & & \ddots & \\ 0 & & & & & & \delta_1^{-1} \end{pmatrix},$$

where $\delta_i \in C^*$. Then K' is the generalized direct product of H' and L' such that $H' \cap L'$ is the finite subgroup of L' consisting of all matrices of the above form with $\delta_i = +1$ or -1. By Corollary 2.2 we have

$$\begin{aligned} \operatorname*{ind}_{H' \uparrow K'} \tilde{\chi} &\cong \tilde{\chi}|_{H'} \cdot \operatorname*{ind}_{H \cap L' \uparrow L'} \tilde{\chi}|_{H' \cap L'} \\ &\cong \tilde{\chi}|_{H'} \cdot \int' \hat{x} d\hat{x}, \text{[6]} \end{aligned}$$

where $\int'$ is taken over the set of all $\hat{x} \in \hat{L}'$ such that $\hat{x}|_{H' \cap L'} = \tilde{\chi}|_{H \cap L'}$. Therefore

[7] This can be seen by using a similar result for $SL(n, C)$ proved in [3].

$$\operatorname*{ind}_{H'\uparrow K'} \tilde{\chi} \cong \int' \tilde{\chi}\,|_{H'} \cdot \hat{x}\, d\hat{x}$$
$$\cong \int_{A_{\tilde{\chi}}} \lambda\, d\lambda\,,$$

where $A_{\tilde{\chi}}$ is the set of all one-dimensional representations of K' which extends $\tilde{\chi}$ and $d\lambda$ is the transform of the Haar measure of $A_{1_{H'}}$ (viewed as a locally compact abelian group) by the translation

$$\lambda \longmapsto \lambda_0\lambda,\ \ \lambda \in A_{1_{H'}},\ \ \lambda_0$$

is a fixed element of $A_{\tilde{\chi}}$. In summary, we have by Theorems 1.1 and 1.5

$$T^{\chi}\,|_{G_{n_1,n_2}} \cong \operatorname*{ind}_{K'\uparrow G_{n_1,n_2}} \big(\operatorname*{ind}_{H'\uparrow K'} \tilde{\chi}\big)$$
$$\cong \int_{A_{\tilde{\chi}}} \operatorname*{ind}_{K'\uparrow G_{n_1,n_2}} \lambda\ d\lambda\,. \tag{12}$$

Since K' is also the generalized direct product of D_{n_1,n_2} and $K_{1,\cdots,1,n_1-n_2} \times D_{n_2}$ we can write by using Corollary 2.2 and Theorem 1.2:

$$\operatorname*{ind}_{K'\uparrow G_{n_1,n_2}} \lambda \cong \lambda\,|_{D_{n_1,n_2}} \operatorname*{ind}_{K_{1,\cdots,1,n_1-n_2}\uparrow SL(n_1,C)} \lambda\,|_{K_{1,\cdots,1,n_1-n_2}} \chi_{D_{n_2}} \operatorname*{ind}_{\uparrow SL(n_2,C)} \lambda\,|_{D_{n_2}}\,. \tag{13}$$

It remains to apply Lemma 3.3 or Corollary 3.2 and carry out the computations. We have

THEOREM 3.5. *Let $n = n_1 + n_2$, $n_1, n_2 \geqq 2$. Then the restriction of the element $T^{(m_2,\cdots,\rho_n)}$ of the non-degenerate principal series of $SL(n, C)$ to G_{n_1,n_2} is equivalent to*

$$\sum_{\Sigma_2^n k_i \equiv \Sigma_2^n m_i \pmod n} \int \cdots \int_\infty U^{(k_2,\cdots,k_n;\,\sigma_2,\cdots,\sigma_n)}\, d\sigma_2 \cdots d\sigma_n$$

where $U^{(k_2,\cdots,\sigma_n)}$ is an element of the principal series of G_{n_1,n_2}.

Proof. Using the explicit parametrization of the set $A_{\tilde{\chi}}$ occurring in (12) we see that the restriction of $T^{(m_2,\cdots,\rho_n)}$ to G_{n_1,n_2} is equivalent to

$$\sum_{k_2,\cdots,k_{n_2+1}} \int \cdots \int \operatorname*{ind}_{K'\uparrow G_{n_1,n_2}} (k_2, \cdots\cdot \sigma_n)\ d\sigma_2 \cdots d\sigma_{n_2+1}\,, \tag{14}$$

where $k_{n_2+2}, \cdots, k_n;\ \sigma_{n_2+2}, \cdots, \sigma_n$ depend linearly on $k_2, \cdots, k_{n_2+1}$; $\sigma_2, \cdots, \sigma_{n_2+1}$ by some simple formula. Put $\lambda = (k_2, \cdots, k_n; \sigma_2, \cdots, \sigma_n)$. Then Lemma 3.3 (or Corollary 3.2) and (13) show that:

$$\operatorname*{ind}_{K' \uparrow G_{n_1,n_2}} \lambda \cong \Sigma' \Sigma'' \int \cdots \int \infty \lambda_{D_{n_1,n_2}} T^{(h'_2,\cdots,\tau'_{n_1})} \tag{15}$$
$$\times T^{(h'_{n_1+2},\cdots,\tau'_n)} d\tau'_2 \cdots d\tau'_{n_1} d\tau'_{n_1+2} \cdots d\tau'_n ,$$

where Σ' extends over all $(h'_2, \cdots, h'_{n_1})$ such that $\sum_2^{n_1} h'_i \equiv \sum_2^{n_1} k_i \pmod{n_1}$, and Σ'' over all $(h'_{n_1+2}, \cdots, h'_n)$ such that $\sum_{n_1+2}^{n} h'_i \equiv \sum_{n_1+1}^{n} k_i \pmod{n_2}$. By Corollary 2.2 $\lambda_{D_{n_1 \cdot n_2}} T^{(h'_2,\cdots,\tau'_{n_1})} \times T^{(h'_{n_1+2},\cdots,\tau'_n)}$ is equivalent to an element $U^{(h_2,\cdots,\tau_n)}$ of the principal series of $G_{n_1 n_2}$ where $(h_2, \cdots, \tau_n)$ can be easily computed in terms of h'_i and τ'_i. Using this parametrization, (15) becomes

$$\operatorname*{ind}_{K' \uparrow G_{n_1,n_2}} \lambda \cong \tilde{\Sigma} \int \cdots \int \infty U^{(h_2,\cdots,\tau_n)} d\tau_2 \cdots d\tau_{n_1} d\tau_{n_1+2} \cdots d\tau_n , \tag{16}$$

where $\tilde{\Sigma}$ extends over all $(h_2, \cdots, h_n)$ such that

$$\begin{cases} \sum_{n_1+1}^{n} h_i \equiv \sum_{n_1+1}^{n} k_i \pmod{n_2} \\ n_2 \sum_2^{n_1} h_i - n_1 \sum_{n_1+1}^{n} h_i = n_2 \sum_2^{n_1} k_i - n_1 \sum_{n_1+1}^{n} k_i \end{cases} \tag{17}$$

and

$$\tau_{n_1+1} = \frac{n_2}{n_1}\left(\sum_2^{n_1} \tau_i - \sum_2^{n_1} \sigma_i\right) + \sum_{n_1+1}^{n} \sigma_i - \sum_{n_1+2}^{n} \tau_i .$$

Thus applying Theorem 1.5 to (14) and taking (16) into account we get

$$T^{(m_2,\cdots,\rho_n)} |_{G_{n_1 n_2}} \cong \sum_{k_2,\cdots,k_{n_2+1}} \tilde{\Sigma} \int \cdots \int$$
$$\infty U^{(h_2,\cdots,\tau_n)} d\sigma_2 \cdots d\sigma_{n_2+1} d\tau_2 \cdots d\tau_{n_1} d\tau_{n_1+2} \cdots d\tau_n .$$

Fix $k_2, \cdots, k_{n_2+1}, h_2, \cdots, h_n, \sigma_2, \cdots, \sigma_{n_2}$. Then the mapping

$$(\tau_2, \cdots, \tau_{n_1}, \sigma_{n_2+1}, \tau_{n_1+2}, \cdots, \tau_n) \longmapsto (\tau_2, \cdots, \tau_n)$$

is a measure preserving homeomorphism of $\mathbf{R}^{n-1}$ onto itself. Since each component in the above decomposition is independent of $\sigma_2, \cdots, \sigma_{n_2}$ and the multiplicity is already everywhere infinite, the decomposition itself is equivalent to

$$\sum_{k_2,\cdots,k_{n_2+1}} \tilde{\Sigma} \int \cdots \int \infty U^{(h_2,\cdots,\tau_n)} d\tau_2 \cdots d\tau_n .$$

Now it is easy to see $(k_2, \cdots, k_{n_2+1}, h_2, \cdots, h_n) \mapsto (h_2, \cdots, h_n)$ maps the set of all $(k_2, \cdots, k_{n_2+1}, h_2, \cdots, h_n)$ satisfying (17) *onto* the set of all $(h_2, \cdots, h_n)$ such that $\sum_2^n h_i \equiv \sum_2^n m_i \pmod{n}$.

We now come to the general case. Let $\hat{D}_n^+$ be a fundamental domain defined in § 2.

THEOREM 3.6 *Let $n = n_1 + \cdots + n_r$, $r \geqq 2$. In the case $r = 2$ we also assume $n_1, n_2 \geqq 2$. Then the restriction of $T^{(m_2,\cdots,\rho_n)}$ to $G_{n_1,\cdots,n_r}$ is equivalent to $\Sigma \int \cdots \int \infty\, U^{(k_2,\cdots,\sigma_2)}\, d\sigma_2 \cdots d\sigma_n$ where $U^{(k_2,\cdots,\sigma_n)}$ belongs to the principal series of $G_{n_1,\cdots,n_r}$ and $\Sigma \int \cdots \int$ is the summation integral over the set of all $(k_2, \cdots, \sigma_n) \in \hat{D}_n^+$ such that $\sum_2^n k_i \equiv \sum_2^n m_i \pmod n$.*

Proof. We shall use an induction on r. Put $m = n_1 + \cdots + n_{r-1}$. Then $G_{n_1,\cdots,n_r} \subset G_{m,n_r}$. Thanks to Theorem 3.5 it is sufficient to decompose the restriction to $G_{n_1,\cdots,n_r}$ of any element U^ρ of the principal series of G_{m,n_r}. On the other hand Corollary 2.2. gives

$$U^\rho = \rho\,|_{D_{m,n_r}} (T^{\rho_1} \times T^{\rho_2})$$

where $\rho_1 = \rho\,|_{K_m}$, $\rho_2 = \rho\,|_{K_{n_r}}$ and T^{ρ_1}, T^{ρ_2} are the corresponding elements of the non-degenerate principal series of $SL(m, C)$ and $SL(n_r, C)$ respectively. Therefore

$$U^\rho\,|_{G_{n_1,\cdots,n_r}} = \rho\,|_{D_{m,n_r}} (T^{\rho_1}\,|_{G_{n_1,\cdots,n_{r-1}}} \times T^{\rho_2}) .$$

By induction hypothesis, $T^{\rho_1}\,|_{G_{n_1,\cdots,n_{r-1}}}$ is decomposed in terms of the principal series of $G_{n_1,\cdots,n_{r-1}}$, hence we have decomposed $U^\rho\,|_{G_{n_1,\cdots,n_r}}$ in terms of representations of the form $\rho\,|_{D_{m,n_r}} (U^\sigma \times T^{\rho_2})$, where U^σ is some element of the principal series of $G_{n_1,\cdots,n_{r-1}}$. In fact those representations occurring in the decomposition are elements of the principal series of $G_{n_1,\cdots,n_r}$ as seen easily by Corollary 2.2.

Again the detailed computation is based on some change of variables similar to that in the proof of Theorem 3.5 and will not be repeated here.

COROLLARY 3.7. *The restriction of every element of the non-degenerate principal series of $SL(n, C)$ to $SL(n_1, C) \times \cdots \times SL(n_r, C)$ is equivalent to the regular representation.*

4. Application to the decomposition of some tensor products. It is known that the character of $D_{n_1,\cdots,n_r}$ is parametrized by $Z^{r-1} \times R^{r-1}$.[8] Let $\tilde{\chi} = (k_2, \cdots, \sigma_r)$ be such a character, then $\tilde{\chi}$ extends in an obvious manner to a one-dimensional representation of the sub-

[8] See [3] for an explicit description and the proof of the irreducibility of the degenerate principal series.

group $H_{n_1,\cdots,n_r}$ consisting of all block matrices $g=(g_{ij})_{1\leq i,j\leq r}$ such that $g_{ij}=0$ for $i>j$. More explicitly

$$\tilde{\chi}(g)=\det(g_{2,2})^{k_2+i\sigma_2}\,|\det g_{2\,2}|^{-k_2}\cdots(\det g_{r,r})^{k_r+i\sigma_r}\,|\det g_{r,r}|^{-k_r}\,,$$

for $g=(g_{i,j})\in H_{n_1,\cdots,n_r}$. This representation induces to an irreducible representation of G^8 belonging to the $(n_1,\cdots,n_r)$-degenerate principal series of G and denoted by $\tilde{T}^{\tilde{\chi}}$. Let T^{χ} be any element of the non-degenerate principal series. The problem is to decompose $T^{\chi}\otimes\tilde{T}^{\tilde{\chi}}$ into irreducible repesentations of G. Since $H_{n_1,\cdots,n_r}\supset K_n$, the complement of the double coset $K_n s_0 H_{n_1,\cdots,n_r}$ in G has Haar measure zero.[7] Recall that

$$s_0=\begin{pmatrix}0 & & & 1\\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ \pm 1 & & & 0\end{pmatrix}\in SL(n,C)\,.$$

It is clear that $K_n\cap s_0H_{n_1,\cdots,n_r}s_0^{-1}=K_{n_r,\cdots,1}$. Put $\chi'(k)=\chi(k)\tilde{\chi}(s_0^{-1}ks_0)$, for $k\in K_{n_r,\cdots,n}$. Theorems 1.4 and 1.1 give us

$$T^{\chi}\otimes\tilde{T}^{\tilde{\chi}}\cong\operatorname*{ind}_{K_{n_r,\cdots,n_1}\uparrow G}\chi'\cong\operatorname*{ind}_{G_{n_r,\cdots,n_1}\uparrow G}U^{\chi'}\,. \tag{18}$$

LEMMA 4.1. *Let $n=n_1+\cdots+n_r$ be as in Theorem 3.6. Then*

$$\operatorname*{ind}_{G_{n_1,\cdots,n_r}\uparrow G}U^{(m_2,\cdots,\rho_n)}\cong\Sigma\int\cdots\int_{\infty}T^{(k_2,\cdots,\sigma_n)}d\sigma_2\cdots d\sigma_n\,, \tag{19}$$

where $\Sigma\int\cdots\int$ is the summation-integral over the set of all $(k_2,\cdots,\sigma_n)\in\hat{D}_n^0$ such that $\sum_2^n k_i\equiv\sum_2^n m_i$ (mod n).

Proof. Corollary 1.10 together with Theorem 3.6 prove that (19) is valid for every $m_2,\cdots,m_n$ and for almost all $\rho_2,\cdots,\rho_n$. On the other hand, let $H'_{n_1\cdots,n_r}=s_0H_{n_r,\cdots,n_1}s_0^{-1}\supset Z_n$. Then the complement of $H'_{n_1,\cdots n_r}\cdot K_n$ in G has Haar measure zero. Thus by Theorems 1.3 and 1.1:

$$\operatorname*{ind}_{G_{n_1,\cdots,n_r}\uparrow H'_{n_1,\cdots,n_r}}U^{(m_2,\cdots,\rho_n)}\cong T^{(m_2,\cdots,\rho_n)}|_{H'_{n_1,\cdots,n_r}}\,.$$

On the other hand

$$T^{(m_2,\cdots,\rho_n)}|_{H'_{n_1,\cdots,n_r}}\cong T^{(m'_2,\cdots,\rho'_n)}|_{H'_{n_1,\cdots,n_r}}$$

if and only if $\sum_2^n m_i\equiv\sum_2^n m'_i$ (mod n). This equivalence can be proved by using a slight modification of the proof of Theorem 3 of [4].

Therefore

$$\operatorname*{ind}_{G_{n_1,\cdots,n_r} \uparrow G} U^{(m_2,\cdots,\rho_n)} \cong \operatorname*{ind}_{G_{n_1,\cdots,n_r} \uparrow G} U^{(m_2',\cdots,\rho_n')}$$

if and only if $\sum_2^n m_i \equiv \sum_2^n m_i' \pmod n$.

THEOREM 4.2. *The tensor product of an element $T^{(m_2,\cdots,\rho_n)}$ of the non-degenerate and an element $\widetilde{T}^{(k_2,\cdots,\sigma_r)}$ of the $(n_1, \cdots, n_r)$-degenerate principal series of $SL(n, C)$ is equivalent to $\Sigma \int \cdots \int \varepsilon T^{(h_2,\cdots,\tau_n)}\, d\tau_2 \cdots d\tau_n$ where $\Sigma \int \cdots \int$ is the summation-integral over the set of all $(h_2, \cdots, h_n; \tau_2, \cdots, \tau_n) \in \hat{D}_n^0$ such that $\sum_2^n h_i \equiv \sum_2^n m_i + \sum_2^r n_i k_i \pmod n$.*

The multiplicity $\varepsilon = \infty$ if (a) $r > 2$ or (b) $r = 2$ and $n_1, n_2 \geqq 2$. Otherwise $\varepsilon = 1$.

Proof. It is enough to apply Lemma 4.1 in the first case $(\varepsilon = \infty)$ or Corollary 3.2 in the second case $(\varepsilon = 1)$ to obtain the decomposition of the induced representation occurring in the right hand side of (18).

In the special case $r = n$, $\widetilde{T}^{(k_2,\cdots,\sigma_r)}$ is another element of the non-degenerate principal series and hence

COROLLARY 4.3. *The tensor product $T^{(m_2,\cdots,\rho_n)} \otimes T^{(k_2,\cdots,\sigma_n)}$ of two elements of the nondegenerate principal series of $SL(n, C)$ can be decomposed as follows: $\Sigma \int \cdots \int \varepsilon T^{(h_2,\cdots,\tau_n)}\, d\tau_2 \cdots d\tau_n$, where $\Sigma \int \cdots \int$ is the summation-integral over the set of all $(h_2, \cdots, r_n) \in \hat{D}_n^0$ such that $\sum_2^n h_i \equiv \sum_2^n (m_i + k_i) \pmod n$, and $\varepsilon = 1$ if $r = 2$, $\varepsilon = \infty$ if $r > 2$.*

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