PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 3, 1971

ON THE MULTIVALENCE OF A CLASS OF MEROMORPHIC FUNCTIONS

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The estimates are given for the radius of the largest disk about the origin on which all the functions of the form $a/z+\phi(z)$ are p-valent, where $\phi(z)$ is an analytic function defined in the unit disk and of modulus less than unity there. Similar results are obtained concerning the starlikeness of the image of circles about the origin.

Suppose a polynomial of degree n assumes p-times a certain value in the center of a circle and does not take this value elsewhere in this circle. The author determines the largest concentric circle in which the polynomial is p-valent. The same problem is then considered in a more general setting and similar results are obtained.

1. Introduction. In this paper we shall deal with two questions. The first part deals with the multivalence of meromorphic functions of the form

$$f(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}\,, \tag{1}$$

where $A_k > 0$ and $a_k \in S$, $k = 1, 2, \cdots, n$; S being the closed unit disk.

Actually most of the results will be deduced for a more general class of functions, namely those representable in the form

$$g(z) = \frac{a}{z} + \phi(z),\ a > 0\,, \tag{2}$$

defined for all z such that $0 < |z| < 1$ and such that the function $\phi(z)$ is analytic and of modulus less than unity in the unit disk.

The relation between the functions of the form (1) and (2) becomes clearer if we apply a result due to J. L. Walsh and the author [11] (see loc. cit. Theorem C), according to which the function $f(z)$ defined by (1) can also be written as

$$f(z) = \frac{A}{z - \alpha(z)}\,, \tag{3}$$

where $A = \sum_{k=1}^{n} A_k$ and where $\alpha(z)$ is an analytic function defined for all z such that $|z| > 1$ and satisfying $|\alpha(z)| < 1$ there. Obviously the functions of the type (2) can now be obtained from (3) by simple transformations.

R. Distler [5] determined the domain of univalence for functions of the form (1), where S is an arbitrary set. He has shown that this domain is the interior of the director set of the closed convex hull C of S, that is, the set of all points such that C subtends an angle of $\pi/2$ at each of these points. For example the director set of the line segment $[-1, 1]$ is the unit circle and the director set of this circle is the circle $|z| = 2^{1/2}$. Goodman [7] (p. 1043) mentions that the problem of the determination of the domain of p-valence ($p \geqq 2$), starlikeness and convexity of these functions remains open.

In the first part of this note we shall determine the domain of univalence of the functions of the form (2) and we shall estimate the domain of p-valence of functions of the form (1) and (2). The results abtained are probably not sharp in the case of $p \geqq 2$. We also give an estimate of the domain of starlikeness for functions of type (2). Often for simplicity we shall assume $a = 1$.

In the second part, two related problems in the theory of polynomials are considered.

Problem 1. If a polynomial of degree n assumes p times ($1 \leqq p < n$ and counting multiplicities) a certain value in the center of a circle and does not take this value elsewhere in this circle, find the largest concentric circle in which the polynomial is p-valent.

Problem 2. If a polynomial of degree n assumes p times in a circle ($1 \leqq p < n$) the values it has at the center, find the largest concentric circle in which the polynomial is necessarily p-valent. For simplicity we shall assume that the circle in question is the unit circle.

In this case it is easy to show that there exist numbers depending only on p and n which solve these problems. For general polynomials Problem 1 was solved by M. Biernacki [1], Chapter 11, who also gave an estimate to the solution of Problem 2.

We shall generalize Problems 1 and 2 to lacunary polynomials. We shall solve Problem 1 and give lower and upper bounds for Problem 2.

2. Multivalence of certain meromorphic functions.

Here we need the following result due to Minami [8], (see also [2], Chapter 11, Theorem 44).

THEOREM A. *Let $\phi(z)$ be a regular function defined in a convex domain D in the complex plane and let a be an arbitrary constant. Furthermore suppose that*

(1) *For $z \in D$, $\phi^{(p)}(z) \in A$, where A is a convex domain;*

(2) *There exists a polar set E* (a set of the form $\theta_1 \leqq \arg z \leqq \theta_2$,

$0 \leqq r_1 \leqq |z| \leqq \infty$, $\theta_1 \leqq \theta_2$ are real numbers) *such that the polar region* E_1 *described by the expression* $\omega = (-1)^{p+1}p!az^{-p-1}$ *as* $z \in E$ *is disjoint from* A.

Then the function

$$g(z) = \frac{a}{z} + \phi(z)$$

is completely p-valent (that is, $g(z) + c_0 + c_1 z + \cdots + c_{p-1}z^{p-1}$ *is at most p-valent for arbitrary constants* c_i; $i = 0, 1 \cdots, p-1$) *in the intersection* $D \cap E$.

We are now in a position to establish

THEOREM 1. *The function* $g(z)$ *defined by* (2) *is univalent in the disk*

$$|z| < \left(\frac{a}{1+a}\right)^{1/2}.$$

This estimate is sharp for $a = 1$ and for all a the best estimate of the radius of the disk about the origin for which all functions of type $g(z)$ are univalent does not exceed the number

$$\operatorname{Min}\left[\frac{(2a)^{1/2}}{1+a^{1/2}}, a^{1/2}\right].$$

Proof. Let ρ be a number such that $0 < \rho < 1$ and let ρ_1 be a positive number such that $\rho_1^2 < a(1-\rho^2)$. We now apply Theorem A with $D: |z| < \rho$, $A: |z| < (1-\rho^2)^{-1}$, $E: |z| < \rho_1$ and $E_1: |z| \geqq a\rho_1^{-2}$. It follows that $|\phi'(z)| \leqq (1-|z|^2)^{-1}$ for all $|z| < 1$ and thus by Theorem A $g(z)$ is univalent in the disk

$$(4) \qquad |z| < \operatorname{Min}\{\rho, [a(1-\rho^2)]^{1/2}\}.$$

The result now follows by taking the maximum of the righthand side of inequality (4) *as* ρ varies in the interval $(0, 1)$.

To prove the second part of the theorem, consider the function $\phi(z)$ defined by

$$\frac{1}{z-1} + \frac{1}{z-i} = \frac{2}{z-\alpha(z)}$$

and

$$\phi(z) = \alpha\left(\frac{1}{z}\right) = \frac{z+c}{i-cz},$$

where $c = (-1+i)/2$.

In this case $g'(z) = 0$ if z satisfies the relation

$$2iz^2 + a[2 - (1 + i)z]^2 = 0 .$$

If $a > 0$, the solutions of the last equation are of modulus

$$(2a)^{1/2}/(1 + a)^{1/2} .$$

If a is a complex number, by choosing an appropriate argument for a this modulus can be diminished to $(2\,|a|)^{1/2}/(1 + |a|^{1/2})$. However, this does not involve loss of generality, since a rotation of the independent variable will not affect the domain of univalence in the context of the theorem. Also, the example $a/z + z$ shows that the radius of univalence cannot exceed $a^{1/2}$.

COROLLARY. *When $a = 1$, we obtain a result due to Čakalov* [4], *namely that $f(z)$ as defined by* (1) *is univalent in the domain $|z| > 2^{1/2}$, and this domain is maximal for the class of all such functions. With regard to the question of p-valence ($p \geqq 2$) of the functions of the form* (1) *or* (2), *we have*

THEOREM 2. *Let $g(z)$ be a function as defined by* (2). *Then $g(z)$ is completely p-valent in the disk $|z| < r_p$ where r_p is the unique positive root (<1) of the equation*

$$x^{(p+3)/2} = ap!(1 - x^{2/p})^p .$$

Proof. The proof of this is similar to that of Theorem 1, once the inequality $|\phi^{(p)}(z)| < r^{(-p(p-1))/2}(1 - r^2)^{-p}$ valid for $|z| < r^p (0 < r < 1)$ is established by induction. We shall omit the details.

We shall now sharpen the results obtained thus far for functions of the class (1) or (2) by considering the symmetry condition on the points a_k. More precisely, by imposing the condition:

(5) $$\sum_{k=1}^{n} a_k^j = 0, j = 1, 2, \cdots, l$$

or the corresponding conditions

(6) $$|\alpha(z)| \leqq |z|^{-l} \quad \text{for} \quad |z| > 1$$

and

$$|\varphi(z)| \leqq |z|^l \quad \text{for} \quad |z| < 1$$

where $\alpha(z)$ and $\phi(z)$ are defined by (3) and (2) respectively. (See loc. cit., Theorem C, in § 3.)

In this case it is known [6], Chap. VIII, Section 1, Theorem 5, that

$$(7)\qquad |\phi'(z)| \leqq \frac{l|z|^{l-1}}{1-|z|^{2l}}(1-|\phi(z)|^2) \leqq \frac{l|z|^{l-1}}{1-|z|^{2l}}$$

We shall prove

THEOREM 3. *Let $f(z)$ and $g(z)$ be as defined by* (3) *and* (2) *respectively and let conditions* (6) *be satisfied. Then* (a) *The function $f(z)$ is univalent in the region $|z|\, r_1(l)$ where $r_1(l)$ is the largest positive root of the equation*

$$x^{2l+2} - r^2(l)x^{2l} - 1 = 0$$

and where $r(l)$ $(1 < r(l) < r_1(l))$ is the largest positive root of the equation

$$y^{2l} - ly^{l-1} - 1 = 0 \, .$$

(b) *The function $g(z)$ is univalent in the disk $|z| < r_0(l)$, where $r_0(l)$ is the positive root of the equation*

$$lz^{l+1} = a(1 - z^{2l}) \, .$$

Proof. (a) Let $F(z) = (A/f(z)) = z - \alpha(z)$, where $|\alpha(z)| \leqq |z|^{-l}$ for $|z| > 1$. If $F(z_1) = F(z_2)$ where $|z_1| > 1$, $|z_2| > 1$, then $|z_1 - z_2| \leqq 2r_1^{-l}$ with $r_1 = \min(|z_1|, |z_2|)$. Also $\mathrm{Re}F'(z) > 1 - |\alpha'(z)| > 0$ if $|z| > r(l)$, where $r(l)$ is as described above. Furthermore, it is clear that if $|z_i| \geqq r_1(l)$ $(i = 1, 2)$, then the line segment joining z_1 to z_2 lies outside the disk $|z| \leqq d$ where $d^2 = r_1^2 - r_1^{-2l}$. Thus the function $F(z)$ is univalent in the region $|z| > r_1(l)$ if $r_1(l)$ is such that $d \geqq r(l)$.
(b) The proof is analogous to the proof of Theorem 1. We notice that if $a > 0$

$$r_0(l) > \left[\frac{(l^2 + 4a^2)^{1/2} - l}{2a}\right]^{1/l} .$$

Obviously the method discussed here can be applied to obtain estimates on the radius of the disk about the origin in which all functions in Theorem 3(b) are completely p-valent.

It seems, however, that these bounds are not sharp for $p > 1$. The question of the exact bound remains open even in the apparently simpler case of polynomials which we shall take up in the next section.

We conclude this section with one estimate on the number R such that the image of $|z| = R$ by functions of type (1) is starlike with respect to the origin. For simplicity we shall assume that $a = 1$.

THEOREM 4. *Let $f(z)$ be a function defined by* (1). *Then if $R > (27/11)^{1/2}$ then the image of $|z| = R$ is starlike with respect to the origin.*

Proof. By (1) and (3)

$$g(z) = \frac{A}{f(1/z)} = \frac{1}{z} - \alpha\left(\frac{1}{z}\right) = \frac{1}{z} + \beta(z)$$

defined for $0 < |z| < 1$ with $\beta(z)$ analytic and $|\beta(z)| \leqq 1$ in the unit disk. Let

$$h(z) = \frac{1}{g(z)} .$$

$h(z)$ is analytic in the unit disk and $h(0) = 0$. We investigate the starlikeness of $h(z)$. To this effect we notice that the number

$$\mathrm{Re}\left\{z\frac{h'(z)}{h(z)}\right\} = \mathrm{Re}\left\{\frac{1 - z^2\beta'(z)}{1 + z\beta(z)}\right\}$$

will be positive if and only if

$$\mathrm{Re}\,[1 + \overline{z\beta} - z^2\beta' - z\,|z|^2\,\bar{\beta}\beta'] > 0 . \tag{8}$$

Let $|z| = r$, $|\beta(z)| = a$. Then $|\beta'(z)| \leqq (1 - a^2)/1 - r^2)$ and (8) reduces to

$$ar + r^2\frac{1 - a^2}{1 - r^2} + ar^3\frac{1 - a^2}{1 - r^2} < 1$$

or

$$\phi(a) = ar + r^2(2 - a^2) - a^3r^3 < 1 .$$

Now $\phi(a)$ is positive and increasing for $0 \leqq a \leqq (3r)^{-1}$ so that $\phi(a) < \phi(1) < 1$ if $0 \leqq a < 1$ and $0 \leqq r \leqq 1/3$ if, however, $r > 1/3$ then

$$\phi(a) \leqq \phi\left(\frac{1}{3r}\right) = 2r^2 + \frac{5}{27} < 1$$

for $0 \leqq a < 1$ and $r^2 < 11/27$. The theorem is established.

It is perhaps worthwhile to mention that the function $f(z)$ of the type (1) possesses the property

$$\mathrm{Re}\left[\frac{1}{f(z)}\right]' > 0 \quad \text{for} \quad |z| > \sqrt{2} .$$

It is known (Noshiro-Warschawski-Wolff [2] p. 17) that an analytic function $\psi(z)$ in a convex domain is univalent if $\mathrm{Re}\,\psi'(z) > 0$ there. However, the geometric properties of the range that characterize these functions $\psi(z)$, even in simple domains as the unit disk, are not known. (See [9], p. 317.)

3. Multivalence of lacunary polynomials. There are two related problems concerning the valence of polynomials in a disk on the basis of the knowledge of the distribution of one of its values.

PROBLEM 1. If a polynomial of degree n assumes p times $(1 \leqq p < n)$ a certain value (counting multiplicities) in the center of a circle and does not take this value elsewhere in this circle, find the largest concentric circle in which the polynomial is p-valent.

PROBLEM 2. If a polynomial of degree n assumes p times $(1 \leqq p < n)$ the values it has at the center, find the largest concentric circle in which the polynomial is necessarily p-valent.

For simplicity we shall assume that the circle in Problems 1 and 2 is the unit circle. In this case it is easy to show that there exist numbers depending only on p and n which are solutions to these problems. For arbitrary polynomials a sharp bound for Problem 1 was given by M. Biernacki [1], p. 627, who also gave estimates for the solution of Problem 2. (See [1], p. 632, and [3], p. 92.)

We shall here generalize Problems 1 and 2 to lacunary polynomials, for which we solve Problem 1 and give bounds for the solution of Problem 2.

The following results shall be applied in the proofs.

THEOREM B. [10], Theorem 5. *Let*

$$P(z) = a_p z^p + a_{p-s} z^{p-s} + \cdots + a_0 ,$$

$Q(z) = b_q z^q + b_{q-t} z^{q-t} + \cdots + b_0$, $a_p b_q \neq 0$, $q > p$, $s \geqq 1$, $t \geqq 1$, *have their zeros in the disks* $|z| \leqq R_1$ *and* $|z| \leqq R_2$ *respectively. Let* $r = \operatorname{Min}(s, t)$. *Then at least* p *zeros of the polynomial*

$$P(z) + \lambda Q(z)$$

lie in the disk

$$|z| \leqq \operatorname{Max}\left\{\left(\frac{qR_1^r + pR_1^r}{q-p}\right)^{1/r}, R_2\right\}. \tag{9}$$

REMARK. In [10] the bound (9) was deduced but no discussion about its sharpness was made. However, following the analysis M. Biernacki [1], pp. 625-627, made on the location of the polynomial

$$(X + P)^p + a(X - Q)^q = 0 \tag{10}$$

where $q > p$ are integers, P and Q positive real numbers and a is such that the equation (10) has a double real root, one deduces that the polynomial:

$$(X^r + R_1^r)^\alpha + a(X^r - R_2^r)^\beta ,$$

where

$$r = s = t,\ p = \alpha r,\ q = \beta r,\ a = (-1)^{\alpha+\beta+1}\frac{\alpha^\alpha}{\beta^\beta}\left(\frac{\beta-\alpha}{R_1^r + R_2^r}\right)^{\beta-\alpha}$$

has exactly p zeros in the disk (9) and r double zeros on its circumference. One sees, therefore, that if $s = t$ then the bound of Theorem B is sharp.

THEOREM C. [11], Lemma 2(a). *If* $|\alpha_k| \leqq r$ *and* $m_k > 0$, $k = 1, 2, \cdots, s$, *there exists an analytic function* $\alpha(z)$ *defined for all* $|z| > r$ *satisfying*

$$\sum_{k=1}^{s}\frac{m_k}{z-\alpha_k} = \sum_{k=1}^{s}\frac{m_k}{z-\alpha(z)}$$

and such that $|\alpha(z)| \leqq r$ for $|z| > r$. *If in addition* $\sum_{k=1}^{s} m_k\alpha_k^l = 0$ *for* $l = 1, 2, \cdots, p$, then $|\alpha(z)| \leqq r^{p+1}/|z|^p$ for $|z| > r$.

We are now in a position to prove

THEOREM 5. *Suppose that the polynomial*

$$f(z) = z^p(1 + a_r z^r + \cdots + a_{n-p}z^{n-p})$$

$a_{n-p} \neq 0, 1 \leqq p < n, 1 \leqq r \leqq n - p$, *does not vanish in the unit disk except for a p-fold zero at the origin. Then* $f(z)$ *is necessarily p-valent in the disk*

$$|z| \leqq \left(\frac{p}{n}\right)^{1/r}.$$

Furthermore the polynomials.

$$z^p(1 + z^r)^k$$

for $n = k_l + p, k = 1, 2 \cdots$ *satisfy the hypotheses of the theorem and are not p-valent in any disk about the origin of radius larger than* $(p/n)^{1/r}$.

Proof. Write $f(z) = z^p g(z)$. The polynomial $h(z) = z^{n-p}g(1/z) = a_{n-p} + a_{n-p-1}z + \cdots + a_r z^{n-p-r} + z^{n-p}$ has all its zeros in the disk $|z| < 1$.

The equations $h(z) + az^n$ and $f(z) + a$ are reciprocal to each other. Therefore if the polynomial $h(z) + az^n$ has at least $(n - p)$ zeroes in a disk $|z| < 1/\rho$ for arbitrary a, then $f(z)$ is necessarily p-valent in the disk $|z| < \rho$. By Theorem B with ρ replaced by $(n - p), s = r$, $Q(z) = z^n$, $R_2 = 0$, $t = n$, $R_1 = 1$ and $q = n$, we deduce that $h(z) + az^n$ has at least $(n - p)$ zeros in the disk $|z| < (n/p)^{1/r}$. This proves the first part of the theorem.

To establish the second part of the theorem, consider the polynomial

$$f(z) = g(z^r)^{1/r} = z^p(1 + z^r)^k$$

where $g(z) = z^p(1 + z)^{n-p}$, $n = kr + p$, $k = 1, 2, \cdots$. It is known [1], p. 628, that the polynomial $g(z)$ is p-valent in the disk $|z| < p/n$, but it takes at least $(p + 1)$ times the value $a = (-1)^p p^p(n - p)^{n-p}/n^n$ in any larger disk about the origin. Therefore the polynomial $f(z)^r$ takes the value a at least $(p + 1)r$ times in any disk about the origin whose radius is greater than $(p/n)^{1/r}$. It now follows easily that $f(z)$ must take at least one of the r possible values $a^{1/r}$ at least $(p + 1)$ times in this disk. The proof is complete.

With regard to Problem 2, we have

THEOREM 6. *Suppose that the polynomial*

$$f(z) = a_q z^q + \cdots + a_n z^n ,$$

$a_q \neq 0$, $1 \leqq q < n$, *vanishes exactly* p *times* (counting multiplicities) *in the disk* $|z| < 1$ *and assume, furthermore, that the* $(n - p)$ *zeros of* $f(z)$, *which lie outside the unit disk, say* $z_1, \cdots, z_{n-p}$ *possess the symmetry property*

$$\sum_{i=1}^{n-p} z_i^{-j} = 0, j = 1, 2, \cdots, k - 1 , \tag{11}$$

where $k = 1, 2, \cdots$. *When* $k = 1$, *we shall agree that no conditions of the form* (11) *are imposed. Then the polynomial* $f(z)$ *is p-valent in the disk*

$$|z| < 2pA^{-1}e^{-2p} ,$$

where

$$A = \text{Max}\left[\left(\frac{2n - p + 1}{p + 1}\right)^{1/k}, \frac{n - p}{k}\right].$$

REMARK. Actually $A \leqq A' = \text{Max}\,\{(n-p)^{1/k}, (n-p)/k)\}$ if $p \leqq n-2$ and $p \neq 1, 2$ or if $p \neq 1$ and $n \geqq 5$. The case $p = 1$ has been solved in Problem 1 and the case $p = n - 1$ has been solved by G. Szego [12], Theorem 12′, p. 46.

Proof. Write $g(z) = z^{-q}f(z)$ and $z^{n-q}g(1/z) = P(z)Q(z)$, where the polynomials $P(z)$ and $Q(z)$ are of degrees $(n - p)$ and $(p - q)$ respectively, and such that all the zeros of $P(z)$ lie in the unit circle. Moreover, by hypothesis the first $(k - 1)$ moments of these zeros vanish. We may assume $0 \leqq k \leqq n - p - 1$. The equations $P(z)Q(z) + bz^n = 0$ and $f(z) + b = 0$ are reciprocal to each other. By an argument similar to that given in the proof of Theorem 5, the reciprocal of any number R such that the disk $|z| < R$ contains at least $(n - p)$ zeros of the polynomial $P(z)Q(z) + bz^n$ for arbitrary b will provide an estimate sought in statement of the theorem.

Now simple calculations show:

$$\operatorname*{Max}_{|z|=R}\left[\frac{\theta}{\partial\theta}\arg(z-a)\right] = \begin{cases} \dfrac{R}{R-|a|} & \text{if } |a| < R \\ \dfrac{R}{R+|a|} & \text{if } |a| \geqq R \end{cases}$$

and

$$\operatorname{Re}\left[\frac{z}{z-\alpha(z)}\right] \leqq \frac{R^k}{R^k - 1}$$

if $|z| = R > 1$ and $|\alpha(z)| \leqq R^{-(k-1)}$.

These estimates combined with Theorem C yield the fundamental inequality valid for all $R > 1$:

$$\begin{aligned} \frac{\theta}{\partial\theta}\arg\left[\frac{P(z)Q(z)}{z^n}\right] &= \operatorname{Re}\left[\frac{zP'(z)}{P(z)}\right] + \operatorname{Re}\left[\frac{zQ'(z)}{Q(z)}\right] - n \\ &\leqq (n-p)\frac{R^k}{R^k - 1} + \frac{sR}{R - r} + \frac{p - q - s}{2} - n\,, \end{aligned} \tag{12}$$

where s $(0 \leqq s \leqq p - q)$ is the exact number of zeros of $Q(z)$ in the disk $|z| \leqq r$ $(r > 1)$ and R is any number $(R > r)$ such that $(p - q - s)$ zeros of $Q(z)$ all lie in $|z| > R$.

Now if R is such as to make the expression on the righthand side of (12) negative (that such a number exists can be seen by letting $R \to \infty$), then as can be easily verified $1/R$ becomes a good estimate for the radius of the disk sought in the theorem. Indeed in this case

$$\begin{aligned} \mathop{\Delta}_{|z|=R} \arg(PQ + bz^n) &= \mathop{\Delta}_{|z|=R} \arg(PQ) \\ &+ \mathop{\Delta}_{|z|=R} \arg\left(1 + \frac{bz^n}{PQ}\right) \geqq \mathop{\Delta}_{|z|=R} \arg(PQ) = n - p + s\,. \end{aligned}$$

Thus the polynomial $PQ + bz^n$ has at least $(n - p + s)$ zeros in the disk $|z| \leqq R$ for arbitrary complex numbers b. Since moving a zero lying in the unit disk to the origin does not affect the numbers r and R and increases q by one, we may assume, in view of (12), that $q = 1$. We are then led to the consideration of the inequality

$$(n-p)\frac{R^k}{R^k-1}+\frac{s}{2}\frac{R+r}{R-r}+\frac{p-2n-1}{2}<0\,. \tag{13}$$

First it is immediate that this inequality has a solution with $R>r>1$. Secondly, denoting the left-hand side of (13) by $M(R)$ it is easily seen that if $M(R_1)=0$ for some $R_1(R_1>r>1)$, then $M(R)<0$ for all $R>R_1$. Substituting $1+\rho\,(\rho>0)$ for R in the second summand on the left-hand side of (13) and substituting $1+k\rho$ for R^k in the first summand on the left-hand side of (13) one obtains the inequality

$$\rho^2-\rho\left[\frac{2(n-p)}{k(p-s+1)}-1+r\frac{p+s+1}{p-s+1}\right]+\frac{2(n-p)(r-1)}{k(p-s+1)}>0 \tag{13'}$$

Neglecting the positive constant term in (13′) and taking into account that the first two summands in (13) are montonically decreasing in R it follows that inequality (13) will be satisfied if

$$R>a_s+rb_s\,;$$

where

$$a_s=\frac{2(n-p)}{(p+1-s)k},\ b_s=\frac{p+s+1}{p-s+1}\,.$$

We note that $a_s\leqq(n-p)/k$ and $b_s\leqq s+1$. $s=0$, we can find directly that a good estimate is

$$R=a_0=\left(\frac{2n-p+1}{p+1}\right)^{1/k}\,.$$

If the disk $|z|<a_0$ does not contain at least $(n-$ p) zeros of the polynomial $PQ+bz^n$, then at least one of the zeros of Q lies in the disk $|z|\leqq a_0$. So assume, then, $s=1$ and $r=a_0$ and let

$$R_1=a_1+a_0b_1\,.$$

If the disk $|z|<R_1$ (which we know contains at least one zero of Q) contains exactly one zero of Q, then the disk $|z|\leqq R_1$ must contain at least $(n-p)$ zeros of the polynomial $PQ+bz^n$, since in this case we can set $s=1,r=R_1-\varepsilon,R=R_1$ with $\varepsilon>0$ sufficiently small. So assume that the disk $|z|<R_1$ contains at least two zeros of $Q(z)$. We then continue the same argument, assuming $s=2,r=R_1$ and let

$$R_2=a_2+R_1b_2\,.$$

In this way, either $|z|<R_2$ already contains at least $(n-p)$ zeros of $PQ+bz^n$ or else the disk $|z|\leqq R_2$ contains at least three zeros of Q. We finally arrive at

$$R_{p-1}=a_{p-1}+R_{p-2}b_{p-1}$$

such that the disk $|z| < R_{p-1}$ must contain at least $(n - p)$ zeros of the polynomial $PQ + bz^n$ because the alternative implies that the polynomial $Q(z)$ must be of degree at least p, which is impossible by the definition of $Q(z)$. It remains to estimate the number R_{p-1}. First we have

$$R_{p-1} = a_{p-1} + a_{p-2}b_{p-1} + a_{p-3}b_{p-1}b_{p-2} + \cdots + a_1 b_{p-1}b_{p-2}\cdots b_2 + a_0 b_{p-1}b_{p-2}\cdots b_1 .$$

If $A = \mathrm{Max}_{0 \leqq i \leqq p-1} a_i$, we obtain

$$R_{p-1} \leqq A[1 + b_{p-1} + b_{p-1}b_{p-2} + \cdots + b_{p-1}b_{p-2}\cdots b_1] \leqq A\left[1 + \frac{2p}{2!} + \frac{(2p)^2}{3!} + \frac{(2p)^3}{4!} + \cdots + \frac{(2p)^{p-1}}{p!}\right] \leqq A\frac{e^{2p}}{2p} .$$

This completes the proof.

REMARK 1. A slightly simpler calculation could be made, assuming (except for several simple cases mentioned after the statement of Theorem 6) $A = A'$ and furthermore starting with the basic inequality

$$R \geqq A' + r(s + 1) .$$

Then we can set

$$R_1' = A' + 2a_0 \leqq 3A'$$
$$R_2' = A' + 3R_1' \leqq 4R_1' \leqq 3.4A'$$
$$R_3' = A' + 4R_2' \leqq 5R_2' \leqq 3.4.5A'$$

etc. This leads to the inequality

$$R_{p-1}' \leqq \frac{(p + 1)!}{2} A' .$$

Thus $f(z)$ is p-valent in the disk

$$|z| < \frac{2}{A'(p + 1)!}$$

where

$$A' = \mathrm{Max}\left\{(n - p)^{1/k}, \frac{n - p}{k}\right\} ,$$

valid for at least all $n \geqq 5$ except for the known cases $p = 1$ and $p = n - 1$.

M. Biernacki [1], p. 632, obtained the estimate $((n - p + 1)/2)(p + 1)!$ for R_{p-1} which is greater considerably than either R_{p-1} or the radius obtained in Theorem 6, except, in the latter case, for several small values of p. It should be noted that M. Biernacki has improved on his result cited above in a later work [3], improving on his previous result for $p > 1$ namely by replacing R_{p-1} as given above by

$$\frac{(p+1)(p+2)\cdots(2p-1)(2n-2p+1)}{(p-1)!}.$$

It seems, however, that all these estimates are too small. In view of Theorem 5, the exact estimate cannot be greater than $(p/n)^{1/r}$, which is exact for $p = 1$. For $p = n - 1$ and $k = 1$, the exact bounds as given by G. Szegö [12] are 1/2 for odd n and $1/2 \cos \pi/2n$ for even n. In this connection we mention a generalization of Szegö's theorem to lacunary polynomials obtained in [11], p. 417.

THEOREM D. *If all the zeros of* $P(z) = a_0 + a_q z^q + \cdots + a_n z^n$, $1 \leqq q \leqq n$, *lie in the region* $|z| \geqq r$, *then all the zeros of the polynomial* $Q(z) = a_0 + a_q z^q + \cdots + a_{n-1} z^{n-1}$ *lie in the region* $|z| \geqq rx(q)$, *where* $x(q)$ *is the positive root of the equation* $x^q + x - 1 = 0$.

The last result allows us to obtain a lower bound for the exact radius sought in problem 2 for lacunary polynomials.

THEOREM 7. *If the polynomial of degree* n

$$f(z) = a_1 z + a_2 z^2 + \cdots + a_{n-q} z^{n-q} + a_n z^n$$

has $(n-1)$ *zeros in the unit disk, then* $f(z)$ *is at most* $(n-1)$*-valent in the disk* $|z| < x(q)$ *where* $x(q)$ *is as defined in Theorem D.*

Proof. By hypothesis the polynomial

$$Q(z) = z^n f\left(\frac{1}{z}\right) = a_n + a_{n-q} z^q + \cdots + a_1 z^{n-1}$$

has one zero in $|z| \leqq 1$. We have to show that the polynomial

$$g(z) = a_0 + a_1 z + \cdots + a_{n-q} z^{n-q} + a_n z^n$$

has at most $(n-1)$ zeros in $|z| < x(q)$ for arbitrary a_0. Assume the countrary. That is, assume that the polynomial $g(z)$ has all its zeros in the disk $|z| < x(q)$. Then the polynomial

$$P(z) = z^n g\left(\frac{1}{z}\right) = a_n + a_{n-q} z^q + \cdots + a_1 z^{n-1} + a_0 z^n$$

has all its zeros in $|z| \geqq 1/x(q)$. By Theorem D with $r = (1/x(q)) + \varepsilon$, it follows that the polynomial $Q(z)$ has all its zeros in $|z| > 1$ which is a contradiction. It is shown in [11], p. 417, that the exact bound

$x_0(q)$ in Theorem D satisfies $x(q) \leqq x_0(q) \leqq 2^{-1/q}$. The exact value $x_0(q)$ is, to the author's knowledge, still unknown.

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Received August 13, 1970 and in revised form April 19, 1971. The author acknowledges partial support of grant NSF GP 11881.

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