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OSCILLATORY PROPERTIES OF SOLUTIONS OF EVEN ORDER DIFFERENTIAL EQUATIONS

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Consider the following nth order nonlinear differential equation

(1) $$x^{(n)} + f(t, x, x', \cdots, x^{(n-1)}) = 0 .$$

All functions considered will be assumed continuous and all the solutions of (1), continuously extendable throught the entire nonnegative real axis. A nontrivial solution of (1) is called oscillatory if it has zeros for arbitrarily large t and equation (1) is called oscillatory if all of its solutions are oscillatory. A nontrivial solution of (1) is called nonoscillatory if it has only a finite number of zeros on $[t_0, \infty)$ and equation (1) is called nonoscillatory if all of its solutions are nonoscillatory. In this paper, theorems on oscillation and nonoscillation are presented.

Recently, J. S. W. Wong [8] posed a definition called strongly continuous with which he proved some theorems to (1) for $n = 2$. The proof is based on that of his earlier results [7]. We introduce more general definition. A function $f(t, x_1, \cdots, x_n)$ is called generalized strongly continuous from the left at x_{1c} if $f(t, x_1, x_2, \cdots, x_n)$ is jointly continuous in t and $x_i (i = 1, 2, \cdots, n)$ and for $\varepsilon > 0$ there exist $\delta > 0$, $T \geqq 0$, and $x_\delta \in [x_{1c} - \delta, x_{1c}]$ such that for all $x_1 \in [x_{1c} - \delta, x_{1c}]$, and for all x_i satisfying $|x_i - k_i| \leqq \delta$ (k_i is any constant) for $i = 2, \cdots, T$,

$$(1 - \varepsilon)f(t, x_\delta, k_2, \cdots, k_n) \leqq f(t, x_1, \cdots, x_n) \leqq (1 + \varepsilon)f(t, x_{1c}, k_2, \cdots, k_n) ,$$

for all $t \geqq n$.

Generalized strong continuity from the right is defined analogousely. A function $f(t, x_1, \cdots, x_n)$ is said to be generalized strongly continuous if it is generalized strongly continuous both from the left and from the right. If $f = f(t, x_1)$, then our definition is the same as Wong's one. For example, $f(t, x_1, \cdots, x_n) = a(t)f(x_1, \cdots x_n)$ is generalized strongly continuous.

2. Oscillation and nonoscillation theorems.

THEOREM 1. *Assume that n is even and that*

(α) *$f(t, c, k_2 \cdots, k_n)$ is bounded for any constant c and $k_i (i = 2, \cdots, n)$ and $x_1 f(t, x_1, \cdots, x_n) > 0$ $(x_1 \neq 0)$.*

Let $f(t, x_1, \cdots, x_n)$ be generalized strongly continuous from the left *for* $x_1 > 0$ *and generalized strongly continuous from the right for* $x_1 < 0$. *Then, a necessary and sufficient condition for equation* (1) *to have a bounded nonoscillatory solution is*

$$\left|\int^{\infty} t^{n-1} f(t, c, k_2, \cdots, k_n)dt\right| < \infty \tag{2}$$

($c(\neq 0)$ and $k_i (i = 2, \cdots, n)$ are some constants).

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1), which must eventually be of one sign. Then we may assume that $x(t) > 0$ for $t \geqq T > 0$. Since $x(t) > 0$, then $f(t, x, x', \cdots, x^{(n-1)}) > 0$ for $t \geqq T$, we see from (1) and the assumption that $x(t)$ is bounded:

$$x^{(n)} \leqq 0,\ x^{(n-1)} \geqq 0,\ x^{(n-2)} \leqq 0, \cdots, x' \geqq 0\ ,$$
$$\lim_{t\to\infty} x^{(i)}(t) = 0, \qquad i = 1, 2, \cdots, n-1\ .$$

From this and the fact $x(t)$ is positive and bounded implies that $x(t)$ tends to a finite limit $L > 0$. Integrating (1), we obtain for sufficiently large t

$$x^{(n-i)}(t) = (-1)^{n-i-1}\int_t^{\infty}\frac{(s-t)^{i-1}}{(i-1)!}f(s, x(s), x'(s), \cdots, x^{(n-1)}(s)ds$$
$$(\text{for, } i = 1, 2, \cdots, n-1)\ .$$

In particular,

$$x(t) = L - \int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}f(s, x(s), x'(s), \cdots, x^{(n-1)}(s))ds\ .$$

By the generalized strong continuity of $f(t, x_1, x_2, \cdots, x_n)$ implies that there exist $0 < \delta < L$, and $L_\delta \in [L - \delta, L]$ such that for all x_i satisfying $|x_i - k_i| \leqq \delta (i = 2, \cdots, n)$ and for all $x_1 \in [L - \delta, L]$,

$$f(t, x_1, x_2, \cdots, x_n) \geqq \frac{1}{2}f(t, L_\delta, k_2, \cdots k_n)\ .$$

Choose T sufficiently large, we can restrict the solution $x(t)$ to satisfy $L - \delta \leqq x(t) \leqq L$ for all $t \geqq T$, and that

$$|x^{(i)}(t) - 0| \leqq \delta\ (i = 1, 2, \cdots, n-1)\ ,$$

for all $t \geqq T$.
Thus, we obtain for $t \geqq T$,

$$0 < \frac{1}{2}f(t, L_\delta, 0, \cdots, 0) < f(t, x(t), x'(t), \cdots, x^{(n-1)}(t))\ .$$

Accordingly, we obtain

$$x(t) = L - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x(s), x'(s), \cdots, x^{(n-1)}(s))ds \tag{3}$$
$$\leqq L - \frac{1}{2(n-1)!}\int_t^\infty (s-t)^{n-1} f(s, L_\delta, 0, \cdots, 0)ds .$$

Since $x(t)$ is bounded, we obtain

$$\int_t^\infty (s-t)^{n-1} f(s, L_\delta, 0, \cdots, 0)ds < \infty , \tag{4}$$

which implies

$$\int_t^\infty s^{n-1} f(s, L_\delta, 0, \cdots, 0)ds < \infty .$$

Conversely, we show that if (2) holds for some constant $c > 0$ and $k_i (i = 2, \cdots, n)$, then there exists a nonnegative continuous bounded solution to the following integral equation:

$$\begin{aligned} x_{n-1}(t) &= k_n + \int_t^\infty f(s, x_0(s), x_1(s), \cdots, x_{n-1}(s))ds \\ x_{n-2}(t) &= k_{n-1} - \int_t^\infty (s-t) f(s, x_0(s), x_1(s), \cdots, x_{n-1}(s))ds \\ x_{n-3}(t) &= k_{n-2} + \int_t^\infty \frac{(s-t)^2}{2!} f(s, x_0(s), x_1(s), \cdots, x_{n-1}(s))ds \\ &\vdots \\ x_0(t) &= c - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x_0(s), x_1(s), \cdots, x_{n-1}(s))ds . \end{aligned} \tag{5}$$

We define $E = \{0, 1, 2, \cdots, n-1\}$. Let a positive number T be chosen such that

$$\max_{i \in E} \frac{1}{(n-1-i)!}\int_t^\infty (s-t)^{n-1-i} f(s, c, k_2, \cdots, k_n)ds \leqq \frac{c}{M} \tag{6}$$

where $M (> 2)$ is some constant.

We define. for N a positive interger $N \geqq T$: for $t \geqq N$,

$$\begin{aligned} x_{n-1,N}(t) &= k_n \\ x_{n-2,N}(t) &= k_{n-1} \\ &\vdots \\ x_{1,N}(t) &= k_2 \\ x_{0,N}(t) &= c \end{aligned} \tag{7}$$

and for $T \leqq t \leqq N$,

$$
\begin{aligned}
x_{n-1,N}(t) &= k_n + \int_{t+(1/N)}^{\infty} f(s, x_{0,N}(s), x_{1,N}(s), \cdots, x_{n-1,N}(s))ds \\
x_{n-2,N}(t) &= k_{n-1} - \int_{t+(1/N)}^{\infty} \left(s - t - \frac{1}{N}\right) f(s, x_{0,N}(s), \cdots, x_{n-1,N}(s))ds \\
&\vdots \\
x_{1,N}(t) &= k_2 + \int_{t+(1/N)}^{\infty} \frac{(s - t - (1/N)^{n-2}}{(n-2)!} f(s, x_{0,N}(s), \cdots, x_{n-1,N}(s))ds \\
x_{0,N}(t) &= c - \int_{t+(1/N)}^{\infty} \frac{(s - t - (1/N)^{n-1}}{(n-1)!} f(s, x_{0,N}(s), \cdots, x_{n-1,N}(s))ds \, .
\end{aligned} \tag{8}
$$

This formula defines $x_{i,N}(t)$ for $i = 0, 1, \cdots, n-1$, successively on the intervals $[N - (K/N), N - (K-1)/N]$ for $k = 1, 2, \cdots, N(N-T)$: hence $x_{i,N}(t)$, $i = 0, 1, \cdots, n-1$, are defined on $[T, \infty)$.

For $N - (1/N) \leqq t < \infty$, we have by (6)

$$
\begin{aligned}
|x_{i,N}(t) - k_{i+1}| &\leqq \int_{t+(1/N)}^{\infty} \frac{(s - t - (1/N))^{n-1-i}}{(n-1-i)!} f(s, x_{0,N}(s), \cdots, x_{n-1,N}(s))ds \\
&\leqq \int_{t+(1/N)}^{\infty} \frac{(s - t - (1/N))^{n-1-i}}{(n-1-i)!} f(s, c, k_2, \cdots, k_n)ds \\
&\leqq \frac{c}{M} (i = 1, 2, \cdots, n-1)
\end{aligned}
$$

and also

$$
0 < c - \frac{c}{M} \leqq x_{0,N}(t) \leqq c \, .
$$

By easy induction, we have

$$
\begin{aligned}
0 \leqq |x_{i,N}(t) - k_{i+1}| &\leqq \frac{c}{M} i = 1, 2, \cdots, n-1 \, , \\
0 < c - \frac{c}{M} &\leqq x_{0,N}(t) \leqq c
\end{aligned} \tag{9}
$$

on the entire interval $[T, \infty)$. Consequently, for $t \geqq T$, since f is generalized strongly continuous and $f(t, c, k_2, \cdots, k_n)$ is bounded, we have

$$
\begin{aligned}
|x_{n-1',N}(t)| &= f\left(t + \frac{1}{N}, x_{0,N}\left(t + \frac{1}{N}\right), \cdots, x_{n-1,N}\left(t + \frac{1}{N}\right)\right) \\
&\leqq \frac{3}{2} f\left(t + \frac{1}{N}, c, k_2, \cdots, k_n\right) \\
&\leqq \frac{3}{2} K \ (K \text{ is constant}) \, ,
\end{aligned} \tag{10}
$$

$$|x'_{i,N}(t)| = |x_{i+1,N}(t) - k_{i+2}| \leqq \frac{c}{M}, \qquad (i = 0, 1, \cdots, n-2).$$

Since the family $\{x_{i,N}(t)\}(i = 0, 1, 2, \cdots, n-1)$, is uniformly bounded and equicontinuous on $[T, B]$ (B is arbitrary), we extract from $\{x_{i(N}(t)\}$ $(i = 0, 1, \cdots, n-1)$, a uniformly convergent subsequence $\{x_{i,k}(t)\}$,

$$\lim_{k\to\infty} x_{i,k}(t) = \bar{x}_i, \qquad (\text{on } [T, B], \text{ for } i = 0, 1 \cdots, n-1).$$

For any large number $B > T$, we may write (8) in the form

$$x_{i,k}(t) = d + (-1)^{i+1}\left\{\int_{t+(1/k)}^{B} \frac{(s - t - (1/k))^{n-1-i}}{(n-1-i)!} f(s, x_{0,k}(s), \cdots, x_{n-1,k}(s))ds + \phi_k(B)\right\}, \tag{11}$$

where,

$$\phi_k(\mathrm{B}) = \int_B^\infty \frac{(s - t - (1/k))^{n-1-i}}{(n-1-i)!} f(s, x_{0,k}(s), \cdots, x_{n-1,k}(s))ds,$$
$$(d = c \text{ for } i = 0,\ d = k_{i+1} \text{ for } i = 1, 2 \cdots, n-1).$$

For fixed B we let k tend to infinity in (11) and obtain

$$\liminf_{k\to\infty} \phi_k(B) \leqq (-1)^i(-\bar{x}_i + d + (-1)^{i+1}\int_t^B \frac{(s-t)^{n-1-i}}{(n-1-i)!} f(s, \bar{x}_0, \cdots, \bar{x}_{n-1})ds) \leqq \limsup_{k\to\infty} \phi_k(B).$$

From (6) and (9) and f is generalized strongly continuous, we obtain

$$\begin{aligned} 0 \leqq \phi_k(B) &= \int_B^\infty \frac{(s - t - (1/k))^{n-1-i}}{(n-1-i)!} f(s, x_{0\,k}(s), \cdots, x_{n-1,k}(s))ds \\ &\leqq \frac{3}{2}\int_B^\infty \frac{(s-t)^{n-1-i}}{(n-1-i)!} f(s, c, k_2, \cdots, k_n)ds < \infty. \end{aligned} \tag{12}$$

By (2), the integral in (12) tend to zero as $B \to \infty$. Thus, we conclude that $\bar{x}_i(t)$ $(i = 0, 1, \cdots, n-1)$, is a solution of (5) and also $\bar{x}_0(t)$ is a bounded nonoscillatory solution of (1).

THEOREM 2. *Assume that n is even and that*

(β) $x_1 f(t, x_1, x_2, \cdots, x_{n-1}, \gamma) > 0$ $(x_1 \neq 0)$, where γ is constant.

Let $f(t, x_1, x_2, \cdots, x_{n-1}, \gamma)$ be generalized strongly continuous from the left for $x_1 > 0$, and generalized strongly continuous from the right for $x_1 < 0$. Then, a necessary and sufficient conditior for equation

$$x^{(n)} + f(t, x, x', \cdots, x^{(n-2)}, \gamma) = 0 \ (\gamma \text{ is constant}), \tag{1'}$$

to have a bounded nonoscillatory solution is

$$\left|\int^{\infty} t^{n-1} f(t, c, k_2, \cdots, k_{n-1}, \gamma) dt\right| < \infty \tag{2'}$$

($c(\neq 0)$ and $k_i\,(i = 2, \cdots, n-1)$ are some constant).

Proof. The necessity follows from the necessary part of the proof of Theorem 1.

Conversely, we show that if (2′) holds for some constant $c > 0$ and $k_i (i = 2, \cdots, n-1)$, then there exists a nonnegative continuous bounded solution to the following integral equation:

$$\begin{aligned} x_{n-2}(t) &= k_{n-1} - \int_t^{\infty} (s-t) f(s, x_0(s), \cdots, x_{n-2}(s), \gamma) ds \\ x_{n-3}(t) &= k_{n-2} + \int_t^{\infty} \frac{(s-t)^2}{2!} f(s, x_0(s), \cdots, x_{n-2}(s), \gamma) ds \\ &\vdots \\ x_0(t) &= c - \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} f(s, x_0(s), \cdots, x_{n-1}(s), \gamma) ds\,. \end{aligned} \tag{13}$$

Let a positive number T be chosen such that

$$\max_{i \in E} \int_T^{\infty} \frac{(s-t)^{n-1-i}}{(n-1-i)!} f(s, c, k_2, \cdots, k_{n-1}, \gamma) ds \leqq \frac{c}{M} \tag{14}$$

($M(> 2)$ is some constant).
We define, for N a positive integer $N \geqq T$: for $t \geqq N$,

$$\begin{aligned} x_{n-2,N}(t) &= k_{n-1} \\ &\vdots \\ x_{1,N}(t) &= k_2 \\ x_{0,N}(t) &= c \end{aligned}$$

and for $T \leqq t \leqq N$

$$\begin{aligned} x_{n-2,N}(t) &= k_{n-1} - \int_{t+(1/N)}^{\infty} \left(s - t - \frac{1}{N}\right) f(s, x_{0,N}(s), \cdots, x_{n-2,N}(s), \gamma) ds \\ &\vdots \\ x_{1,N}(t) &= k_2 + \int_{t+(1/N)}^{\infty} \frac{(s-t-(1/N))^{n-2}}{(n-2)!} f(s, x_{0,N}(s), \cdots, x_{n-2,N}(s), \gamma) ds \\ x_{0,N}(t) &= c - \int_{t+(1/N)}^{\infty} \frac{(s-t-(1/N))^{n-1}}{(n-1)!} f(s, x_{0,N}(s), \cdots, x_{n-2,N}(s), \gamma) ds\,. \end{aligned}$$

As same as the proof of Theorem 1, $x_{i,N}(t)$ $(i = 0, 1, \cdots, n-2)$, are defined on $[T, \infty)$ and that for $i = 0, 1, \cdots, n-3$,

$$|x'_{i,N}(t)| = |x_{i+1,N}(t) - k_{i+2}| \leqq \frac{c}{M}$$

and for $i = n - 2$,

$$|x'_{i,N}(t)| = \int_t^\infty f(s, x_{0,N}(s), \cdots, x_{n-2,N}(s), \gamma)ds \leqq \frac{c}{M} .$$

Hence the family $\{x_{i,N}(t)\}$ $(i = 0, 1, \cdots, n-2)$ is uniformly bounded and equicontinuous on $[T, B]$ (B is arbitrary). Using an argument similar to that given in the proof of Theorem 1, we hove a bounded nonoscillatory solution of (1′).

REMARK. For $n = 2$, Theorem 2 coincides with Theorem 3 in [8].

COROLLARY 1. *Suppose that n is even and that*

$$x_1 f(t, x_1, \cdots, x_n) > 0 \qquad (x_1 \neq 0) .$$

Let $f(t, x_1, \cdots, x_n)$ be generalized strongly continuous from the left for $x_1 > 0$ and generalized strongly continuous from the right for $x_1 < 0$ and that

$$\left|\int_{t_0}^\infty t^{n-1} f(t, c, k_2, \cdots, k_n)dt\right| = +\infty (c(\neq 0) \text{ and } k_i (i = 2, \cdots, n)$$

are any constants). *Then, every bounded solution of* (1) *is oscillatory.*

Proof. The proof of Corollary 1 follows immediately from the necessary part of Theorem 1.

COROLLARY 2 [1, Theorem 1]. *Consider*

$$x^{(2n)} + p(t)g(x, x', x^{(2)}, \cdots, x^{(2n-1)}) = 0 \tag{15}$$

under the following assumption:

(i) p: $I \to R_+ = (0, +\infty)$, $I = [t_0, +\infty)$, $t_0 \geqq 0$, $p \in C[t_0, +\infty)$,

and

$$\int_{t_0}^{+\infty} t^{2n-1} p(t)dt = +\infty \tag{A}$$

is satisfied;

(ii) g: $R^{2n} \to R = (-\infty, \infty)$, $x_1 g(x_1, x_2, \cdots, x_{2n}) > 0$ *for* $x_1 \neq 0$,

and continuous on R^{2n};
then, under the above conditions, every bounded solution of (15) *is oscillatory.*

Proof. The function $p(t)g(x_1, x_2, \cdots, x_{2n})$ is generalized strongly continuous. Hence, Corollary 2 is included in Corollary 1.

COROLLARY 3 [4, Theorem 1]. *Under the assumption*
(α') *$p(t)$ is bounded and eventually nonnegative*:

$$x_1 g(x_1, x_2, \cdots, x_{2n}) > 0 \quad (x_1 \neq 0), \qquad \text{for} \quad (x_1, x_2, \cdots, x_{2n}) \in R^{2n} ,$$

a necessary and sufficient condition that (15) *have a bounded nonoscillatory solution is*

$$\int^{\infty} t^{2n-1} p(t) dt < \infty . \tag{16}$$

Proof. The function $p(t)g(x_1, x_2, \cdots, x_n)$ is generalized strongly continuous, hence Corollary 3 is included in Theorem 1.

COROLLARY 4 [4, Theorem 2]. *Under the assumption*
(β') *$p(t)$ is eventually nonegative and $x_1 g(x_1, x_2, \cdots, x_{2n-1}, c) > 0$ $x_1 \neq 0$, for $(x_1, x_2, \cdots, x_{2n-1}, c) \in R^{2n}$ where c is constant, a necessary and sufficient condition that the differential equation*

$$x^{(2n)} + p(t)g(x, x', \cdots, x^{(2n-2)}, c) = 0 , \qquad n \geqq 1 , \tag{17}$$

c is constant, have a bounded nonoscillatory solution is (16).

Proof. The proof follows immediately from Theorem 2.

THEOREM 3. *Consider*

$$x^{(2n)} + p(t)g(x) = 0 , \tag{18}$$

under the following assumptions:
(i) p: $I \to R_+ = (0, +\infty)$, $I = [t_0, +\infty)$, $t_0 \geqq 0$, $p \in C[t_0, +\infty)$,
(ii) g: $R \to R = (-\infty, -\infty)$, $g \in C'(-\infty, +\infty)$, $xg(x) > 0$ for $x \neq 0$, *and continuous on R;*
$g'(x) \geqq 0$ for $|x| \in [K, +\infty)$ (K is some positive constant), and

$$\int_{\varepsilon}^{+\infty} \frac{1}{g(u)} du < +\infty , \qquad \int_{-\varepsilon}^{-\infty} \frac{1}{g(u)} du < +\infty$$

for every $\varepsilon > 0$;
then a necessary and sufficient condition that every solution of (18) *is oscillatory is*

$$\int_{t_0}^{+\infty} t^{2n-1} p(t) dt = +\infty . \tag{19}$$

Proof. From Theorem 2, if $\int_{t_0}^{+\infty} t^{2n-1}p(t)dt < +\infty$, then (18) is nonoscillatory. Hence we conclude that if every solution of (18) is oscillatory, then $\int_{t_0}^{+\infty} t^{2n-1}p(t)dt = +\infty$.

Conversely $\int_{t_0}^{+\infty} t^{2n-1}p(t)dt = +\infty$, then every solution of (18) is oscillatory in the case $n > 1$ [1] and $n = 1$ [6; 7].

THEOREM 4. *Assume that n is even and that* (β). *Let* $f(t, x_1, \cdots, x_{n-1}, \gamma)$ *be generalized strongly continuous from the left for* $x_1 > 0$, *and generalized strongly continuous from the right for* $x_1 < 0$. Then, *a necessary and sufficient condition for every bounded solution of* (1′) *to be oscillatory is*

$$\left|\int^{\infty} t^{n-1}f(t, c, k_2, \cdots, k_{n-1}, \gamma)dt\right| = +\infty \tag{20}$$

($c(\neq 0)$ and $k_i(i = 2, \cdots, n-1)$ are any constant).

Proof. Assume that (20) does not hold, then (2′) holds for some $c \neq 0$ and $k_i(i = 2, \cdots, n-1)$. Hence by Theorem 2, equation (1′) has a bounded nonoscillatory solution, so clearly condition (20) is necessary. Conversely, let $x(t) > 0$ be a nonoscillatory solution of (1′). In view of the arguments of Theorem 1, $x(t)$ must be nondecreasing and the limit is finite. Hence the argument in the proof of Theorem 1 is applicable, which shows that leads a contradiction.

THEOREM A [5]. *If in addition to the hypothesis of Corollary* 3 (or Corollary 4), *for some* $r > 1$ *and* n *is even,*

$$\liminf_{|x_1| \to +\infty} \frac{|g(x_1, x_2, \cdots, x_n)|}{|x_1|^r} > 0 \tag{21}$$

then a necessary and sufficient condition that all solution of (15) (or (17)) *be oscillatory is*

$$\int^{\infty} t^{n-1}p(t)dt = +\infty\ . \tag{22}$$

THEOREM B [2]. *Consider*

$$x^{(n)} + p(t)g(x, x', \cdots, x^{(n-1)}) = 0 \tag{23}$$

with n even, and moreover,

(i) p: $I \to R_+ = (0, +\infty)$, $I = [t_0, +\infty)$, $t_0 \geqq 0$.

(ii) g: $R^n \to R = (-\infty, +\infty)$, *and such that Condition* (*G*): $x_1 g(x_1, x_2, \cdots, x_n) > 0$ *for every* $(x_1, x_2, \cdots, x_n) \in R^n$ *with* $x_1 \neq 0$, *and for every* $(x_1, x_2, \cdots, x_n) \in R^n$, *and every* $\lambda \geqq K$ (=fixed positive constant), $g(-x_1, -x_2, \cdots, -x_n) = -g(x_1, x_2, \cdots, x_n)$, *and* $g(\lambda, \lambda x_2, \cdots, \lambda x_n) = \lambda^\gamma g(1, x_2, \cdots, x_n)$, where $\gamma = q/r$, *q r odd positive integers relatively prime*;

then under any one of the following conditions, all solutions of (23) *are oscillatory*:

(a) $$0 < \gamma < 1\,, \quad \int_{t_0}^{\infty} t^{\gamma(n-1)} p(t)dt = +\infty\,;$$

(b) $$\gamma = 1\,, \quad \int_{t_0}^{\infty} t^{n-1-\varepsilon} p(t)dt = +\infty\,,$$

for some ε with $0 < \varepsilon < 1$;

(c) $$\gamma > 1\,, \quad \int_{t_0}^{\infty} t^{n-1} p(t)dt = +\infty\,.$$

Kartsatos [2, Remark 3] posed a problem that under what additional assumptions on the function g, the conditions of Theorem B are also necessary for the theorem to hold. In case $0 < \gamma < 1$, the condition (a) is also necessary for theorem B to hold. When $\gamma = 1$ (this is the linear case), it is well known that condition (b) can not be necessary. Consider the Euler equation. Thus, we answer the problem for $\gamma > 1$.

THEOREM 5. *In addition to the assumptions of Theorem A, assume $p(t)$ being bounded. Then* (c) *is necessary for all solutions of* (23) *to be oscillatory.*

Proof. As $p(t)$ being bounded, the proof follows immediately from Theorem 1.

THEOREM 6. *Consider the equation*

$$x^{(n)} + p(t)g(x, x', \cdots, x^{(n-2)}, \delta) = 0 \text{ (for } n \text{ even)}, \tag{24}$$

where δ is constant. Then (c) *is necessary for all solutions of* (24) *to be oscillatory.*

Proof. The proof follows immediately from Theorem 2.

REMARKS. Theorem 5 and Theorem 6 are proved also from Theorem A, since from Condition (G), we see

$$\liminf_{|x_1| \to +\infty} \frac{|g(x_1, x_2, \cdots, x_{n-1})|}{|x_1|^r} = \liminf_{|x_1| \to +\infty} \frac{|x_1|^r g(1, (x_2/x_1), \cdots, (x_{n-1}/x_1))}{|x_1|^r} > 0\,.$$

In case $0 < \gamma < 1$, Ličko and Švec [3] proved Theorem 6 with

$g = x^{\gamma}$ $(0 < \gamma < 1)$.

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