PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 3, 1971

NEW CONCEPTS IN THE THEORY OF TOPOLOGICAL SPACE—SUPERCONDENSED SET, SUBCONDENSED SET, AND CONDENSED SET.

YOSHINORI ISOMICHI

A set which satisfies $A^{ai} = A^{i}$ ($A^{ia} = A^{a}$) is named a supercondensed (subcondensed) set. A supercondensed and subcondensed set is named a condensed set. Theorems about these new concepts are discussed in this paper.

Sets of the topological space are classified into three classes by comparing A^{ai} and A^{ia}.

The theory of topological space is usually constructed upon the axioms of *neighborhood*. One defines "open set", "closed set", "open kernel", and "closure" using the concept of neighborhood. Furthermore one introduces the concept of *accumulation point* and defines "perfect set", "set which is dense in itself", and "isolated set".

Accumulation points are classified by their cardinal numbers of their neighborhoods. In particular, a point every neighborhood of which contains points of the set more than $\aleph_1$ is called a "condensation point". (A set which coincides with its all condensation points is called a condensed set, hitherto. But in this paper the terminology "condensed set" is used for a different meaning.)

In this paper we define the following new concepts; *supercondensed set*, *subcondensed set*, and *condensed set*, not using the concepts of accumulation point and cardinal number, but using the concepts of open kernel and closure only. So it may be advocated that these new concepts are fairly basic ones. Further we introduce concepts of "*border of a set*" (which is different from "boundary of a set") and discuss about the relations between these new concepts. Furthermore we classify sets of the topological space into three classes.

In the following discussion we assume that A, B, C, etc. are the subsets of the topological space X. The open kernel of a subset A is denoted by A^{i}, closure and complemet of A are denoted by A^{a}, A^{c}, respectively.

LEMMA 1. (Kuratowski Theorem 6). *For any A of X, the following relations hold.*

$$A^{aiai} = A^{ai}$$
$$A^{iaia} = A^{ia}$$

LEMMA 2. (Halmos Lemma 4). *For any A and B of X, the following relations hold.*

$$A^{iai} \cap B^{iai} = (A \cap B)^{iai} ,$$
$$A^{aia} \cup B^{aia} = (A \cup B)^{aia} .$$

DEFINITION 1. A set S of X is said to be supercondensed if

$$S^{ai} = S^{i} .$$

A set I of X is said to be subcondensed if

$$I^{ia} = I^{a} .$$

DEFINITION 2. A set C of X is said to be condensed if

$$C^{ai} = C^{i} ,$$

and

$$C^{ia} = C^{a} .$$

REMARK 1. Any closed set is supercondensed, and any open set is subcondensed.

REMARK 2. By Definition 1 and 2, a set is condensed if and only if the set is supercondensed and subcondensed.

THEOREM 1. *The complement of a supercondensed set is subcondensed. The complement of a subcondensed set is supercondensed.*

Proof. Let S be a supercondensed set. Then we have

$$S^{aic} = S^{ic} . \qquad (*)$$

Considering the properties of complement we get

$$S^{aic} = S^{aca} = S^{cia} ,$$
$$S^{ic} = S^{ca} .$$

Using these equalities, we have the following equality.

$$(S^{c})^{ia} = (S^{c})^{a} .$$

This relation implies that the set S^{c} is subcondensed. The second part of the theorem is proved in the similar manner.

COROLLARY TO THEOREM 1. *The complement of a condensed set* *is also condensed.*

Proof. If C is a condensed set, then C^c is a subcondensed set by the first half of Theorem 1. And if C is a condensed set, then C^c is a supercondensed set by the second half of Theorem 1. Connecting the both propositions, we have proved that C^c is a condensed set.

THEOREM 2. *A set A of X is supercondensed if and only if*

$$A^{ai} \subset A .$$

A set B of X is subcondensed if and only if

$$B^{ia} \supset B .$$

Proof. Let A be a supercondensed set. Then we have

$$A^{ai} = A^i \subset A .$$

This is a relation that we need. Conversely, we assume the relation

$$A^{ai} \subset A .$$

Taking the open kernel of both sides, we get

$$A^{ai} \subset A^i .$$

Considering the self-evident relation

$$(A^a)^i \supset A^i$$

we obtain

$$A^{ai} = A^i .$$

This relation implies that A is a supercondensed set. The second half of this theorem can be proved in the similar manner.

COROLLARY TO THEOREM 2. *A set A of X is condensed if and only if*

$$A^{ai} \subset A \subset A^{ia} .$$

This corollary is evident from the definition of the condensed set.

DEFINITION 3 (Kuratowski). A set K of X is called a regular open set, if

$$K^{ai} = K .$$

A set H of X is called a regular closed set, if

$$H^{ia} = H .$$

REMARK 1. The whole space X and the empty set $\varnothing$ are regular open and also regular closed.

REMARK 2. The complement of a regular open set is regular closed. The complement of a regular closed set is regular open. ($K^c = K^{aic} = K^{cia}$)

REMARK 3. (Stone Theorem 25). The intersection of two regular open sets is regular open. The union of two regular closed sets is regular closed.

REMARK 4. For any set A, A^{ai} is a regular open set, and A^{ia} is a regular closed set. (by Lemma 1)

THEOREM 3. *A set is regular open if and only if the set is supercondensed and open. A set is regular closed if and only if the set is subcondensed and closed.*

Proof. Let K be a regular open set. Then, K satisfies the following relation.

$$K^{ai} = K . \tag{†}$$

Using this relation we get

$$K^i = K^{aii} = K^{ai} = K .$$

This implies that K is a open set. From eq. (†), we get

$$K^{ai} = K^i .$$

This relation implies that K is a supercondensed set.

Let a set A be supercondensed and open. Then we have

$$A^{ai} = A^i = A .$$

This implies that A is a regular open set. The second part of the theorem can be proved in the similar manner.

COROLLARY TO THEOREM 3. *A set is regular open if and only if the set is condensed and open. A set is regular closed if and only if the set is condensed and closed.*

Proof. This is evident from Remark 1 of Definition 2.

THEOREM 4. *A set A is condensed if and only if there is a regular open set U such that*

$$U \subset A \subset U^a .$$

A set B is condensed if and only if there is a regular closed set V such that

$$V^i \subset B \subset V .$$

Proof. Let us assume A is condensed. We take

$$U = A^{ai} .$$

Then

$$U \subset A .$$

And

$$U^a = A^{aia} \supset A^{ia} \supset A .$$

So U satisfies

$$U \subset A \subset U^a .$$

If we assume that there is a regular open set U which satisfies

$$U \subset A \subset U^a ,$$

then

$$A^{ai} \subset U^{aai} = U^{ai} = U ,$$
$$A^{ai} \supset U^{ai} = U .$$

So

$$U = A^{ai} .$$

On the other hand

$$A^{ia} \subset U^{aia} = U^a$$
$$A^{ia} \supset U^{ia} = U^a .$$

So

$$U^a = A^{ia} .$$

Then we have

$$A^{ai} \subset A \subset A^{ia} .$$

The remaining part of the theorem is dual to the proved part.

THEOREM 5. *A set A is condensed if and only if the boundary of A coincides with*

$$A^{ia} - A^{ai} = A^{ia} \cap (A^c)^{ia} .$$

REMARK. The boundary of A, which is denoted by A^f, is defined by

$$A^f = A^a - A^i = A^a \cap (A^c)^a .$$

Proof. We assume that A is a condensed set. By the corollary to Theorem 1, A^c is a condensed set. So we have the following equalities by the Definition 1.

$$A^{ia} = A^a$$
$$(A^c)^{ia} = (A^c)^a .$$

Therefore

$$A^{ia} \cap (A^c)^{ia} = A^a \cap (A^c)^a = A^f .$$

Next we prove the sufficiency. We assume

$$A^f = A^{ia} \cap (A^c)^{ia} .$$

Then we have

$$\begin{aligned} A^a &= A^i \cup A^f && \text{(Definition)} \\ &= A^i \cup \{A^{ia} \cap (A^c)^{ia}\} && \text{(Condition)} \\ &\subset A^i \cup A^{ia} \\ &= A^{ia} . \end{aligned}$$

Considering the evident relation

$$A^a \supset (A^i)^a$$

we have

$$(*) \qquad A^a = A^{ia} .$$

The same relation holds for A^c, because the boundary of A^c coincides with that of A. So we have

$$(A^c)^a = (A^c)^{ia} .$$

Considering

$$A^{ca} = A^{ic}$$
$$A^{cia} = A^{aca} = A^{aic}$$

we get

$$A^{ic} = A^{aic} .$$

Taking the complement of both sides we obtain

$$A^{i} = A^{ai} .$$

This relation and relation (*) imply that the set A is condensed.

DEFINITION 4. The inner kernel of the boundary of a set is called the border of the set.

REMARK. The border of a set A, which is denoted by A^b, is expressed by

$$A^b = (A^f)^i .$$

THEOREM 6. *The border of any set A is regular open and is expressed by*

$$A^b = A^{ai} - A^{ia} = A^{ai} \cap (A^c)^{ai} .$$

REMARK. This theorem corresponds formally to the following proposition:
"The boundary of any set A is closed and is expressed by

$$A^f = A^a - A^i = A^a \cap (A^c)^a ."$$

Proof. Considering the fact that the boundary of A is closed, we have

$$A^b = A^{fi} = (A^f)^{ai} .$$

Applying Remark 4 of Definition 3 we conclude that A^b is a regular open set.

Next we prove the remaining part of the theorem. We have the following relation.

$$\begin{aligned} A^{ai} - A^{ix} &= A^{ai} \cap (A^c)^{ai} \\ &= (A^a)^i \cap (A^{ca})^i \\ &= (A^a \cap A^{ca})^i && \text{(Property of open kernel)} \\ &= (A^f)^i && \text{(Definition of boundary)} \\ &= A^b && \text{(Definition of border)} \end{aligned}$$

THEOREM 7. *A set is supercondenced if and only if*

(1) *the open kernel of the set is regular open, and*

(2) *the border of the set is empty.*

A set is subcondenced if and only if

(1′) *the closure of the set is regular closed, and*

(2′) *the border of the set is empty.*

REMARK. For any set A of X, $A^{bb} = \phi$ because A^b is regular open so condensed.

Proof. Let A be a supercondensed set. Then

$$A^{ai} = A^i . \qquad (^*_*)$$

So we get

$$A^{aiai} = A^{iai} .$$

The left side of above equality becomes

$$A^{aiai} = A^{ai} \qquad \text{(Lemma 1)}$$
$$= A^i . \qquad \text{(by eq. } (^*_*))$$

Then we get, the relation

$$A^i = (A^i)^{ai} .$$

This implies that A^i is a regular open set. Using eq. $(^*_*)$ we get

$$A^{ai} = A^i \subset A^{ia} .$$

Then

$$A^b = A^{ai} - A^{ia} = \phi .$$

Next we prove the sufficiency. Suppose that A satisfies the following two conditions;

$$(\dagger) \qquad A^{iai} = A^i$$
$$A^{ia} \supset A^{ai} . \text{ (This is equivalent to } A^b = \varnothing)$$

Taking the open kernel of the second relation, we get

$$A^{iai} \supset A^{ai} .$$

Combining this relation and relation (†) we have

$$A^i \supset A^{ai} .$$

Considering the self-evident relation

$$A^i \subset A^{ai}$$

we get

$$A^i = A^{ai} .$$

This relation implies that A is a supercondensed set. The remaining part of the theorem is dual to the proved proposition.

COROLLARY TO THEOREM 7. *A set is condensed if and only if*
(1) *the open kernel of the set is regular open,*
(2) *the closure of the set is regular closed, and*
(3) *the border of the set is empty.*
This proposition is evident from definition of the condensed set.

DEFINITION 5. Any set of X can be classified into one of the following three classes.
Class I $\{A: A^{ai} \subset A^{ia}, A \subset X\}$
Class II $\{B: B^{ai} \supsetneqq B^{ia}, B \subset X\}$
Class III $\{C: C^{ai}, C^{ia}$ *are non-comparable*, $C \subset X\}$

REMARK 1. A set belongs to class I if and only if the border of the set is empty. So a supercondensed set and a subcondensed set belong to Class I.

REMARK 2. The Boolean subring A_R which was defined by Stone in the T_0 space coincides with class I, because A_R is composed of all the sets whose boundaries are nowhere dense.

THEOREM 8. *A set and its complement belong to the same class.*

Proof. We assume that A belongs to class I. Then the following relation holds.

$$A^{ai} \subset A^{ia} .$$

Taking the complement of both sides we obtain

$$A^{aic} \supset A^{iac} .$$

Using the property of the complement we get the relation

$$(A^c)^{ia} \supset (A^c)^{ai} .$$

This implies that A^c also belongs to class I. We can prove similarly the fact that the complement of a set of class II belongs to class II.

Next we prove about class III. In this case we must say that if C^{ai} and C^{ia} are noncomparable then $(C^c)^{ai}$ and $(C^c)^{ia}$ are also noncomparable. We use reduction absurdum for proving this proposition. Let $(C^c)^{ai}$ and $(C^c)^{ia}$ be comparable. Then C^c belongs to class I or II. So $C^{cc} = C$ must belong to class I or II by the proved part of this thorem. This is a contradiction.

THEOREM 9. *If a set A belongs to class I then*

$$A^{ai} = A^{iai}$$

and

$$A^{ia} = A^{aia} .$$

On the other hand, if one of these two equalities holds then the set A belongs to class I.

Proof. Suppose that A belongs to class I. Then the following relation holds.

$$A^{ai} \subset' A^{ia} .$$

Taking the open kernel of both sides we have

$$A^{ai} \subset A^{iai} .$$

Considering the self-evident relation

$$A^{ai} \dot{\supset} (A^{i})^{ai}$$

we get

$$A^{ai} = A^{iai} .$$

In the similar way we have

$$A^{ia} = A^{aia} .$$

Next, we prove the fact that if

$$A^{ai} = A^{iai}$$

then the set A belongs to class I. Clearly,

$$A^{ia} \supset (A^{ia})^{i} = A^{ai} .$$

This implies that A belongs to class I. We can prove the proposition for

$$A^{ia} = A^{aia}$$

in the similar manner.

THEOREM 10. *Any sets A, B of class I satisfy the following relations,*

$$A^{ai} \cap B^{ai} = (A \cap B)^{ai}$$
$$A^{ia} \cup B^{ia} = (A \cup B)^{ia}$$

Proof.

$$(A \cap B)^{ai} \subset (A^a \cap B^a)^i = A^{ai} \cap B^{ai} .$$

On the other hand

$$\begin{aligned} A^{ai} \cap B^{ai} &= A^{iai} \cap B^{iai} && \text{(Class I)} \\ &= (A \cap B)^{iai} && \text{(Lemma 2)} \\ &\subset (A \cap B)^{ai} , \end{aligned}$$

so

$$A^{ai} \cap B^{ai} = (A \cap B)^{ai} .$$

The second relation of the theorem is dual to the first one.

THEOREM 11. *The intersection and the union of any two sets of class I belong to class I.*

Proof. Let A, B belong to class I. Then

$$\begin{aligned} (A \cap B)^{ai} &= A^{ai} \cap B^{ai} && \text{(Theorem 10)} \\ &= A^{iai} \cap B^{iai} && \text{(Class I)} \\ &= (A \cap B)^{iai} . && \text{(Lemma 2)} \end{aligned}$$

So, $A \cap B$ belongs to class I by Theorem 9. The second part of the theorem is dual to the first one.

Acknowledgement. The author wishes to express his thanks to the referee for many valuable comments, especially, for his suggestion to Theorem 4. He is indebted to Dr. Akira Igarashi and Mr. Hidemitsu Ogawa for criticizing on this paper.

REFERENCES

1. N. Bourbaki, *Elements of Mathematics, General Topology,* Part I (117–138), (Addison Wesley 1966).

2. P. R. Halmos, *Lectures of Boolean Algebras* (12-17) (Van Nostrand, 1963).
3. C. Kuratowski, *Sur l'operation $\bar{A}$ de l'Analysis Situs*, Fundamenta Mathematicae, **3** (1922), 182-199.
4. M. H. Stone, *Application of the theory of Boolean Rings to General Topology*, Trans. Amer. Math. Soc., **41** (1937), 375-481.

Received November 23, 1970.

ELECTROTECHNICAL LAB. TOKYO.