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HOMOTOPY GROUPS OF PL-EMBEDDING SPACES, II

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THEOREM. For $i \leqq m-2$ and $n \leqq m-3$, $\pi_i PL(S^n, S^m)$ is isomorphic to $\pi_i V^{PL}_{m,n}$, the homotopy groups of the PL-Stiefel manifold of n-planes in Euclidean m-space.

E. C. Zeeman [10] conjectured that the homotopy groups, $\pi_i PL(S^n, S^m)$, $m \geqq n+i+3$, of the space of PL-embeddings of the n-sphere into the m-sphere were trivial. As indicated in [4], results of M. C. Irwin [5] and C. Morlet [7] can be used to verify this conjecture. In the theorem above, we generalize this result.

In particular, we have the following [2].

COROLLARY. *$\pi_i PL(S^n, S^m) = 0$ for $i < m-n$.*

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We shall assume familiarity with the Δ-set theory of C. P. Rourke and B. J. Sanderson [9] (or equivalently, the quasisimplicial theory of C. Morlet [8]). Let Δ^i be the standard i-simplex and let $\partial_k: \Delta^i \to \Delta^{i-1}$ be the kth face map. We shall consider the following Δ-sets which are easily seen to be Kan Δ-sets. We indicate an i-simplex from each. All maps commute with the projection along the second factor and $\partial_k f$ is defined to be the restriction to the product of the appropriate set and $\partial_k \Delta^i$.

$PL(S^n, S^m)$	$f: S^n \times \Delta^i \to S^m \times \Delta^i$ is a PL-embedding.
$PL(S^n, S^m \bmod X)$	$f: S^n \times \Delta^i \to S^m \times \Delta^i$ is a PL-embedding such that $f \mid X \times \Delta^i$ is the identity, $X \subseteq S^n$.
$\mathrm{Aut}(S^m)$	$f: S^m \times \Delta^i \to S^m \times \Delta^i$ is a PL automorphism.
$\mathrm{Aut}(S^m \bmod X)$	$f: S^m \times \Delta^i \to S^m \times \Delta^i$ is a PL-automorphism such that $f \mid X \times \Delta^i$ is the identity, $X \subseteq S^m$.
PL_m	Germ of a PL-automorphism $f: R^m \times \Delta^i \to R^m \times \Delta^i$ such that $f \mid 0 \times \Delta^i$ is the identity; R^m is Euclidean m-space and 0 is the origin.
$PL_{m\ n}$	Germ of a PL-automorphism $f: R^m \times \Delta^i \to R^m \times \Delta^i$ such that $f \mid R^n \times \Delta^i$ is the identity; $R^n = R^n \times 0 \subseteq R^n \times R^{m-n} = R^m$.

The quotient complex $PL_m / PL_{m,n} = V^{PL}_{m,n}$ is the PL-Stiefel manifold introduced by A. Heafliger and V. Poenaru [1].

PROPOSITION 1. *$PL_{m,n} \subseteq PL_m \xrightarrow{p} V^{PL}_{m,n}$ is a Kan fibration where p is the natural projection.*

Let $S^n \subseteq S^m$ be the standard inclusion and define $r\colon \mathrm{Aut}(S^m) \to PL(S^n, S^m)$ by $r(f) = f|S^n \times \Delta^i$ where f is an i-simplex of $\mathrm{Aut}(S^m)$. The following was proved by C. Morlet [8].

PROPOSITION 2. $\mathrm{Aut}(S^m \bmod S^n) \subseteq \mathrm{Aut}(S^m) \xrightarrow{r} PL(S^n, S^m)$ *is a Kan fibration.*

Let x and y be distinct points of S^n and define similar to r the map $r'\colon \mathrm{Aut}(S^m \bmod x, y) \to PL(S^n, S^m \bmod x, y)$. One can similarly prove the following.

PROPOSITION 3. $\mathrm{Aut}(S^m \bmod S^n) \subseteq \mathrm{Aut}(S^m \bmod x, y) \xrightarrow{r'} PL(S^n, S^m \bmod x, y)$ *is a Kan fibration.*

Let $h\colon S^m - x \to R^m$ be a PL-homeomorphism such that h is onto, $h(S^n - x) = R^n$ and $h(y) = 0$. Define $q\colon \mathrm{Aut}(S^m \bmod x, y) \to PL_m$ by $q(f) =$ germ of $(h \times id.)f(h \times id.)^{-1}$. Note that $q(\mathrm{Aut}(S^m \bmod S^n)) \subseteq PL_{m,n}$. Let $q' = q\,|\,\mathrm{Aut}(S^m \bmod S^n)\colon \mathrm{Aut}(S^m \bmod S^n) \to PL_{m,n}$.

PROPOSITION 4. *q and q' are homotopy equivalences.*

The first part was proved by N. H. Kuiper and R. K. Lashof [6] and the second part can be proved similarly, also, from [6] we have the following.

PROPOSITION 5. *The inclusion* $\mathrm{Aut}(S^m \bmod x, y) \subseteq \mathrm{Aut}(S^m)$ *induces isomorphisms* $\pi_i\mathrm{Aut}(S^m \bmod x, y) \to \pi_i\mathrm{Aut}(S^m)$ *for* $i \leqq m - 2$.

Let f be an i-simplex in $PL(S^n, S^m \bmod x, y)$. By J. F. P. Hudson [3], there exists an i-simplex f' in $\mathrm{Aut}(S^m \bmod x, y)$ such that $r'(f') = f$. Define $q''\colon PL(S^n, S^m \bmod x, y) \to V^{PL}_{m,n}$ by $q''(f) = pq(f')$.

PROPOSITION 6. *q'' is a well defined Δ-map such that the following diagram is commutative.*

$$\begin{array}{ccccc}
\mathrm{Aut}(S^m \bmod S^n) & \subseteq & \mathrm{Aut}(S^m \bmod x, y) & \xrightarrow{r'} & PL(S^n, S^m \bmod x, y) \\
\downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q''} \\
PL_{m,n} & \subseteq & PL_m & \xrightarrow{p} & V^{PL}_{m,n}
\end{array}$$

Proof. Suppose $F'' \in \mathrm{Aut}(S^m \bmod x, y)$ such that $r'(F'') = f$. Hence there exists $g \in \mathrm{Aut}(S^m \bmod S^n)$ such that $F'' = gf'$. Therefore, $q(F'') = q(gf') = q(g)q(f')$ and $pq(F'') = pq(f')$ since $q(g)$ is in $PL_{m,n}$.

Proof of Theorem. It follows from the above propositions that q'' induces isomorphisms $\pi_i PL(S^n, S^m \bmod x, y) \to \pi_i V^{PL}_{m,n}$ for all i. Note that the following diagram is commutative.

$$\begin{array}{ccccc} \mathrm{Aut}(S^m \bmod S^n) \subseteq & \mathrm{Aut}(S^m \bmod x, y) & \xrightarrow{r'} & PL(S^n, S^m \bmod x, y) \\ \| & \cap\!\!| & & \cap\!\!| \\ \mathrm{Aut}(S^m \bmod S^n) \subseteq & \mathrm{Aut}(S^m) & \xrightarrow{r} & PL(S^n, S^m). \end{array}$$

Hence, from the above propositions, the inclusion induces isomorphisms $\pi_i PL(S^n, S^m \bmod x, y) \to \pi_i PL(S^n, S^m)$ for $i \leqq m - 2$, from which the theorem follows.

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