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REFLEXIVE OPEN MAPPINGS

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This paper is a study of reflexive open mappings. From the definition of reflexive open mapping and some of its elementary properties it is clear that it is a generalization of an open mapping. Classes of such mappings are identified and conditions under which such mappings are open mappings are given. Some applications to such mappings in plane regions are given.

1. Introduction. A function f from a topological space X to another such space Y is said to be reflexive open if for every open set U of X the set $f^{-1}(f(U))$ is also open. An immediate observation is that all open mappings and all one-to-one functions are necessarily reflexive open. In this paper an outline of the essential properties of such functions is given and particular attention is given to those conditions which necessarily imply the function is an open function. We show that openness of a reflexive open mapping will follow from additional conditions on the mapping such as quasi-compactness, or, closedness, or by a compactness condition on the domain space, and finally by requiring the domain and range to be certain subsets of Euclidean 2-space. The last condition mentioned is of interest in topological analysis.

2. Notation. A mapping is a continuous function. A generalized continuum is a connected, locally compact, separable metric space. A simple closed arc is written $[ab]$ while the symbols $[ab)$ and (ab) indicate a closed arc less the point b or less the points a and b. A region is an open connected set. A topological ray is a homeomorphic image of a ray in the real line and a topological line is a homeomorphic image of the real line. Other notation and definitions are as in [6]. The cyclic theory used is that of reference [6]. A function $f(X) = Y$ is a local homeomorphism at a point x in X if there exists an open set U about x such that $f(U)$ is open in Y and also $f|U$ is a homeomorphism onto $f(U)$.

DEFINITION. A function $f(X) = Y$ is reflexive open if $f^{-1}(f(U))$ is open whenever U is open.

3. Elementary Properties. In this section we show various properties of reflexive open mappings, most of which are similar to properties of open mappings.

THEOREM 3.1. *Let $f(X) = Y$ be a function. These are equivalent.*
(a) *f is reflexive open.*
(b) *If $A \subset X$ then $f^{-1}(f(\bar{A})) \subset \overline{f^{-1}(f(A))}$.*
(c) *If $A \subset X$ then $\overline{f^{-1}(f(\bar{A}))} = \overline{f^{-1}f(A))}$.*
(d) *If Q is an inverse set so is $\bar{Q}$.*

Proof. All of these are straightforward. As a sample, for the case b implies c, $\overline{f^{-1}(f(A))} \subset \overline{f^{-1}(f(\bar{A}))}$, and $f^{-1}(f(\bar{A})) \subset \overline{f^{-1}(f(A))}$ by hypothesis so equality holds.

THEOREM 3.2. *If $f(X) = Y$ is a function where X is a metric space these are equivalent.*
(a) *f is reflexive open.*
(b) *If $x_n \to x$, then $f^{-}(f(x)) \subset \liminf f^{-1}(f(x_n))$.*
(c) *If $x_n \to x$, then $f^{-1}(f(x)) \subset \limsup f^{-1}(f(x_n))$.*

Proof. Consider the case a implies b. If U is an open set that meets $f^{-}(f(x))$ then $f^{-1}(f(U))$ contains x_n for all but finitely many n so that $U \cap f^{-}(f(x_n)) \neq \phi$ for all but finitely many n. Evidently b implies c. If (x_n) is a sequence in $X - f^{-1}(f(U))$ tending to a point x_0 in $f^{-1}(f(U))$, since each $f^{-1}(f(x_n))$ misses both $f^{-1}(f(U))$ and U it follows that U is not open, so c implies a.

COROLLARY 3.2. *If $f(X) = Y$ is also continuous, then reflexive openness is equivalent to $f^{-1}(f(x)) = \lim f^{-1}(f(x_n))$.*

Proof. By continuity, $\limsup f^{-1}(f(x_n)) \subset f^{-1}(f(x))$.

THEOREM 3.3. *If $f(X) = Y$ is a reflexive open function, then $g = f \mid Q$ is also reflexive open if Q is open in X or if $f^{-1}(f(Q)) = Q$.*

Proof. If Q is open, and U is an open set in X, $g^{-1}(g(U \cap Q)) = f^{-1}(f(U \cap Q)) \cap Q$, which is evidently open in Q.
If Q is an inverse set, then

$$\begin{aligned} g^{-1}(g(U \cap Q)) &= g^{-1}(f(U) \cap f(Q)) \\ &= f^{-1}(f(U)) \cap Q\ . \end{aligned}$$

A function is said to be reflexive closed if $f^{-1}(f(A))$ is closed whenever A is closed. It is evident from the definitions that a mapping generates an upper (lower) semi-continuous decomposition iff the mapping is reflexive closed (reflexive open). We state the next three known results (see [9]) in these terms.

Let $f(X) = Y$ be a continuous function, where X and Y are metric spaces. Let G be the decomposition of X into the sets $f^{-1}(y)$, $y \in Y$.

THEOREM 3.5. *Let M be the hyperspace of the decomposition G. The natural mapping $m(X) = M$ associated with this decomposition is open* (closed) *if and only it f is reflexive open* (reflexive closed).

THEOREM 3.6. *If $f(X) = Y$ is reflexive open, then f factors uniquely as $f = hm$ where $m(X) = M$ is an open mapping and $h(M) = Y$ is a one-to-one mapping.*

THEOREM 3.7. *If $f(X) = Y$ is a reflexive open mapping which is quasi-compact, then f is open.*

COROLLARY 3.7.1. *If f is closed and reflexive open, then f is an open mapping.*

COROLLARY 3.7.2. *If f is reflexive open and X is compact then f is open.*

THEOREM 3.8. *If $f(X) = Y$ is a reflexive open mapping where X is a locally connected generalized continuum and Y is a metric, then the middle space M of the factoring $f = hm$ is also a locally connected generalized continuum.*

Proof. Let p and q be distinct elements of M. There are distinct points x and y of X such that $m(f^{-1}(f(x)) = p$ and $m(f^{-1}(f(y)) = q$ and disjoint neighborhoods U and V in Y about $f(x)$ and $f(y)$ respectively. Thus $m(f^{-1}(U))$ and $m(f^{-1}(V))$ are open sets about p and q respectively which are disjoint, since $m(f^{-1}(U)) \cap m(f^{-1}(V)) \neq \phi$ implies that $U \cap V \neq \phi$.

Now M is locally compact, hence regular, and also connected and locally connected, since m is an open mapping. Using Theorem 4, [5], and its corollary it follows that M is also a separable metric space.

THEOREM 3.9. *Let $f(X) = Y$ be a reflexive open mapping where X and Y are metric spaces. If K is a connected set in Y for which $f^{-1}(K)$ is compact, then any component of $f^{-1}(K)$ maps onto K.*

Proof. Suppose V is an open set about a component Q of $f^{-1}(K)$. There is an open set U about Q with $U \subset V$ such that $Fr(U) \cap f^{-1}(K) = \phi$. Let $S = U \cap f^{-1}(K) = \bar{U} \cap f^{-1}(K)$. Now $f \mid f^{-1}(K)$ is closed, reflexive open by 3.4 and by the corollary to 3.7 even open, so $f(S)$ is both open and closed in K, and thus $f(S) = K$ so that $f(V) \supset K$. Since V is arbitrary it follows that $f(Q) = K$.

THEOREM 3.10. *Let $f(X) = Y$ be a reflexive open mapping where X and Y are metric spaces. If A is a conditionally compact inverse set in X, then $f \mid A$ is open, closed, and quasicompact and if $f(A)$ is perfect, then $f \mid A$ is compact.*

Proof. Since A is an inverse set we have $f(\bar{A} - A) \subset f(\bar{A}) - f(A) = \overline{f(A)} - f(A)$, and thus by Theorem 7, [9], $f \mid A$ is a closed mapping, hence an open mapping by 3.7. By Theorem 4, [9], the conclusion follows.

THEOREM 3.11. *Let $f(X) = Y$ be a reflexive open mapping where X and Y are locally connected generalized continua. If R is a region in Y for which $f^{-1}(R)$ is conditionally compact and Q is a component of $f^{-1}(R)$, then $f(Q) = R$.*

Proof. Let x and y be distinct elements in R with $x \in f(Q)$. Let $a \in Q \cap f^{-1}(x)$. Let $[xy]$ be an arc in R. By 3.10, $f \mid f^{-1}(R)$ is compact, so $f^{-1}([xy])$ is a compact subset of $f^{-1}(R)$ and if C is the component of $f^{-1}([xy])$ which contains a then by 3.9, $f(C) = [xy]$ which implies that $f(Q) = R$.

THEOREM 3.12. *Suppose $f: X \rightarrow Y$ is an additive function, not necessarily continuous, where X and Y are normed linear spaces. Then f is reflexive open.*

Proof. Due to the algebraic structure of X and Y the symbol $A - B$ here will mean the set of all $(a - b)$ such that $a \in A$ and $b \in B$.

Suppose $x_n \rightarrow x$ and $z \in f^{-1}(f(x))$. If N is an open neighborhood of z then $N - z$ is a neighborhood of 0 and since $(x_n - x) \rightarrow 0$. if n is large we have $(x_n - x) \in N - z$, so that $z + (x_n - x) \in N$ if n is large. Thus $f^{-1}f(z + (x_n - x)) = f^{-1}f(x_n)$ meets N if n is large so that $f^{-1}(f(x)) \subset \liminf f^{-1}(f(x_n))$ and by Theorem 3.1 f is reflexive open.

THEOREM 3.13. *There exists a reflexive open function which is discontinuous at each point in its domain.*

Proof. Let X be the real numbers and let H be a Hamel basis for X. Choose $x_0 \in H$. Define

$$f(x_0) = \tfrac{1}{2}$$
$$f(x) = 1 \qquad \text{if } x \in H \text{ and } x \neq x_0 .$$

Now if $x \in X - H$ there is a finite collection of rational numbers $\{r_1, r_2, \cdots r_n\}$ such that $x = \sum_1^n r_i x_i$ with each $x_i \in H$, so we define $f(x) = \sum_1^n r_i f(x_i)$.

A check of the definition shows that f is additive; hence f is reflexive open. Since f is discontinuous at x_0, f is discontinuous at each point in X.

THEOREM 3.14. *If $f(X) = Y$ is a reflexive open mapping where X and Y are locally compact metric spaces and X is separable, then there is an open set $X_0 \subset X$ such that $f(X_0)$ is an open dense set in Y and the restriction of f to X_0 is open.*

Proof. Let Y_0 be the set of all points y in Y for which there exists a compact set K in X such that y is interior to $f(K)$. If $V =$ interior $f(K)$, evidently $V \subset Y_0$ so Y_0 is open.

Let $\{K_n\}$ be a sequence of compact sets in X such that $X = \bigcup_1^\infty K_n$ and also $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$. If $y \in Y$ and U is a conditionally compact open set about y then $\bar{U} = \bigcup_1^\infty (\bar{U} \cap f(K_n))$, and thus there is an open set 0 such that $0 \cap \bar{U}$ is nonempty and lies in some $f(K_m) \cap \bar{U}$. We can assume that 0 lies in U so that $0 \subset f(K_m)$; thus U meets Y_0 and Y_0 is dense in Y. Let $X_0 = f^{-1}(Y_0)$.

Let $g = f \mid X_0$, and let (y_n) be a sequence of distinct points in $g(X_0)$ such that $y_n \to y$, $y \in g(X_0)$. Let K be the compact set in X such that $y \in V = \text{int } f(K)$. Suppose $x \in g^{-}(y)$ and x is not in lim inf $g^{-1}(y_n)$. Then there is an open set U about x for which infinitely many $g^{-1}(y_n)$ lie outside U, and hence there is a sequence (x_{n_m}), each $x_{n_m} \in (g^{-1}(y_{n_m}) \cap K) - U$, and a point $x_0 \in K$ such that $x_n \to x_0$. By continuity $f(x_0) = y$, and $x_0 \in g^{-1}(y)$. Since X_0 is open g is reflexive open, and by Corollary 3.2 we have $g^{-1}(y) = g^{-1}(g(x_0)) = \lim g^{-1}(g(x_{n_m}))$, so that U must meet infinitely many $g^{-1}(g(x_{n_m}))$ which is contradictory. Thus $g^{-1}(y) \subset \lim\inf g^{-1}(y_n) \subset \lim\sup g^{-1}(y_n) \subset g^{-1}(y)$ so that $g^{-1}(y) = \lim g^{-1}(y_n)$ and g is open.

COROLLARY 3.14.1. *If $f(X) = Y$ is a one-to-one mapping, then there exists an open set $X_0 \subset X$ such that $f(X_0)$ is an open dense subset of Y and $f \mid X_0$ is a homeomorphism.*

Proof. A one-to-one mapping is reflexive open; thus $f \mid X_0$ is open, and thus a homeomorphism.

4. **Main Lemma.** Let M be a locally connected generalized continuum with the property that for each x in M there is a conditionally compact region R containing x such that the complement of R is connected. Let Y be an unbounded locally connected generalized

continuum in the plane E^2, where Y has no local cut points and where any simple closed curve in Y bounds a two-cell in Y.

LEMMA 4. *If $h(M) = Y$ is a one-to-one continuous mapping, then M has exactly one noncompact cyclic element. Furthermore, if the noncompact cyclic element has no local cut points, then h is a homeomorphism.*

The proof will be accomplished by assuming h is not a homeomorphism and then establishing a sequence of statements describing the structure of M and a sequence of statements which lead to a contradiction. It is convenient to consider two cases. First suppose Y is an open subset of E^2.

(1) The space M has exactly one noncompact cyclic element.

Proof. Suppose that each true cyclic element of M is compact. By a theorem of G. T. Whyburn, [8], as stated by E. Duda, [2], there is a topological ray α in M which is a closed subset of M and such that h is a local homeomorphism at each point of α. Because Y has no local cut points h is not a local homeomorphism at cut points of M. Therefore α lies in a single true cyclic element and this implies the false statement that α is compact.

Now suppose that C_p and C_q are distinct true cyclic elements of M where C_p is noncompact. There is a point t which is a cut point of M separating $C_p - \{t\}$ and $C_q - \{t\}$, and there is a conditionally compact region R of t whose complement is connected. This implies that $M - \{t\}$ has no more than one non-conditionally-compact component, which must contain $C_p - \{t\}$ and therefore C_q is compact.

(2) Every compact true cyclic element of M is a topological closed two cell.

Proof. See Duda, [2].

(3) For each point x in the noncompact cyclic element C_p there is a topological closed two cell W in C_p containing a neighborhood V of x relative to C_p and the open two cell Q of W is an open subset of M.

Proof. There is a conditionally compact region R of x with $M - R$ connected. In $R \cap C_p$ there is a region V of x, relative to C_p, with $\bar{V}$ a locally connected continuum without separating points and such that $\bar{V} \subset R \cap C_p$. The set $h(\bar{V})$ is a cyclicly connected locally connected continuum so by 2.5 page 107 [6], the boundary B of the unbounded component of $E^2 - h(\bar{V})$ is a simple closed curve. Furthermore if K is the closed two cell bounded by B then $h(\bar{V}) \subset K$. The connected set $h(M - R)$ misses B and is not in K for this would imply that the range of h, $h(R) \cup h(M - R)$, is a conditionally compact set. Since $K \subset h(R)$, $W = h^{-1}(K)$ is a closed disk in $R \cap C_p$. The closed two cell $W \supset \bar{V}$, and the open two cell Q of W is open in M since Q is the inverse image of the interior of K.

(4) If H is the set of points of C_p which lie in an open two cell, then H is connected, dense in C_p, and open in M.

Proof. By (3) H is dense in C_p and open in M. Let A be a component of H and suppose H is not connected. Since $C_p = \bar{H} = \bar{A} \cup (\overline{H - A})$ and C_p is connected there exists $q \in \bar{A} \cap (\overline{H - A})$. Take a topological closed two-cell W about q as in (3). Since q is interior to W it follows that $A \cap W$ and $(H - A) \cap W$ are nonempty. Since A is open in M we have $A \cap Q \neq \phi$ where Q is the open two cell of W and therefore $Q \subset A$. But similarly $(H - A)$ must meet Q and this is false.

(5) Every component of $C_p - H$ is a topological line.

Proof. By (3) if $x \in C_p - H$ there is a topological closed two cell W in C_p and a region V of x, relative to C_p, with $\bar{V} \subset W$. Since x must lie in the boundary curve J of W there is an arc (axb) in $V \cap J$. If $z \in (axb) \cap H$ there is an open two cell Q about z with $Q \subset V \cap H$, but since the boundary curve of Q lies in $\bar{V}$ it follows that $Q \subset W$ so that $z \notin J$. It is evident that $C_p - H$ is a one-manifold, thus by [6, p 194] every component of $C_p - H$ is a topological line or a simple closed curve.

Now if J is a simple closed curve in $C_p - H$ then $h(J)$ is a simple closed curve which is the boundary curve of a closed topological two cell K in $h(M)$. Suppose $h(H) \subset K$. Then $h(C_p) \subset K$ and hence if N is a component of $h(M) - K$ it follows that N is the union of no more than countably many sets of the form $h(D_n)$ where D_n is a component of $M - C_p$. For each such D_n we have $\bar{D}_n \cap C_p = x$, a single point, and since $M - \{x\}$ has at most one nonconditionally compact component it follows that each $\bar{D}_n$ is compact. Because N cannot be the union of countably many (more than one) closed (in N) pairwise disjoint sets there is some n such that $N = h(D_n)$, and thus $\bar{N}$ is compact. By the local connectedness of $h(M)$ the union of the conditionally compact components of $h(M) - K$ is also conditionally compact which implies the false conclusion that $\overline{h(M)}$ is compact.

Thus $h^{-1}(K) \cap C_p = J$ and it follows that the open two-cell Q of K is the union of no more than countably many sets of the form $h(D_n)$. By the same argument as before we have that $Q = h(D_n)$ for some n and thus $\bar{D}_n \cap C_p = J$ which is not a single point.

NOTATION. For any set A in $C_p - H$ let $N(A)$ be the union of A plus all the components D of $M - C_p$ such that $\bar{D} \cap A \neq \phi$. For a component D of $M - C_p$ define $\partial \bar{D}$ to be $\bar{D}$ less all the open two cells of the compact true cyclic elements C_x which lie in $\bar{D}$. Define $\partial N(A)$ to be the union of A plus all the sets $\partial \bar{D}$ such that $\bar{D} \subset N(A)$.

(6) If A is a continuum in $C_p - H$, then $N(A)$ is a continuum and further, $h(\partial N(A)) \subset \overline{h(H)}$.

Proof. Since each $x \in A$ lies in a conditionally compact region R for which $M - R$ is connected there is a finite subcover $\{R_1, R_2, \cdots, R_k\}$ of A with $M - R_i$ connected, and $\bar{R}_i$ compact, $i = 1, 2, 3, \cdots, k$. Now C_p does not lie in any R_i so it follows that each R_i contains any component D of $M - C_p$ that it meets; hence $N(A) \subset \bigcup_1^k R_i$. In addition, if $x \notin N(A)$ there is an open region U about x such that $\bar{U} \cap A = \varnothing$. Therefore U cannot contain any component D which lies in $N(A)$ and since U meets at most finitely many such components D it follows that $x \notin \overline{N(A)}$. Thus $N(A)$ is compact, and obviously connected.

If $h(x) \notin \overline{h(H)}$ choose a region V about $h(x)$ such that $\bar{V} \cap \overline{h(H)} = \phi$. Then $\bar{V} \subset h(M - C_p)$ so that $V \subset \bigcup_1^\infty h(\bar{D}_n)$, each D_n being a component of $M - C_p$. By the argument used in (5) there is an integer n such that $h(\bar{D}_n) \supset V$, and thus a compact true cyclic element $C_x \subset \bar{D}_n$ such that x lies in the open two cell of C_x. It follows that $h(\partial N(A)) \subset \overline{h(H)}$.

(7) If L is a component of $C_p - H$ the restriction of the function h to $N(L)$ is a homeomorphism.

Proof. If not, there is a sequence (x_n) of distinct points in $N(L)$ with $Ux_n = \bar{U}x_n$ and a point x_0 in $N(L) - Ux_n$ such that $h(x_n) \to h(x_0)$. For $n = 0, 1, 2, \cdots$ let $x'_n \in L$ such that $x_n \in N(x'_n)$. There is a closed topological ray R in L with origin x'_0 which contains infinitely many x'_n. For simplicity of notation we shall assume that R contains all the x'_n.

Let $B = \partial N(R - x'_0)$. It follows from local connectedness that $\bar{B} = B \cup x'_0$. If $y \in \overline{h(B)} - h(B \cup x'_0)$ and $x = h^{-1}(y)$ we can choose a conditionally-compact open set U containing x such that $\bar{U} \cap \bar{B} = \phi$, and since $y \in h(\bar{U}) \cap \overline{h(B)}$ it follows that $\overline{h(B)} - h(B \cup x'_0)$ is an F_σ, and hence $\overline{h(B)} - h(B)$ is also an F_σ. In addition since $\partial N(R - x'_0)$ is a countable union of closed arcs plus countably many compact sets $\partial\bar{D}$ so is $h(B)$. Thus by the usual category argument there is an open set $0 \subset Y$ such that $0 \cap \overline{h(B)}$ is not empty and lies in one of the closed sets in $h(B)$. Suppose $d \in \partial\bar{D}$ such that $h(d) \in 0 \cap \overline{h(B)}$. We let $t = \bar{D} \cap (R - x_0')$ and let $[y_0y_1] = h([td])$ where $[td]$ is an arc in $\bar{D}$ and $y_0 = h(t)$. Choose a closed two cell W in C_p such that y_0 is interior to an arc $[ay_0b]$ which lies in $h(W) \cap h(R - x_0')$. If $d \notin (R - x_0')$, let $0'$ be an open region in Y about $h(d)$ with $0' \subset 0$ such that $0' \cap h(W) = \phi$, and choose $y_2 \in 0' \cap h(H)$. Then if $y_3 \in h(W) \cap h(H)$ there is an arc $[y_2y_3]$ in $h(H)$. There is an arc $[y_0y_3]$ in $y_0 \cup (h(W) \cap h(H))$. Let p be the first element in $[y_0y_3]$ from y_0 to y_3 which lies in $[y_2y_3]$, then $[y_0py_2]$ is an arc in $y_0 \cup h(H)$. There is an arc $[y_1y_2]$ in $0'$, and if q is the first element of $[y_1y_2]$ from y_1 to y_2 that lies in $[y_0py_2]$ then q lies in $[py_2]$ and $[y_1q] \cup [qpy_0]$ is an arc. Now let r be the first element of $[y_0y_1]$ from y_0 to y_1 that lies in the half-open arc $[y_1qpy_0)$. Then r lies in $[y_1qp)$, and $[y_0r] \cup [rpy_0]$ is a simple closed curve J contained in $h(\bar{D}) \cup Q \cup h(H)$ such that $J \cap h(R - x_0') = y_0$.

Choose a closed topological two cell $W' \subset W$ such that $h(W') \cap [ay_0b]$ contains a subarc $[a'y_0b']$ and such that $h(W') - J = E \cup F$, two components. Now E must lie in one of the two components S and T of $E^2 - J$, say S, therefore $F \subset T$. It follows that $h(R - x_0')$ meets both S and T, and we may assume that $h(x_0') \in S$. If $x_n' \in [x_0't]$ for infinitely many integers n then infinitely many different x_n lie in the compact set $N[x_0't]$, which implies the false statement that (x_n) has a convergent subsequence. But if for at most finitely many n we have $x_n' \in [x_0't)$ then at most finitely many $h(x_n)$ lie in S which is contradictory to $h(x_n) \rightarrow h(x_0)$.

If $0 \cap \overline{h(B)} \subset h(R - x_0')$, a simple closed curve J can be constructed so that $h(R - x_0')$ meets both components of $E^2 - J$ and meets J at a single point $h(t) \in 0$. This case leads to the same contradiction so the statement is proven.

The ray α of [8] can be chosen so as to meet $C_p - H$ at a single point t, and such that $h(\alpha)$ is the boundary of a closed 2-cell K_0 in Y. If L_0 is the component of $C_p - H$ which contains t it follows that $L_0 \cap h^{-1}(K_0)$ is a closed topological ray R_0 whose origin is t, and furthermore $h(N(R_0))$ lies in the compact set $K_0 \cup h(N(t))$. If L_n is any other component of $C_p - H$, then $L_n \cap \alpha = \phi$ so that $N(L_n) \cap \alpha = \phi$, and hence $h(N(L_n)) \subset K_0$ or $h(N(L_n)) \cap K_0 = \phi$. Let $\mathfrak{A}$ be the collection of sets $A_n = h(N(L_n))$, where $h(L_n) \subset K_0$, and the set $A_0 = h(N(R_0))$. Evidently $\mathfrak{A}$ is a countable collection of pairwise disjoint sets.

The remaining statements lead to a contradiction. It is convenient to make the following definitions:

For each $A_n \in \mathfrak{A}$, with $n > 0$, let L_n be the component of $C_p - H$ which determines A_n. There is a homeomorphism $g_n(L_n) = X$, the real line, and thus $L_n = R_n^+ \cup R_n^-$ where $R_n^+ = g_n^{-1}\{x \mid x \geqq 0\}$ and $R_n^- = g_n^{-1}\{x \mid x \leqq 0\}$. We set $A_n^+ = h(N(R_n^+))$, $A_n^- = h(N(R_n^-))$ and also set $A_0^+ = A_0$. For $n = 0, 1, 2, 3, \cdots$ let $\mathscr{C}_n{}^+$ be the collection of all $A \in \mathfrak{A}$ such that $A \neq A_n$ and for which there is $\{A_{n_1}, A_{n_2}, \cdots A_{n_k}\} \subset \mathfrak{A} - \{A_n\}$ with $\bar{A}_n^+ \cap A_{n_1} \neq \phi$, $\bar{A}_{n_1} \cap A_{n2} \neq \phi$, $\cdots \bar{A}_{n_k} \cap A \neq \phi$. For $n = 1, 2, 3, \cdots$ we define $\mathscr{C}_n{}^-$ similarly, using A_n^-. For $n = 0, 1, 2, \cdots$ let $\mathscr{C}_n = \mathscr{C}_n{}^+ \cup \{A_n\} \cup \mathscr{C}_n{}^-$, where $\mathscr{C}_0{}^-$ is defined to be the empty set, and let F_n^+ be the closure of $\{y \mid \text{there exists } A \in \mathscr{C}_n{}^+ \text{ with } y \in A\}$, F_n^- similarly, and $F_n = F_n^+ \cup A_n \cup F_n^-$.

(8) If $n \neq m$ and $A_m \in \mathscr{C}_n$ then $A_n \notin \mathscr{C}_m$.

Proof. If not, there are two subcollections of $\mathfrak{A}$, $\{A_{n_1}, A_{n_2}, \cdots A_{n_k}\}$ where $A_m = A_{n_k}$, and $\{A_{m_{k+1}}, A_{m_{k+2}}, \cdots A_{m_{k+r=s}}\}$ where $A_n = A_{m_s}$ and a set $\{y_1, y_2, \cdots, y_s\}$ such that $y_1 \in \bar{A}_{m_s} \cap A_{n_1}$ and $y_{i+1} \in \bar{A}_{ni} \cap A_{n_{i+1}}$, if $1 \leqq i < k$, $y_{k+1} \in \bar{A}_{n_k} \cap A_{m_{k+1}}$, and $y_{i+1} \in \bar{A}_{m_i} \cap A_{m_{i+1}}$, if $k < i < s$. For $1 \leqq i \leqq s$ let $x_i' \in C_p - H$ such that $h^{-1}(y_i) \in N(x_i')$ and let R_i be the closed topological ray in $C_p - H$ with origin x_i' and with

$$y_{i+1} \in \overline{h(N(R_i))} \text{ if } i < s$$

and

$$y_1 \in \overline{h(N(R_s))} \;.$$

As in the proof of (7) the set $\bigcup_1^s \overline{h(\partial N(R_i - x_i'))} - \bigcup_1^s h(\partial N(R_i - x_i'))$ is an F_σ, and $\bigcup_1^s h(\partial N(R_i - x_i'))$ is a countable union of closed arcs plus countably many sets of the type $h(\partial \bar{D})$, so there exists a region 0 with $0 \cap \bigcup_1^s \overline{h(\partial N(R_i - x_i'))}$ nonempty and contained in $h(\partial N(R_j - x_j'))$ for some j, $1 \leqq j \leqq s$. The region 0 can be chosen sufficiently small so as to have a vacuous intersection with the compact set $\bigcup_1^s h(N(x_i'))$. By the construction used in (7) there is a simple closed curve J which meets $h(R_j - x_j')$ at a single point $h(t)$. The set $E = \bigcup_{i \neq j} h(N(R_i - x_i'))$ is a connected set which misses J, so E lies in one of the two components S and T of $E^2 - J$, say S. The point $y_j \notin J$ and is in the closure of one of the sets determining E, so $y_j \in S$, hence $h(x_j') \in S$ which implies that $h(N([x_0't)))\subset S$. But $h(N(R_j - [x_j't])) \subset T$ and $\overline{h(N(R_j - [x_j't]))} \cap E \neq \phi$ which is contradictory.

(9) For any integer n there is an integer m such that $A_m \in \mathscr{C}_n$, hence $F_m \subset F_n$ and such that if $p \neq m$ then A_p does not meet both F_m^+ and F_m^-.

Proof. For each $A_i \in \mathscr{C}_n$ let L_i be the set in $C_p - H$ such that

$h(N(L_i)) = A_i$. If $X = \cup L_i$ then from (5) the complement of X is open in $C_p - H$; thus $N(X) = \cup N(L_i)$ is closed, so that if $K_n = \cup \partial N(L_i)$ then $\overline{h(K_n)} - h(K_n)$ is an F_σ, and $h(K_n)$ is a countable closed arc sum plus countably many closed sets $h(\partial \bar{D})$. Thus there exists an open region 0 such that $0 \cap \overline{h(K_n)}$ is nonempty and lies in some $h(\partial N(L_m)) \subset h(K_n)$. Since $A_m = h(N(L_m)) \in \mathscr{C}_n$ it follows that $F_m \subset F_n$.

Suppose $p \neq m$ and assume that $y \in A_p \cap F_m^+$ and $z \in A_p \cap F_m^-$. If $y = z$ let $\gamma = y = z$ and if $y \neq z$ let γ be an arc $[yz]$ in A_p. We can assume that $0 \cap \gamma = \phi$. By the construction used in (7) there is a simple closed curve J such that $J \cap F_m \subset A_m$ with $J - F_m \subset 0 \cup h(H)$, and also $h(L_m)$ meets both components S and T of $E^2 - J$. Note that $m \neq 0$ since $F_m^- \neq \phi$. Suppose $A \in \mathfrak{A}$ such that $A \neq A_m$ and A meets $\bar{A}_m^+$. Thus $A \in \mathscr{C}_m{}^+$; and furthermore $A \subset F_m^+$ so that $A \cap J = \phi$, so we can suppose that $A \subset S$. This implies that each A in $\mathscr{C}_m{}^+$ also lies in S, so that $F_m^+ \subset \bar{S}$. Suppose now that $A \neq A_m$ and $A \cap \bar{A}_m^- \neq \phi$. Then since $\bar{A}_m^- - A_m \subset T$ and $A \cap J = \phi$, it follows that $A \subset T$. Consequently each A in $\mathscr{C}_m^-$ lies in T so that $F_m^- \subset \bar{T}$. From $\{y \cup z\} \subset (F_m - A_m)$ we have $\{y \cup z\} \cap J = \phi$ so that $y \neq z$. Thus γ is a nondegenerate arc with $\gamma \cap S \neq \phi, \gamma \cap T \neq \phi$ and $\gamma \cap 0 = \phi$. Therefore γ meets $(J - (F_m \cup 0))$ which lies in $h(H)$ and $A_p \cap h(H) = \phi$, which is contradictory.

(10) Completion of the proof: The collection $\mathscr{C}_0{}^+$ is not empty, otherwise $h(R_0)$ is a topological ray which is a closed bounded subset of the plane. Similarly $\mathscr{C}_n{}^+$ and $\mathscr{C}_n{}^-$ are not empty for $n > 0$. Let $A_n \in \mathscr{C}_0{}^+$, then $A_0 \notin \mathscr{C}_n$ by (8) and by (9) there is an integer m such that $F_m \subset F_n$ and furthermore A_0 does not meet both members of $\{F_m^+, F_m^-\}$. Let B_0 be a member of $\{F_m^+, F_m^-\}$ that misses A_0.

Let A_{n_1} be the member of $\mathfrak{A}$ with lowest subscript such that $A_{n_1} \cap B_0 \neq \phi$. We assert that there is an integer p for which $F_p \subset B_0$ such that A_{n_1} does not meet both F_p^+ and F_p^-. To show this, first suppose that $B_0 = F_m^+$. Now if $A_{n_1} \in \mathscr{C}_m{}^+$ (and hence $A_{n_1} \subset B_0$) then $n_1 \neq m$. We choose some integer q with $q \neq n_1$ for which $A_q \in \mathscr{C}_{n_1}$. From (9) there is an integer p such that $A_p \in \mathscr{C}_q$ and if $l \neq p$ then A_l does not meet both members of $\{F_p^+, F_p^-\}$. From $A_q \in \mathscr{C}_{n_1} \subset \mathscr{C}_m$ it follows that $F_p \subset F_m$ and from $A_{n_1} \subset B_0$ it follows that $F_p \subset B_0$. Now if $p = n_1$ then $A_{n_1} \in \mathscr{C}_q$ and $A_q \in \mathscr{C}_{n_1}$ with $q \neq n_1$ which contradicts (8). Thus $p \neq n_1$ and A_{n_1} does not meet both members. If $A_{n_1} \notin \mathscr{C}_m{}^+$ we choose q to be any integer for which $A_q \in \mathscr{C}_m{}^+$ and let p be the integer from (9) such that $A_p \in \mathscr{C}_q, F_p \subset F_q$, and such that if $l \neq p$ then A_l does not meet both F_p^+ and F_p^-. From $\mathscr{C}_p \subset \mathscr{C}_q \subset \mathscr{C}_m{}^+$ we note that $n_1 \neq p$, so A_{n_1} does not meet both members. In addition, from $\mathscr{C}_p \subset \mathscr{C}_m{}^+$ we get $F_p \subset F_m^+ = B_0$. It is then evident that if $B_0 = F_m^-$ the argument is the same except for notation so that the assertion is valid. Let B_1 be a member of $\{F_p^+, F_p^-\}$ that misses A_{n_1}. From $B_1 \subset F_p \subset B_0$ it follows that $B_1 \cap (\bigcup_0^{n_1} A_n) = \phi$.

In this manner we exhibit a collection of sets

$$B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_j \supset \cdots$$

where each B_j is a nonvoid compact set, and such that $\bigcup_0^\infty A_n$ misses $\bigcap_0^\infty B_j$. This is contradictory since $\bigcup_0^\infty A_n \supset \bigcap_0^\infty B_j \neq \phi$.

The second case, where Y is not open in E^2, follows from the first. If A is a continuum in $C_p - H$ then it is not necessarily true that $h(\partial N(A)) \subset \overline{h(H)}$. (See (6)). However, let Q_0 be the open two cell of K_0 and suppose A is a continuum in $C_p - H$ such that $h(A) \subset Q_0$. Then $h(N(A))$ lies in Q_0 as well, and the proof of (6) shows that $h(\partial N(A)) \subset \overline{h(H)}$. From this observation it follows by the same proof as in (7) that if L is a component of $C_p - H$ such that $h(L) \subset Q_0$ then $h | N(L)$ is topological. Furthermore, $h | N(R_0)$ is topological, where R_0 is the topological ray in $C_p - H$ with origin t such that $h(R_0) \subset K_0$. This is evident from the proof of (7), for the ray R of the proof must lie in R_0 and $h(N(R - x_0'))$ must lie in Q_0. Since all the sets $R_i - x_i'$ appearing in the proof of (8) map into Q_0 the proof of (8) is valid, and statement (9) has the same proof, for as was noted, $m \neq 0$, so $h(\partial N(L_m))$ must lie in Q_0. We note that in the proof of (10) it follows that $\bar{A}_0 \neq A_0$ by the same argument, and furthermore $(\bar{A}_0 - A_0) \subset Q_0$, so that $\mathscr{C}_0^+ \neq \phi$, and everything else in the proof is the same. Thus exactly the same contradiction is reached and the theorem is proven.

5. Applications. In this section we suppose that $f(E^2) = Y \subset E^2$ is a reflexive open mapping with compact point inverses and that Y is an unbounded subset of E^2. By Theorem 3.8 there is the factoring $f = hm$ where the middle space $M = m(E^2)$ is a locally connected generalized continuum.

LEMMA 5.1. *If $x \in M$ there exists a conditionally compact region R about x such that $M - R$ is connected.*

Proof. Let $x \in M$ and let $y \in Y$ such that $mf^{-1}(y) = x$. Take J a simple closed curve enclosing $f^{-1}(y)$ such that $J \cap f^{-1}(y) = \phi$ and let R be the component of $M - m(J)$ that contains x. There is a component D of $m^{-1}(R)$ that meets $m^{-1}(x) = f^{-1}(y)$ and $\bar{D}$ is compact since J must enclose D. From the openness of the mapping m we have $m(D) = R$ so that R is conditionally compact. Furthermore, $M - R$ is connected, for if $M - R = A \cup B$, a separation, and we assume that $m(J) \subset A$, then since M is connected so is $R \cup B$, but then $(R \cup B) \cap m(J) = \phi$ implies that $(R \cup B) \subset R$ which is absurd.

With f, Y and M as in Lemma 5.1 we have

LEMMA 5.2. *If $x \in M$ and y is the point in Y such that $mf^{-1}(y) = x$, then x separates M if ond only if $f^{-1}(y)$ separates E^2.*

Proof. If $E^2 - f^{-1}(y)$ is connected so is $m(E^2 - f^{-1}(y)) = M - x$. Now suppose that $E^2 - f^{-1}(y) = B \cup C$ where C is the unbounded component and B is the union of the bounded components. By Lemma 5.1 there is a region R containing x such that $M - R$ is connected and $\bar{R}$ is compact; thus it follows that the component Q of $M - x$ which contains $M - R$ is not conditionally compact (since Y is not bounded). By local connectedness of E^2 we have $\bar{B}$ compact, so $m^{-1}(Q)$ must meet C and therefore $m(C) \subset Q$ so $m^{-1}(Q) \supset C$. If A is a component of $m^{-1}(Q)$ that meets B then $A \subset B$ and by openness of m we have $m(A) = Q$ which gives that $\bar{Q}$ is compact which is false, so that $m^{-1}(Q) = C$, and $m^{-1}(m(C)) = C$. Since $E^2 = C \cup f^{-1}(y) \cup B$ it follows that $m^{-1}(m(B)) = B$, and thus $m^{-1}(m(B)) \cap m^{-1}(m(C)) = B \cap C = \phi$. Thus $m(B) \cap m(C) = \phi$, and hence x separates M, since $M - x = m(B) \cup m(C)$.

THEOREM 5.1. *Let $f(E^2) = Y \subset E^2$ be a reflexive open mapping with compact point inverses where Y is an unbounded locally connected generalized continuum with no local cut points and with the property that each simple closed curve in Y bounds a two cell in Y. Then f is an open mapping.*

Proof. Now $f = hm$ where $m(E^2) = M$ is an open mapping and $h(M) = Y$ is a one-to-one mapping. We assume f is not open, so that h is not a homeomorphism, and reach a contradiction.

By Theorem 3.8 we have that M is a locally connected generalized continuum, and by Lemma 5.1 each point $x \in M$ lies in a conditionally compact region R such that $M - R$ is connected. By the proof of (1) in Lemma 4 there is exactly one noncompact cyclic element C_p of M. Let $q \in C_p$ and let $y \in Y$ such that $q = mf^{-1}(y)$. We assert that q is not a local separating point of C_p. First suppose that q separates M. Using the techniques of the proof of Lemma 5.2 we write $E^2 - f^{-1}(y) = C \cup B$ as before. Both C and B are inverse sets under the mapping m. By Lemma 5.2 $M - q = m(C) \cup m(B)$, a separation, and since $m(\bar{B})$ is compact it follows that $m(C) \cap C_p \neq \phi$ so that $m(\bar{B}) \cap C_p = q$. Let W be a region about q relative to C_p and suppose U' is an open set in M such that $U' \cap C_p = W$. Since $m(\bar{B}) \cap C_p = q$

if Q is a component of $\bar{B}$ then $m(\text{int } Q) \cap C_p = \phi$; otherwise q is both open and closed relative to C_p. Thus if $U = U' \cup (m(\text{int } Q))$ it follows that $U \cap C_p = (U' \cap C_p) \cup [(m(\text{int } Q)) \cap C_p] = [U' \cap C_p] = W$. In addition, $E^2 - Q$ is connected, for if $E^2 - Q = S \cup T$, a separation, with $C \subset S$, then $T \cup Q$ is a connected subset of $\bar{B}$ so that $T \cup Q$ must lie in Q.

Now $m(\bar{B})$ is connected, since each component Q of $\bar{B}$ meets $m^{-1}(q)$, and $m^{-1}(m(\bar{B})) = \bar{B}$ so by Theorem 3.9 or by known results $m(Q) = m(\bar{B})$ for each Q; hence each Q has interior points. From this it follows that there exists no more than finitely many $Q_n \subset \bar{B}$, $n = 1, 2, \cdots$ for otherwise, if $x_1 \in \text{int } Q_1$, since $m(Q_n) = m(\bar{B})$, $m^{-1}(m(x_1))$ meets each Q_n at a point $x_n \in \text{int } Q_n$, and since $\bar{B}$ is compact there must be a point x_0 which is a limit point of $\cup x_n$, but x_0 is in the interior of some Q_m, since $m(x_0) \neq q$.

Thus, for a component Q of $\bar{B}$ and for the particular open set U defined as above for Q, there exists a conditionally compact region $V \subset m^{-1}(U)$ such that $\bar{V} \cap \bar{B} = Q$ and $V - \bar{B} = V - Q$ is connected. Noting again that $\bar{B}$ is an inverse set we see that $m(V - \bar{B}) = m(V) - m(\bar{B})$ so that $m(V - \bar{B}) \cap C_p = m(V) \cap C_p - q$. Since C_p is a cyclic element $m(V - \bar{B}) \cap C_p$ is connected; thus $m(V) \cap C_p - q$ is a region of C_p which lies in W.

Now if q does not separate M, and $W = C_p$ is a region (relative to C_p) about q, then $m^{-1}(U) \supset f^{-1}(y) = m^{-1}(q)$. If Q is a component of $f^{-1}(y)$ there is a conditionally compact region V about Q which lies in $m^{-1}(U)$ such that $Fr(V) \cap f^{-1}(y) = \phi$, and $V - f^{-1}(Y)$ is connected, because $E^2 - f^{-1}(Y)$ is connected. It follows that $m(V - f^{-1}(y)) \cap C_p = m(V) \cap C_p - q$ is a region in C_p which lies in W.

Thus no point of C_p is a local separating point of C_p. Thus by Lemma 4 f is an open mapping since h must be a homeomorphism.

THEOREM 5.2. *If $f(E^2) = Y \subset E^2$ where f and Y are as in Theorem 5.1, then each point inverse $f^{-1}(y)$ consists of finitely many components, none of which separate the plane and none of which has an interior.*

Proof. Let $\pi(E^2) = M_1$ be the natural mapping of the decomposition of E^2 in the components of the point inverses $f^{-1}(y)$, and let $l(M_1) = M$ be defined by $l(x) = m(\pi^{-1}(x))$, where M is the middle space of the factoring $f = hm$. Now π is monotone and closed hence compact; l is light and open and $m = l\pi$, see [1], [4] and [9]. Now M has no cut points since h is a homeomorphism, so by Lemma 5.2 no point inverse separates the plane, and evidently no component of a point inverse separates the plane. Using [3], [7], or Corollary 2.31,

p. 172 [6] it follows that M_1 is a topological plane, but then by [10], or Corollary 3.42, p. 191 [6] each $l^{-1}(x)$ consists of isolated points. Since each $l^{-1}(x)$ is compact it follows that each $l^{-1}(x)$ is finite so that each $f^{-1}(y)$ consists of finitely many components Q. Finally, if for some Q we have int $Q \neq \phi$, then $f^{-1}(f(\text{int } Q)) = f^{-1}(y)$ is both open and closed, which contradicts the connectedness and unboundedness of E^2.

THEOREM 5.3. *Suppose $f(E^2) = Y$ is both a reflexive open and a reflexive closed mapping, where Y is as in Theorem* 5.1. *Then each point inverse $f^{-1}(y)$ is compact, and there is an integer k such that each $f^{-1}(y)$ has at most k components.*

Proof. The mapping m of the factoring $f = hm$ is both open and closed and since M is connected m is even a compact mapping, see [9]; thus if $x \in M$ then $m^{-1}(x)$ is compact and then of course each $f^{-1}(y)$ is also compact.

Following the proof of Theorem 5.2 we factor m as $m = l\pi$ where $\gamma(E^2) = M_1$ is monotone closed and compact and $l(M_1) = M$ is light and open, and M_1 is a topological plane. Since m is closed it follows that so is l; hence the decomposition of M_1 generated by l is upper semi-continuous. Now l has bounded multiplicity k by Corollary 5.21, p. 199 [6]; thus each $f^{-1}(y)$ has at most k components.

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