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SOME EXAMPLES FOR THE FIXED POINT PROPERTY

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Examples are given of polyhedra K and L which have the homotopy invariant fixed point property, in the sense that all polyhedra of the same homotopy type have the fixed point property (in fact K and L have no self-maps of zero Lefschetz number) but for which $K \times L$ fails to have the fixed point property.

Examples have been constructed (see [2,4]) of polyhedra K and L with the fixed point property such that $K \times L$ does not have the fixed point property. However, these examples are not completely satisfactory in the sense that the fixed point property for K can be lost by a minor alteration of K without changing its homotopy type (such as by adding a 2-simplex along two edges). Indeed, this is crucial for the examples.

It would be of much greater interest to give examples of this phenomenon such that K and L have the *homotopy invariant* fixed point property, and this question was essentially asked by Bing [1] and Fadell [2]. We shall give the first such examples in this note.

For completeness, we note that it is known that, for instance, if K is simply connected and satisfies the Shi condition (that $\dim K \geqq 3$ and no point of K separates K locally) then K has the fixed point property if and only if it has no self-maps of zero Lefschetz number; see [2] for references. All the spaces we shall consider are of this type, but we shall not make use of this fact.

The spaces we shall be concerned with are the (reduced) mapping cones $C_\varphi = S^n \bigcup_\varphi e^{m+1}$ of maps $\varphi: S^m \to S^n$ with $m > n$. We treat them as CW-complexes, but they can be assumed to be triangulable. Reduced suspension is denoted by S. Note that $SC_\varphi = C_{S\varphi}$.

THEOREM A. *Suppose that $\varphi: S^m \to S^n$ $(m > n)$ is a suspension. If m and n have the same parity then C_φ has the fixed point property if and only if $[\varphi] \neq 0$ in $\pi_m(S^n)$. If m and n have opposite parity, then C_φ has a self-map of Lefschetz number zero if and only if $[\varphi] \in \pi_m(S^n)$ has odd order.*

Proof. Let $f: C_\varphi \to C_\varphi$ and let us compute the Lefschetz number $L(f)$. We may change f by a homotopy so that f takes $S^n \subset C_\varphi$ into itself. Let D be the image in C_φ, under the characteristic map, of the $(m+1)$-disk of radius $1/2$ in the unit disk e^{m+1}. Then by a well-known approximation argument (either simplicial or smooth), which we shall not give, f may be again altered by a homotopy so that it satisfies the following condition: There are $(m+1)$-disks $D_1, D_2, \cdots, D_r$ in the interior of the $(m+1)$-cell of C_φ such that f takes each D_i homeomorphically onto D (and thus has degree ± 1 there) and $f(C_\varphi - \bigcup D_i) \subset C_\varphi - D$. We may as well also assume that the D_i are all inside D. There is the canonical deformation retraction

$$\psi\colon C_\varphi - \operatorname{int} D \to S^n$$

and we may, and shall, identify φ with $\psi \mid \partial D$. Now let $x \in H_n(C_\varphi)$ and $y \in H_{m+1}(C_\varphi)$ be generators. Let

$$f_*(x) = jx \quad \text{and} \quad f_*(y) = ky \ .$$

Then $j = \deg f_1$ where $f_1\colon S^n \to S^n$ is the restriction of f. Also k is the sum of the degrees (each ± 1) of f on the D_i to D (or, equivalently, of f on ∂D_i to ∂D). Let $\eta\colon \partial D \to C_\varphi$ be the inclusion and consider the composition

$$\psi \circ f \circ \eta\colon \partial D \to S^n \ .$$

By the homotopy addition theorem, the homotopy class of this in $\pi_m(S^n)$ is

$$[\psi \circ f \circ \eta] = \sum [\varphi \circ (f \mid \partial D_i)] = k[\varphi] \ .$$

Since η is homotopic, through $C_\varphi - \operatorname{int} D$, to $\varphi\colon \partial D \to S^n \subset C_\varphi$, we see that $\psi \circ f \circ \eta$ is homotopic to $\psi \circ f_1 \circ \varphi = f_1 \circ \varphi$. Thus

$$k[\varphi] = [\psi \circ f \circ \eta] = [f_1 \circ \varphi] = j[\varphi] \ ,$$

with the last equality holding since φ is a suspension. Thus we conclude that

$$k \equiv j \pmod{\operatorname{ord}\,[\varphi]} \ .$$

Now

$$\begin{aligned} L(f) &= 1 + (-1)^n j + (-1)^{m+1} k \\ &\equiv 1 + [(-1)^n - (-1)^m] j \qquad \pmod{\operatorname{ord}\,[\varphi]} \ . \end{aligned}$$

Thus if $L(f) = 0$, then $\operatorname{ord}\,[\varphi]$ divides $1 + [(-1)^n - (-1)^m]j$ which is *odd*, and hence $\operatorname{ord}\,[\varphi]$ is odd. If, moreover, n and m have the same parity then $0 = L(f) \equiv 1$ so that $[\varphi] = 0$.

If $[\varphi] = 0$, there is a retraction $r\colon C_\varphi \to S^n$. Following this by the antipodal map on S^n and the inclusion $S^n \subset C_\varphi$ gives a fixed point free map on C_φ.

Suppose now that n and m have opposite parity and that $p = \operatorname{ord}[\varphi]$ is odd. For sake of simplicity of argument, let us suppose that n is even and m is odd. Then define j by $p = 1 + 2j$ and put

$k = -(1+j)$ so that $j - k = p$. Let $j\colon S^n \to S^n$ and $k\colon S^m \to S^m$ also stand for maps of degrees j and k respectively. Since $[\varphi \circ k] = k[\varphi] = j[\varphi] = [j \circ \varphi]$ it follows easily that there is a map g of the mapping cylinder $M_\varphi \to M_\varphi$ which is k on the top face S^m and is j on the bottom face S^n. Let $f\colon C_\varphi \to C_\varphi$ be the union of g with the cone on the map k. Then $L(f) = 1 + j + k = 0$ as desired.

Many cases of nonpreservation of the fixed point property under suspension follow from Theorem A. Perhaps the most interesting ones are the following:

THEOREM B. *Let K be the space obtained from $S^k \times S^k$ by identifying (x_0, x) with (x, x_0) for some fixed x_0 and all x. (Thus $K = C_\varphi$ for a map $\varphi\colon S^{2k-1} \to S^k$ representing the Whitehead product $[e, e]$ where $e \in \pi_k(S^k)$ is the class of the identity.) Then for $k \neq 1, 3, 7$, K has the fixed point property but SK does not.*

Proof. $SK = C_{S\varphi}$ and $[S\varphi] = S[e, e] = 0$, as is well-known. (See [3] or [5; pp. 488–502] and note that $\varphi = \alpha_k$ in the latter reference.) Thus SK admits a map without fixed points as noted in the proof of Theorem A. Moreover, $[e, e] = 0$ only for $k = 1, 3, 7$ since these are the only spheres which are H-spaces. If k is odd, then the suspension $\pi_{2k-2}(S^{k-1}) \to \pi_{2k-1}(S^k)$ is onto by [5; pp. 489–501] so that the result follows from Theorem A. Suppose now that k is even. Then the Hopf invariant of φ is 2 (see [3; p. 336] or [5; pp. 488–502]). Thus if $x \in H^k(K)$ and $y \in H^{2k}(K)$ are suitable generators we have that $x^2 = 2y$. If $f\colon K \to K$ has $f^*(x) = nx$, then

$$2f^*(y) = f^*(2y) = f^*(x^2) = f^*(x)^2 = (nx)^2 = 2n^2y\,.$$

Thus the Lefschetz number

$$L(f) = 1 + n + n^2 \neq 0$$

since n is an integer.

Now we come to the main result of this note. See the remarks following the proof for specific instances for which the hypotheses are satisfied.

THEOREM C. *Let n be odd and let k and l be even. Let $[\varphi] \in \pi_{n+k}(S^n)$ and $[\psi] \in \pi_{n+l}(S^n)$ be nonzero suspensions of orders p and q respectively. Suppose that p and q are relatively prime. Then $K = C_\varphi$ and $L = C_\psi$ both have the (homotopy invariant) fixed point property, but $K \times L$ has fixed point free self-maps.*

Proof. K and L have the fixed point property by Theorem A. At least one, say q, of p and q is odd. Then we can find integers

a and b such that

$$2ap + bq = 1 .$$

Since a map $S^n \to S^n$ of degree ap kills $[\varphi]$, it extends to $K = G_\varphi$. That is, there is a map $f\colon K \to S^n$ which has degree ap on $S^n \to S^n$. Similarly, there is a map $g\colon L \to S^n$ which has degree bq on S^n. Let $x \in H_n(S^n)$ be a generator. Since n is odd there exists a map $\tau\colon S^n \times S^n \to S^n$ of bidegree $(2, 1)$; see [6; p. 14]. That is, τ_* takes $x \times 1$ to $2x$ and takes $1 \times x$ to x. Let $\Delta\colon S^n \to K \times L$ be the diagonal $S^n \to S^n \times S^n$ followed by inculsion, and note that $\Delta_*(x) = x \times 1 + 1 \times x$.

Consider the composition $\Delta \circ \tau \circ (f \times g)\colon K \times L \to K \times L$, whose image is in S^n considered as the diagonal in $S^n \times S^n \subset K \times L$. The restriction of $\tau \circ (f \times g)$ to the diagonal $S^n \to S^n$ is just $\tau \circ (f \times g) \circ \Delta$ which, in homology, takes x to $2apx + bqx = x$. Thus $\tau \circ (f \times g) \circ \Delta$ has degree one and, since n is odd, is homotopic to a fixed point free map. By the homotopy extension theorem, $\tau \circ (f \times g)\colon K \times L \to S^n$ is homotopic to a map $\mu\colon K \times L \to S^n$ whose restriction to the diagonal S^n has no fixed points. Then $\Delta \circ \tau \circ (f \times g)$ is homotopic to $\Delta \circ \mu\colon K \times L \to K \times L$. Now $\Delta \circ \mu$ has no fixed points since it has none on the diagonal S^n and its image is in S^n. This completes the proof. (Alternatively, one could compute directly that $\Delta \circ \tau \circ (f \times g)$ has zero Lefschetz number and use known results which imply that this must be homotopic to a fixed point free map, since $K \times L$ is simply connected and satisfies the Shi condition.)

REMARKS. (1) Although such homotopy classes probably exist in profusion, they are not easy to find. The only *stable* class $[\psi] \in \pi_{n+l}(S^n)$, *l even,* of *odd* order appearing in toda's tables [7; p. 186] is for $l = 10$. However, many more cases can be found in his tables in [8]. Of course, classes of even order abound.

(2) Taking $n \geqq 13$ and *odd,* there are examples with both $[\varphi]$ and $[\psi]$ in the stable group $\pi_{n+10}(S^n) \approx Z_6$; see [7].

(3) The example in the least dimension seems to be $[\varphi] \in \pi_9(S^7)$ of order 2 and $[\psi] \in \pi_{17}(S^7)$ of order 3 (which is a suspension since $S\colon \pi_{16}(S^2) \to \pi_{17}(S^7)$ is onto).

(4) In the case $n = 7$ one could use Cayley multiplication, having bidegree $(1, 1)$, rather than τ.

(5) It is of interest to note that the fixed point free map $\Delta \circ \mu$: $K \times L \to K \times L$ can be so chosen that *none* of its iterates has fixed points.

(6) We believe that our examples show that failure for the fixed point property to be preserved by suspensions and products should be regarded as a normal phenomenon.

(7) I have been told that W. Holsztynski also noticed that

mapping cones give examples of the nonpreservation of the fixed point property under suspension.

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