PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 3, 1971

UNIQUELY REPRESENTABLE SEMIGROUPS ON THE TWO-CELL

J. T. BORREGO, H. COHEN, AND E. E. DEVUN

A semigroup S is said to be uniquely representable in terms of two subsets X and Y of S if $X \cdot Y = Y \cdot X = S$, $x_1y_1 = x_2y_2$ is a nonzero element of S implies $x_1 = x_2$ and $y_1 = y_2$, and $y_1x_1 = y_2x_2$ is a nonzero element of S implies $y_1 = y_2$ and $x_1 = x_2$ for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. A semigroup S is said to be uniquely divisible if for each $s \in S$ and every positive integer n there exists a unique $z \in S$ such that $z^n = s$. Theorem. If S is a uniquely divisible semigroup on the two-cell with the set of idempotents of S being a zero for S and an identity for S, then S is uniquely representable in terms of X and Y where X and Y are iseomorphic copies of the usual unit interval and the boundary of S equals X union Y. Corollary. If S is a uniquely divisible semigroup on the two-cell and if S has only two idempotents, a zero and an identity, then the nonzero elements of S form a cancellative semigroup.

A semigroup S is said to be uniquely representable in terms of two subsets X and Y of S if $X \cdot Y = Y \cdot X = S$, $x_1y_1 = x_2y_2$ is a nonzero element of S implies $x_1 = x_2$ and $y_1 = y_2$, and $y_1x_1 = y_2x_2$ is a nonzero element of S implies $y_1 = y_2$ and $x_1 = x_2$ for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. A semigroup S is said to be uniquely divisible if for every $s \in S$ and every positive integer n there exists a unique $z \in S$ such that $z^n = s$.

The primary purpose of this paper is to show that if S is a uniquely divisible semigroup on two-cell with the set of idempotents of S being a zero for S and an identity for S, then S is uniquely representable in terms of X and Y where X and Y are iseomorphic copies of the usual unit interval and the boundary of S equals X union Y. As a corollary to this theorem we shall prove a conjecture of D. R. Brown, that if S is a uniquely divisible semigroup on the two-cell and if S has only two idempotents, a zero and an identity, then the nonzero elements of S form a cancellative subsemigroup of S.

NOTATION. Throughout S will be a uniquely divisible semigroup on the two-cell with $E(S)$ (the set of idempotents of S) $= \{0, 1\}$ where 0 is the zero for S and 1 is the identity for S. It is well known that the boundary of S is the union of two usual threads X and Y with $X \cap Y = \{0, 1\}$ and $S = X \cdot Y = Y \cdot X$. Intervals containing x will represent segments of X and intervals with y shall stand for segments of Y. For a positive integer n, $s^{1/n}$ will denote the unique nth root of s in S.

The authors would like to thank the referee for pointing out the following result due to J. D. Lawson and M. Friedberg and which appears in [2].

LEMMA 1. *If T is a uniquely divisible semigroup with $E(T) = \{0, 1\}$, then T has no zero divisors.*

Proof. Suppose $ab = 0$ for some $a, b \in T$, $a \neq 0$. Then $(ba)^2 = b(ab)a = 0$, hence $ba = 0$. Thus $0 = ab = a^{1/2}(a^{1/2}b) = (a^{1/2}b)a^{1/2} = (a^{1/2}b)(a^{1/2}b)$, so $a^{1/2}b = 0$. It follows that $a^{1/2^n}b = 0$ for all n. Since $\{a^{1/2^n}\} \to 1$, $b = 0$.

Define $f: X \times Y \to S$ onto S by $f(x, y) = xy$. The proofs of the following three lemmas are analogous to the proofs in [3].

LEMMA 2. *If $f(x_1, y_1) = f(x_2, y_2) \neq 0$, then either*
(1) $x_1 = x_2$ *and* $y_1 = y_2$ *or*
(2) $x_1 > x_2$ *and* $y_2 > y_1$ *or*
(3) $x_2 > x_1$ *and* $y_1 > y_2$.

LEMMA 3. *If $s \in S \backslash \{0\}$, then there exist (x_1, y_1), $(x_2, y_2) \in f^{-1}(s)$ such that for all $(x, y) \in f^{-1}(s)$ we have $x_1 \geqq x \geqq x_2$ and $y_2 \geqq y \geqq y_1$.*

LEMMA 4. *If $s \in S \backslash \{0\}$, then $\pi_1(f^{-1}(s))$ is connected.*

LEMMA 5. *If $s \in S \backslash \{0\}$, then $f^{-1}(s)$ is an arc.*

Proof. Let $[x_1, x_2] = \pi_1(f^{-1}(s))$, and define $h: [x_1, x_2] \to f^{-1}(s)$ by $h(x) = (x, y)$ where y is the unique $y \in Y$ (lemma 2) such that $f(x, y) = s$. Now $h: [x_1, x_2] \to f^{-1}(s)$ is a continuous, one-to-one, onto function. Thus $h: [x_1, x_2] \to f^{-1}(s)$ is a homeomorphism, and $f^{-1}(s)$ is an arc.

DEFINITION 6. Let $J = \{(x, y) : (x, y) \in X \times Y$ and $f^{-1}(f(x, y))$ is not a point$\}$.

LEMMA 7. *If $s \in f(J)$, then $Xs = sY$.*

The proof of the above lemma is analogous to the proof of Lemma 10 of [3].

LEMMA 8. *If $\{(x, y): 0 \leqq x < x_0, 0 \leqq y < y_0\} \subset J$, then $\{(x, y): 0 \leqq x \leqq x_0, 0 \leqq y \leqq y_0\} \backslash \{(x_0, y_0)\} \subset J$. Moreover, for each $(x', y') \in \{(x, y): 0 \leqq x \leqq x_0, 0 \leqq y \leqq y_0\} \backslash \{x_0, y_0)\}$ there exists $\bar{x} \in X$ such that $f(\bar{x}, y_0) = f(x', y')$.*

Proof. Let $x_1 \in [0, x_0)$ and fix $x_2 \in (x_1, x_0)$. Then for each $y \in [0, y_0)$

we have $(x_2, y) \in J$. Select an increasing sequence $\{z_n\}$, with $z_n \in [0, y_0)$ and $z_n \to y_0$. Now there exist $x_3 \in X$ and a sequence $\{w_n\}$, with $w_n \in Y$, such that $x_3x_2 = x_1$, and $x_3f(x_2, z_n) = f(x_2, z_n)w_n$. Now $\{z_nw_n\}$ is an increasing sequence, and hence it must converge. Let $z_nw_n \to y_1$. Then $f(x_1, y_0) = f(x_2, y_1)$, and $0 \leqq y_1 < y_0$. Hence $(x_1, y_0) \in J$. A similar argument shows $(x_0, y_1) \in J$ for $y^1 \in [0, y_0)$.

Next let $(x_1, y_1) \in \{(x, y) : 0 \leqq x \leqq x_0, 0 \leqq x \leqq y_0\} \backslash \{(x_0, y_0)\}$. Select $(x_2, y_2) \in \{(x, y): 0 \leqq x \leqq x_0, 0 \leqq y < y_0\}$ such that $f(x_2, y_2) = f(x_1, y_1)$. Now $(x_2, y_0) \in J$. Fix $y_3 \in J$ such that $y_0y_3 = y_2$ By Lemma 7 there exists $x_3 \in X$ such that $x_3f(x_2, y_0) = f(x_2, y_0)y_3$. Letting $x_4 = x_3x_2$ we have $f(x_4, y_0) = f(x_2, y_2) = f(x_1, y_1)$.

COROLLARY 9. *If* $(x, 1), (1, y) \in J$, *then* $x = 0$ *or* $y = 0$.

Proof. Since $(x, 1), (1, y) \in J$ there exist $x_1 \in X, y_1 \in Y$ such that $x_1f(x, 1) = f(x, 1)y$ and $xf(1, y) = f(1, y)y_1$. Thus $x_1x = yy_1$. This is impossible unless $x = 0$ or $y = 0$.

LEMMA 10. *Let* $x \in X\backslash\{1\}, y \in Y$. *Then* yx *can be written as* $x'y'$ *with* $x' \in X\backslash\{1\}, y' \in Y$.

Proof. If $y = 0$ the result is clear. Thus we will assume $y \in Y\backslash\{0\}$. We will divide the proof into several steps.

Step (1). Since $S = Y \cdot X = X \cdot Y$ we know that there exist $x_1 \in X\backslash\{1\}$, $y_1 \in Y$ such that $y_1x_1 \notin X \cup Y$, and thus there exist $x_2 \in X\backslash\{1\}$, $y_2 \in Y$ such that $y_1x_1 = x_2y_2$.

Step (2). Let $y_3 \in Y$ with $y_3 \geqq y_1$. Then there exists $y_4 \in Y$ such that $y_4y_3 = y_1$. Thus $y_4y_3x_1 = y_1x_1 \notin X \cup Y$. Hence $y_3x_1 \notin Y$.

Step (3). We claim that for $y_3 \in [y_1, 1]$ and n a positive integer, $y_3x_1^{1/n} \notin Y$. For if this were not the case there would exist a positive integer n and a $y_3 \in [y_1, 1]$ such that $y_3x_1^{1/n} = y_6 \in Y$. But by Lemma 2, $y_6 < y_3$. Thus there exists $y_7 \in Y\backslash\{1\}$ such that $y_7y_3 = y_6$. Hence $y_3(x_1^{1/n})^n = y_3x_1^{1/n}(x_1^{1/n})^{n-1} = y_6(x_1^{1/n})^{n-1} = y_7y_3(x_1^{1/n})^{n-1} = \cdots = y_7^ny_3 \in Y$. Thus $y_3x_1 \in Y$. This is a contradiction.

Step (4). Let $x \in X\backslash\{1\}$. Then for $y_3 \in [y_1, 1]$ we claim y_3x can be represented as x_8y_8 with $x_8 \in X\backslash\{1\}$, and $y_8 \in Y$. Choose n a positive integer such that $x_1^{1/n} \in [x, 1])$. Then there exists $x_9 \in X$ such that $x_1^{1/n}x_9 = x$. Thus $y_3x = y_3x_1^{1/n}x_9$. However, $y_3x_1^{1/n} \notin Y$, and hence y_3x can be written as x_8y_8 with $x_8 \in X\backslash\{1\}$, and $y_8 \in Y$.

Step (5). Finally, let $x \in X\backslash\{1\}$ and $y \in Y$. If $y = 1$, then $yx = xy$ and $x \in X\backslash\{1\}$ and $y \in Y$. If $y \in Y\backslash\{0, 1\}$, then there exist a positive integer m and $y_3 \in [y_1, 1)$ such that $y = (y_3)^m$. Now $yx = (y_3{}^mx = x'y'$ with $x' \in X\backslash\{1\}$, and $y' \in Y$.

The same argument can be used to show that if $x \in X$ and $y \in Y \backslash \{1\}$, then xy can be written as $y'x'$ with $x' \in X$ and $y' \in Y \backslash \{1\}$.

THEOREM 11. *If $s \in S \backslash \{0\}$, then there exist unique $x \in X$, $y \in Y$ such that $xy = s$.*

Proof. Suppose this is not the case. Then there exist $x_1 \in X \backslash \{0, 1\}$, $y_1 \in Y \backslash \{0, 1\}$ such that $(x_1, y_1) \in J$. From corollary 9 we can assume $\{(1, y): y \in Y \backslash \{0\}\} \cap J = \phi$. Let $x_2 = \sup \{x: (x, y_1) \in J\}$. Now $x_2 \in (0, 1)$ and $\{(x, y): 0 \leqq x \leqq x_2, 0 \leqq y \leqq y_1\} \backslash \{(x_2, y_1)\} \subset J$.

Next take $x_3 \in (x_2, 1)$. Then there exist $x_4 \in X \backslash \{0, 1\}$, $y_4 \in Y$ such that $y_1 x_3 = x_4 y_4$. If $x_4 \in (0, x_2]$, fix $x_5 \in (x_2, x_3)$. If $x_4 \in (x_2, 1)$, fix $x_5 \in (x_2, \min \{x_3, x_2/x_4\})$ where x_2/x_4 represents the unique element p of X such that $px_4 = x_2$. Take $y_2 \in (y_1, 1)$. Then there exist $x_6 \in X$, $y_6 \in Y \backslash \{0, 1\}$ such that $y_2 x_2 = x_6 y_6$. If $y_6 \in (0, y_1]$ fix $y_7 \in (y_1, y_2)$. If $y_6 \in (y_1, 1)$, fix $y_7 \in (y_1, \min \{y_2, y_1/y_6\})$.

For each $x \in [x_2, x_5]$ we have $(xy_1)^2 = x'y'$ with $x' \in (0, x_2]$ and $y' \in (0, y_1]$. By lemma 8 there exists a unique $\bar{x} \in (0, x_2]$ such that $(xy_1)^2 = x'y' = \bar{x} y_1$. Hence we can define a function $x \rightarrow \bar{x}$ from $[x_2, x_5]$ into $(0, x_2]$. The function $x \rightarrow \bar{x}$ defined above is continuous and monotone and thus maps $[x_2, x_5]$ onto an interval $[\bar{x}_2, \bar{x}_5]$.

Also for $y \in [y_1, y_7]$ we have $(x_2 y)^2 = \tilde{x}\tilde{y}$ with $\tilde{x} \in (0, x_2]$ and $\tilde{y} \in (0, y_1]$. Again by lemma 8 there exists a unique $x(y) \in (0, x_2]$ such that $(x_2 y)^2 = \tilde{x}\tilde{y} = x(y) y_1$. Thus we can define a function $y \rightarrow x(y)$ from $[y_1, y_7]$ into $(0, x_2]$ which is continuous and monotone and hence maps $[y_1, y_7]$ onto an interval $[x(y_1), x(y_7)]$.

Now $(x_2 y_1)^2 = \bar{x}_2 y_1$ and $(x_2 y_1)^2 = x(y_1) y_1$. Hence $\bar{x}_2 = x(y_1)$, so the intervals $(\bar{x}_2, \bar{x}_5]$ and $(x(y_1), x(y_6)]$ intersect. Thus there exist $x \in (x_2, x_5]$ and $y \in (y_1, y_7]$ such that $(xy_1)^2 = (x_2 y)^2$. However, $(x, y_1) \notin J$, thus $xy_1 \neq x_2 y$. This is a contradiction.

In the same manner we can show that each element $s \in S \backslash \{0\}$ can be written uniquely as yx with $y \in Y$ and $x \in X$.

LEMMA 12. *Let T be a semigroup without zero divisors, $E(T) = \{0, 1\}$, and which is uniquely representable in terms of two usual threads X and Y. Then $T \backslash \{0\}$ is cancellative.*

Proof. Let $s, s_1, s_2 \in T \backslash \{0\}$ with $s = xy$, $s_1 = x_1 y_1$, $s_2 = x_2 y_2$ with $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$, and suppose $ss_1 = ss_2$. Then $xyx_1y_1 = xyx_2y_2$. Now let $yx_1 = \bar{x}_1 \bar{y}_1$ and $yx_2 = \bar{x}_2 \bar{y}_2$. Thus $x\bar{x}_1\bar{y}_1 y_1 = x\bar{x}_2\bar{y}_2 y_2$. Since T is uniquely representable we get that $\bar{x}_1 = \bar{x}_2$ and thus $x_1 = x_2$. This implies $\bar{y}_1 = \bar{y}_2$ and hence $y_1 = y_2$. Hence $s_1 = s_2$. In the same manner we can show that if $s, s_1, s_2 \in T \backslash \{0\}$ with $s_1 s = s_2 s$, then $s_1 = s_2$. Thus

$T\backslash\{0\}$ is cancellative.

COROLLARY 13. *If S is a uniquely divisible semigroup on the two-cell with $E(S) = \{0, 1\}$, then $S\backslash\{0\}$ is a cancellative semigroup.*

REFERENCES

1. D. R. Brown, Topological semilattices on the two-cell, Pacific J. Math., **15** (1965), 35-46.

2. D. R. Brown, and M. Friedberg, Linear Representations of certain compact semigroups, to appear.

3. E. E. DeVun, Special semigroups on the two-cell, Pacific J. Math., **34** (1970), 639-645.

4. Hildebrant, J. S., On uniquely divisible semigroups on the two-cell, Pacific J. Math., **23** (1967), 91-95.

5. Anne Lester Hudson, Some semigroups on the two-cell, Proc. Amer. Math. Soc., **10** (1959), 648-655.

Received December 10, 1969.

WICHITA STATE UNIVERSITY