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GLOBALIZATION THEOREMS FOR LOCALLY FINITELY GENERATED MODULES

ROGER WIEGAND

Each commutative ring has a coreflection $\hat{R}$ in the category of commutative regular rings. We use the basic properties of $\hat{R}$ to obtain globalization theorems for finite generation and for projectivity of R-modules.

1. Preliminaries. A detailed description of the ring $\hat{R}$ may be found in [8]. Here we list without proofs the facts that will be needed. We assume that everything is unitary, but not necessarily commutative. However, R will always denote an arbitrary *commutative* ring. All unspecified tensor products are taken over R. For each $a \in R$ and each $P \in \text{Spec}(R)$, let $a(P)$ be the image of a under the obvious map $R \to R_P/PR_P$. Then $\hat{R}$ is the subring $\prod_P R_P/PR_P$ consisting of finite sums of elements $[a, b]$, where $[a, b]$ is the element whose P^{th} coordinate is 0 if $b \in P$ and $a(P)/b(P)$ if $b \notin P$. There is a natural homomorphism $\varphi: R \to \hat{R}$ taking a to $[a, 1]$. The ring $\hat{R}$ is regular (in the sense of von Neumann). The statement that $\hat{R}$ is a coreflection means simply that each homomorphism from R into a commutative regular ring factors uniquely through φ.

The map $\text{Spec}(\varphi)$: $\text{Spec}(\hat{R}) \to \text{Spec}(R)$ is one-to-one and onto; for each $P \in \text{Spec}(R)$ we let $\hat{P}$ be the corresponding prime (= maximal) ideal of $\hat{R}$.

If A is an R-module and $P \in \text{Spec}(R)$, then A_P/PA_P and $(A \otimes \hat{R})_{\hat{P}}$ are vector spaces over R_P/PR_P and $\hat{R}_{\hat{P}}$ respectively. The map $\varphi: R \to \hat{R}$ induces an isomorphism $R_P/PR_P \cong \hat{R}_{\hat{P}}$, and, under the identification, A_P/PA_P and $(A \otimes \hat{R})_{\hat{P}}$ are isomorphic vector spaces.

2. Globalization theorems.

LEMMA. *If $A \otimes \hat{R} = 0$ and A_R is locally finitely generated then $A = 0$.*

Proof. For each prime P, $A_P/PA_P = 0$, by the last paragraph of § 1. Since A_P is finitely generated over R_P, Nakayama's lemma implies that $A_P = 0$ for each $P \in \text{Spec}(R)$. Therefore $A = 0$.

THEOREM 1. *Assume $(A \otimes \hat{R})$ is finitely generated over $\hat{R}$, and that A_R is either locally free or locally finitely generated. Then A_R is finitely generated.*

Proof. Assume A_R is locally free. Then, for each prime P, A_P is a direct sum of, say, κ copies of R_P. Then A_P/PA_P is a direct sum of κ copies of R_P/PR_P. But since $(A \otimes \hat{R})$ is finitely generated over $\hat{R}$, A_P/PA_P is finite dimensional over R_P/PR_P. Thus κ is finite, and we conclude that A_R is locally finitely generated.

Now, if A_R is not finitely generated, we can express A as a well-ordered union of submodules A_α, each of which requires fewer generators than A. We will get a contradiction by showing that some $A_\alpha = A$. Let $B_\alpha = \operatorname{Im}(A_\alpha \otimes \hat{R} \to A \otimes \hat{R})$. Since

$$A \otimes \hat{R} = \varinjlim_{\alpha}(A_\alpha \otimes \hat{R}) , \qquad A \otimes \hat{R} = \bigcup_\alpha B_\alpha .$$

Since the B_α are nested and $(A \otimes \hat{R})$ is finitely generated over $\hat{R}$, some $B_{\alpha_0} = A \otimes \hat{R}$, that is, $A_{\alpha_0} \otimes \hat{R} \twoheadrightarrow A \otimes \hat{R}$. Let $C = A/A_{\alpha_0}$. Then $C \otimes \hat{R} = \operatorname{Coker}(A_{\alpha_0} \otimes \hat{R} \to A \otimes \hat{R}) = 0$, and C_R is certainly locally finitely generated. By the lemma, $C = 0$, and $A_{\alpha_0} = A$.

THEOREM 2. *Let A_R be finitely generated and flat, and assume $(A \otimes \hat{R})$ is $\hat{R}$-projective. Then A_R is projective.*

Proof. By Chase's theorem [3, Theorem 4.1] it is sufficient to show that A_R is finitely related. Let $0 \to K \to F \to A \to 0$ be an exact sequence, with F_R free of finite rank. This sequence splits locally, so K is locally finitely generated. Since A_R is flat, the long exact sequence of Tor shows that $0 \to K \otimes \hat{R} \to F \otimes \hat{R} \to A \otimes \hat{R} \to 0$ is exact. This sequence splits, so $(K \otimes \hat{R})$ is finitely generated over $\hat{R}$. By Theorem 1, K_R is finitely generated.

3. Applications. The following result generalizes the well-known fact that over a noetherian ring every finitely generated flat module is projective.

PROPOSITION 1. *If R has* a.c.c. *on intersections of prime ideals then every finitely generated flat R-module is projective.*

Proof. In [8] these rings are characterized as those for which $(A \otimes \hat{R})$ is $\hat{R}$-projective for *every* finitely generated A_R. The conclusion follows from Theorem 2.

Suppose A_R is locally finitely generated. For each prime ideal P let $r_A(P)$ denote the number of generators required for A_P over R_P. By Nakayama's lemma, $r_A(P) = d_A(\hat{P})$, the dimension of $(A \otimes \hat{R})_{\hat{P}}$ as a vector space over $\hat{R}_{\hat{P}}$. Since the map $\hat{P} \to P$ is continuous, it follows that if r_A is continuous on $\operatorname{Spec}(R)$ then d_A is continuous on $\operatorname{Spec}(\hat{R})$. Using these observations we can give easy proofs of the following two theorems:

THEOREM 3 (Bourbaki [1, Th. 1]): *Assume A_R is finitely generated and flat, and that r_A is continuous. Then A_R is projective.*

THEOREM 4 (Vasconcelos [7, Prop. 1.4]): *Assume A_R is projective and locally finitely generated, and that r_A is continuous. Then A_R is finitely generated.*

Proof of Theorem 3. By Theorem 3 we may assume R is regular. A proof of Theorem 3 in this case may be found in [5], but we include one here for completeness. For each $k \geqq 0$ let

$$U_k = \{P \in \operatorname{Spec}(R) \mid r_A(P) = k\} .$$

By hypothesis the sets U_k are clopen, and we let e_k be the idempotent with support U_k. Then $A = A\,e_0 \oplus \cdots \oplus A\,e_n$, and $r_{A e_k}$ is constant on $\operatorname{Spec}(Re_k)$. Therefore we may assume r_A is constant on $\operatorname{Spec}(R)$, say $r_A(P) = n$ for all P. Given a prime P, choose $a_1, \cdots, a_n \in R$ such that $a_1(P), \cdots, a_n(P)$ span A_P. Then $a_1(Q), \cdots, a_n(Q)$ span R_Q for all Q in some neighborhood of P. (Here we need A_R finitely generated.) In this way we get a partition of $\operatorname{Spec}(R)$ into disjoint clopen sets $V_1, \cdots, V_m$ together with elements $a_{ij} \in R$ such that $a_{ij}(P), \cdots, a_{nj}(P)$ span A_P for each $P \in V_j$. Let e_j be the idempotent with support V_j, and set $b_i = \Sigma_j e_j a_{ij}$. Then, if P_R is free on $u_1, \cdots, u_n$, the map $P \to A$ taking u_i to b_i is an isomorphism locally, and therefore globally.

Proof of Theorem 4. By Theorem 1 and the proof of Theorem 3 we can assume R is regular and $r_A(P) = n$ for all P. Write $A = \bigoplus \sum_{i \in I} Re_i$, $e_i^2 = e_i \neq 0$, by [4]. Given $P \in \operatorname{Spec}(R)$, since $(Re_i)_P$ is 0 if $e_i \in P$ and R_P if $e_i \notin P$, we see that there are precisely n indices i for which $e_i \notin P$. For each n-element subset $J \subseteqq I$ let

$$U(J) = \{P \in \operatorname{Spec}(R) \mid e_j \notin P \text{ for each } j \in J\} .$$

These open sets cover $\operatorname{Spec}(R)$, so $\operatorname{Spec}(R) = U(J_1) \cup \cdots \cup U(J_m)$. If $j \notin J_1 \cup \cdots \cup J_m$ then e_j is in every prime ideal, contradicting $e_j \neq 0$. Therefore $|I| \leqq mn$, and A_R is finitely generated.

As a final application we give the following:

PROPOSITION 2. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of flat R-modules Assume A_R is finitely generated and $(B \otimes \hat{R})_{\hat{R}}$ is projective. Then A_R is projective.*

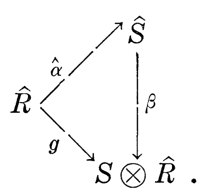
Proof. Since C_R is flat, $0 \to A \otimes \hat{R} \to B \otimes \hat{R} \to C \otimes \hat{R} \to 0$ is exact. Since $\hat{R}$ is semihereditary $(A \otimes \hat{R})$ is R-projective. By Theorem 2, A_R is projective.

If B_R is projective this proposition contains no new information. (In fact, a trivial extension of Chase's Theorem shows that the sequence splits.) On the other hand, if we let M_R be projective, take $f \in R$, and let $B = M_f = \{[m/f^n]\}$, then B_R is not in general projective; but by the second corollary to Theorem 5 (next section), $B \otimes \hat{R}$ is $\hat{R}$-projective.

4. Epimorphisms. Suppose M is a multiplicative subset of R, and let $S = M^{-1}R$. Since $S \otimes \hat{R}_{\hat{P}} = S_P/PS_P$ for each prime P, we see that $S \otimes \hat{R}_{\hat{P}}$ is $\hat{R}_{\hat{P}}$ if $P \cap M = \varnothing$, and 0 if $P \cap M \neq \varnothing$. If we could show that $(S \otimes \hat{R})_{\hat{R}}$ is finitely generated, it would follow easily that $S \otimes \hat{R} = \hat{R}/K$, where K is the intersection of those primes $\hat{P}$ for which $P \cap M = \varnothing$. We give an indirect proof of this fact in a more general setting.

Suppose R and S are commutative rings and that $\alpha\colon R \to S$ is an epimorphism in the category of rings. By a theorem of Silver [6] this is equivalent to the natural map $S \otimes S \to S$ being an isomorphism. It is known [8] that $R \to \hat{R}$ is an epimorphism, and it follows readily that the natural maps $f\colon S \to S \otimes \hat{R}$ and $g\colon R \to S \otimes \hat{R}$ are epimorphisms.

THEOREM 5. *Let R and S be commutative rings and let $\alpha\colon R \to S$ be an epimorphism in the category of rings. Then there is a unique ring homomorphism $\beta\colon \hat{S} \to S \otimes \hat{R}$ making the following diagram commute*:



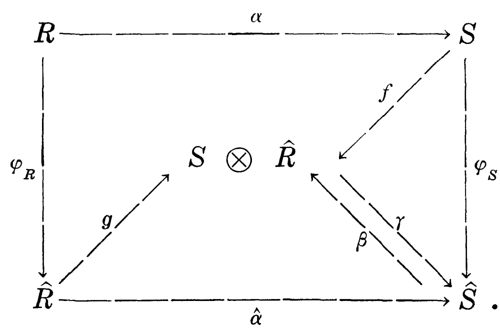
Moreover, β is an isomorphism, and $\hat{\alpha}$ and g are surjections with kernel $K = \cap \{\hat{P} \mid S_P \neq PS_P\}$.

Proof. We first show that $S \otimes \hat{R}$ is regular. Suppose A and B are $(S \otimes \hat{R})$-modules. Then by Silver's Theorem $B = S \otimes_R B$, and by [2, p. 165] we have

$$A \otimes_{S \otimes \hat{R}} B = A \otimes_{S \otimes \hat{R}} (S \otimes_R B) = (A \otimes_S S) \otimes_{\hat{R} \otimes R} B = A \otimes_{\hat{R}} B .$$

It follows that tensor products over $S \otimes \hat{R}$ are exact, and therefore

$S \otimes \hat{R}$ is regular. Hence there is a unique map $\beta: \hat{S} \to S \otimes \hat{R}$ such that $\beta\varphi_S = f$, where $\varphi_S: S \to \hat{S}$ is the natural map. Consider the diagram:



Here γ is defined by the equations $\gamma f = \varphi_S$, $\gamma g = \hat{\alpha}$. Now $\gamma\beta\varphi_S = \gamma f = \varphi_S$ and $\beta\gamma f = \beta\varphi_S = f$. Since φ_S and f are both epimorphisms, we see that $\gamma = \beta^{-1}$. Also, $B\hat{\alpha} = B\gamma g = g$, as required. Uniqueness of β follows from the fact that $\hat{\alpha}$ is an epimorphism (since both α and φ_S are).

Next, we show $\hat{\alpha}$ is onto. To simplify notation, we assume R is regular and $\alpha: R \to S$ is an epimorphism. Then $S \otimes S \xrightarrow{\mu} S$ is an isomorphism. But then $S_P \otimes_{R_P} S_P \to S_P$ is an isomorphism for each $P \in \operatorname{Spec}(R)$. If $s \in S_P$ then $1 \otimes s - s \otimes 1 \in \ker\ \mu_P = 0$. It follows that the dimension of S_P as a vector space over R_P is either 0 or 1. Therefore α_P is surjective for each P, $(\alpha(1) = 1)$, and we conclude that α is surjective.

Finally, we compute $\ker g = K$. If $P \in \operatorname{Spec}(\hat{R})$, then

$$K \subseteq \hat{P} \iff K_{\hat{P}} = 0 \iff \hat{S}_{\hat{P}} \neq 0 \iff S \otimes \hat{R}_{\hat{P}} \neq 0 \iff S_P/PS_P \neq 0 .$$

COROLLARY 1. *Let M be a multiplicative subset of R and let $S = M^{-1}R$. Then $S \otimes \hat{R}$ is a cyclic $\hat{R}$-module, and $S \otimes \hat{R}$ is $\hat{R}$-projective if and only if $\{\hat{P} \mid M \cap P \neq \varnothing\}$ is closed in* $\operatorname{Spec}(\hat{R})$.

Proof. Let K be as in Theorem 5. Then $S \otimes \hat{R} = \hat{R}/K$ is $\hat{R}$-projective if and only if K is a principal ideal, that is, if and only if the set of primes containing K is open in $\operatorname{Spec}(\hat{R})$. But

$$\hat{P} \supseteq K \iff PS_P \neq S_P \iff M \cap P = \varnothing .$$

The next corollary shows that Theorem 2 is false if A_R is not assumed to be finitely generated.

COROLLARY 2. *For each $f \in R$, $R_f \otimes \hat{R}$ is $\hat{R}$-projective.*

Proof. Set $M = \{f^n : n \geqq 0\}$. Then $P \cap M \neq \varnothing$ if and only if $\varphi(f) \in \hat{P}$. Thus K is the principal ideal of $\hat{R}$ generated by $\varphi(f)$, and $\hat{R}/K$ is $\hat{R}$-projective.

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University of Wisconsin