PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 1, 1971

A SEMILATTICE DECOMPOSITION INTO SEMIGROUPS HAVING AT MOST ONE IDEMPOTENT

MOHAN S. PUTCHA AND JULIAN WEISSGLASS

A semigroup S is said to be *viable* if $ab = ba$ whenever ab and ba are idempotents. The main theorem of this article proves in part that S is a viable semigroup if and only if S is a semi-lattice of $\mathscr{S}$-indecomposable semigroups having at most one idempotent.

Furthermore, each semigroup appearing in the decomposition has a group ideal whenever it has an idempotent. Also included as part of the main theorem is the more elementary result that S is viable if and only if every $\mathscr{J}$-class contains at most one idempotent.

Throughout S will denote a semigroup and $E = E(S)$ the set of idemotents of S.

DEFINITION. Let $a, b \in S$. We say $a \mid b$ if there exist $x, y \in S$ such that $ax = ya = b$. The set-valued function $\mathfrak{M}$ on S is defined by $\mathfrak{M}(a) = \{e \mid e \in E, a \mid e\}$. The relation δ on S is defined by $a\ \delta\ b$ if $\mathfrak{M}(a) = \mathfrak{M}(b)$.

Our first goal is to show that if S is viable then δ is a congruence on S and S/δ is the semilattice described above.

LEMMA 1. *Let S be viable. If $ab = e \in E$, then $bea = e$.*

Proof. $(bea)^2 = beabea = bea$. Hence $bea \in E$. But cleary $abe = e \in E$. Hence $bea = abe = e$.

LEMMA 2. *Let S be viable. Suppose $a \in S$ and $e \in E$. Then $a \mid e$ if and only if $e \in S^1aS^1$.*

Proof. If $a \mid e$, then $e \in S^1aS^1$ by definition. Conversely assume $e = sat$ with $s, t \in S^1$. By (1), $ates = e$ and $tesa = e$. Therefore $a \mid e$.

THEOREM 3. *Let S be viable. Then*

(i) *δ is a congruence relation on S containing Green's relation $\mathscr{H}$.*

(ii) *S/δ* is a semilattice and

(iii) *each δ-class contains at most one idempotent and a group ideal whenever it contains an idempotent.*

Proof. (i) Clearly δ is an equivalence relation. We will show that δ is right compatible. Assume $a\ \delta\ b$. If $ac \mid e \in E$, then

$acx = e$ for some $x \in S$. By (1), $cxea = e$. Hence $a|e$. Thus $b|e$, so $yb = e$ for some $y \in S$. Therefore $ybcxea = e$, so $bc|e$ by (2). Hence $\mathfrak{M}(ac) \subseteqq \mathfrak{M}(bc)$. Similary $\mathfrak{M}(bc) \subseteqq \mathfrak{M}(ac)$ and hence $ac\ \delta\ bc$. That δ is left compatible follows analogously. Consequently, δ is a congruence. It is immediate that $\mathscr{H} \subseteqq \delta$.

(ii) To show S/δ is a band, let $a \in S$. If $a^2|e \in E$ then by (2), $a|e$. Hence $\mathfrak{M}(a^2) \subseteqq \mathfrak{M}(a)$. Suppose $a|e \in E$, say $ax = ya = e$, $x, y \in S$. Then $ya^2x = e$. Again using (2), $a^2|e$. Thus, $\mathfrak{M}(a^2) = \mathfrak{M}(a)$ and $a\ \delta\ a^2$. So S/δ is a band. Now let $a, b \in S$. If $e \in \mathfrak{M}(ab)$, then there exist $x, y \in S$ such that $abx = yab = e$. Hence $ya(ba)bx = e$, and by (2), $e \in \mathfrak{M}(ba)$. Therefore $\mathfrak{M}(ab) \subseteqq \mathfrak{M}(ba)$. By symmetry, $\mathfrak{M}(ba) \subseteqq \mathfrak{M}(ab)$. Hence $ab\ \delta\ ba$ and S/δ is a semilattice.

(iii) Suppose, $e_1\ \delta\ e_2$ with $e_1, e_2 \in E$. Then $e_1 \in \mathfrak{M}(e_1) = \mathfrak{M}(e_2)$, so $e_2|e_1$. Similarly $e_1|e_2$. Hence $e_1\ \mathscr{H}\ e_2$ and by [2], Lemma 2.15, $e_1 = e_2$. Thus each δ-class contains at most one idempotent. Now suppose A is a δ-class containing an idempotent e. Let $a \in A$. Since $e \in \mathfrak{M}(e) = \mathfrak{M}(a) = \mathfrak{M}(a^2)$, there exists $x \in S$ such that $a^2x = e$. Now $a\ \delta\ a^2$ implies $ax\ \delta\ a^2x$, so $ax\ \delta\ e\ \delta\ a$. Hence $ax \in A$ and $a(ax) = e$ implies e is a right zeroid of A. Similarly e is a left zeroid and by [2], §2.5, Exercise 6, A has a group ideal.

A semigroup is said to be *$\mathscr{S}$-indecomposable* if it has no proper semilattice decomposition.

COROLLARY 4. *If the viable semigroup S is $\mathscr{S}$-indecomposable then $S/\delta = 1$ and is either idempotent-free or has a group ideal and exactly one idempotent.*

LEMMA 5. *Assume I is an idempotent-free ideal of S. Then S is viable if and only if the Rees factor semigroup S/I is viable.*

Proof. Assume S is viable and that $ab, ba \in E(S/I)$. If $ab \in I$, then $ba = b(ab)a \in I$, so $ab = ba$ in S/I. So we may assume ab and ba are not in I. But then $ab, ba \in E(S)$. Hence $ab = ba$ in S and so in S/I. Therefore S/I is viable. Conversely, let $ab, ba \in E(S)$. Since S/I is viable $ab = ba$ in S/I. But $ab, ba \notin$ I since I is idempotent-free. Hence $ab = ba$ in S and S is viable.

A semigroup S is said to be E-inversive if for every $a \in S$ there exists $x \in S$ such that $ax \in E$.

THEOREM 6. *The following are equivalent.*

(i) *Every $\mathscr{J}$-class of S contains at most one idempotent*

(ii) *S is viable.*

(iii) *S is a smilattice of $\mathscr{S}$-indecomposable semigroups each of* *which contains at most one idempotent and a group ideal whenever it contains an idempotent.*

(iv) *S is a semilattice of semigroups having at most one idempotent.*

(v) *S is viable and E-inversive or an ideal extension of an idempotent-free semigroup by a viable E-inversive semigroup.*

Proof. (i) $\Rightarrow$ (ii) If ab and ba are idempotents then $ab = a(ba)b \in S^1baS^1$. Similarly $ba \in S^1abS^1$. Hence $ab \mathscr{J} ba$, so $ab = ba$.

(ii) $\Rightarrow$ (iii) By Tamura [3], S is a semilattice of $\mathscr{S}$-indecomposable semigroups. Since subsemigroups of viable semigroups are viable, each component is viable. The result follows from (4).

(iii) $\Rightarrow$ (iv) a fortiori

(iv) $\Rightarrow$ (i) Suppose $e, f \in E$ with $e \mathscr{J} f$. Then e and f are in the same component of the given semilattice decomposition. Hence $e = f$.

(ii) $\Rightarrow$ (v) Let $I = \{a \in S \mid \mathfrak{M}(a) = \emptyset\}$. If I is empty then S is E-inversive. Otherwise, I is obviously an idempotent-free δ-class of S. Moreover if $ax \mid e$ or $xa \mid e$, $e \in E$, then by (2), $a \mid e$. Hence, $a \in I$ implies $ax, xa \in I$ so that I is an ideal of S. By (5), S/I is viable. Since S/I has a zero, it is E-inversive. In fact, every nonzero element of S/I divides a nonzero idempotent of S/I.

(v) $\Rightarrow$ (ii) Follows from (5).

REMARK. Observe that the semilattice decomposition of (iii) in general will not be isomorphic to S/δ since in fact S may be idempotent free. Also, $\mathscr{J}$ may be replaced $\mathscr{D}$ in the theorem.

LEMMA 7. *S is an ideal extension of a group by a nil semigroup if and only if S is a subdirect product of a group and a nil semigroup.*

Proof. Suppose S is an ideal extension of a group G by a nil semigroup N. Let e be the identity of G. It is easy to see that e is central in S. It is well known that S is a subdirect product of subdirectly irreducible semigroups S_α $(\alpha \in \Omega)$. Let $\sigma_\alpha \colon S \rightarrow S_\alpha$ be the natural map. Let $e_\alpha = e\sigma_\alpha$. Then e_α is a central idempotent in S_α and hence is zero or 1 (cf. [1]). If $e_\alpha = 0$, then $\sigma_\alpha(G) = 0$ and hence $S_\alpha = \sigma_\alpha(S)$ is a nil semigroup. If $e_\alpha = 1$, then all of S_α is contained in $\sigma_\alpha(G)$ and hence S_α is a group. Consequently each S_α is a nil semigroup or a group. Let $\Omega_1 = \{\alpha \mid \alpha \in \Omega,\ S_\alpha \text{ is nil}\}$ and let $\Omega_2 = \{\alpha \mid \alpha \in \Omega,\ S_\alpha \text{ is a group}\}$. Let $\psi_i = \prod_{\alpha \in \Omega_i} \sigma_\alpha \colon S \rightarrow \prod_{\alpha \in \Omega_i} S_\alpha$ be defined for $i = 1, 2$. One can check that S is a subdirect product of $S\psi_1$ and $S\psi_2$ with $S\psi_1$ a nil semigroup and $S\psi_2$ a group.

Conversely, suppose S is a subdirect of a group G and a nil semigroup N. Consider S embedded in $G \times N$. Let e be the identity of G. There exists $a \in N$ such that $(e, a) \in S$. There exists a positive integer k such that $a^k = 0$. Hence $(e, 0) = (e, a^k) = (e, a)^k \in S$. If $g \in G$, there exists $b \in N$ such that $(g, b) \in S$. Thus $(g, 0) = (e, 0)(g, b) \in S$. Hence $G \times \{0\} \subseteqq S$ and $G \times \{0\}$ is an ideal of S. Let $(g, a) \in S$. Since $a \in N$, there exists a positive integer k such that $a^k = 0$. Hence $(g, a)^k = (g^k, a^k) = (g^k, 0) \in G \times \{0\}$. Therefore S is an ideal extension of the group $G \times \{0\}$ by a nil semigroup.

COROLLARY 8. *The following are equivalent.*

(i) *S is viable and a power of each element lies in a subgroup.*

(ii) *S is a semilattice of semigroups which are ideal extensions of groups by nil semigroups.*

(iii) S is a semilattice of semigroups each of which is a subdirect product of a nil semigroup.

Moreover the decompositions in (ii) *and* (iii) *are the same and coincide with the δ-decomposition as specified in Theorem* 3.

A semigroup S is *separative* if $x^2 = xy = y^2$ $(x, p \in S)$ implies $x = y$.

COROLLARY 9. *The following are equivalent.*

(i) *S is viable, separative and a power of each element of S lies in a subgroup.*

(ii) *S is a semilattice of groups.*

Proof. (i) $\Rightarrow$ (ii) By (8), it suffices to show that if T is separative and an ideal extension of a group G by a nil semigroup, then $T = G$. Let e be the identity of G. Then e is central in T. If $T \neq G$, then there exists $a \in T$, $a \notin G$ with $a^2 \in G$. Then $a^2 = (ae)^2 = a(ae)$. Thus $a = ae \in G$, a contradiction. Hence $T = G$.

(ii) $\Rightarrow$ (i) Obvious.

REFERENCES

1. M. Chacron and G. Thierrin, *σ-reflexive semigroups and rings*, Canad. Math. Bull., (to appear).

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol. **1**, Amer. Math. Soc., Providence, Rhode Island, 1961.

3. T. Tamura, *Another proof of a theorem concerning the greatest semilattice decomposition of a semigroup*, Proc. Japan Academy, **40** (10), 1964, 777-780.

Received October 31, 1970.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA