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GENERATORS OF THE MAXIMAL IDEALS OF $A(\bar{D})$

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Let $A = A(\bar{D})$ be the sup norm algebra of functions continuous in $\bar{D}$ and holomorphic in D, where D is a bounded, strictly pseudoconvex domain in C^n. This paper gives necessary and sufficient local conditions that a subfamily of A generates the maximal ideal $\mathscr{M}_w(\bar{D})$ of functions in A vanishing at $w \in \bar{D}$. In particular, it shows that $\mathscr{M}_w(\bar{D})$ is generated by $z_1 - w_1, \cdots, z_n - w_n$ when $W \in D$.

In [3], Gleason shows that if m is an (algebraically) finitely generated maximal ideal of a commutative Banach algebra A, the maximal ideal space $\mathscr{M}_A$ can be given an analytic structure near m, in terms of which the Gelfand transforms of the elements of A are holomorphic functions.

In a sense, the results of this paper go in the opposite direction. We consider a bounded domain D in C^n, with C^2 strictly pseudoconvex boundary, and study the algebra $A = A(\bar{D})$ of functions continuous on $\bar{D}$ and holomorphic in D. By a recent result, Henkin [4], Kerzman [7], Lieb [9], A equals the closure in $C(\bar{D})$ of the algebra $O(\bar{D})$ of functions holomorphic in some neighbourhood of $\bar{D}$, from which it follows that $\mathscr{M}_A \approx \bar{D}$.

We first fix the notation. If $w \in \bar{D}$, $\mathscr{M}_w$ denotes the maximal ideal of the ring O_w of germs of holomorphic functions at w, while $\mathscr{M}_w(\bar{D})$ is the maximal ideal in A of functions vanishing at w. If f is a function on some neighbourhood of w, f_w denotes the germ of f at w.

THEOREM 1. *Let $w \in D$, and $f_1, \cdots, f_N \in A$. Then $f_1, \cdots, f_N$ generate $\mathscr{M}_w(\bar{D})$ if and only if*

(1) *$f_{1_w}, \cdots, f_{N_w}$ generate $\mathscr{M}_w$, and*

(2) *w is the only common zero of $f_1, \cdots, f_N$ in $\bar{D}$.*

COROLLARY. *If $w \in D$, $z_1 - w_1, \cdots, z_n - w_n$ generate $\mathscr{M}_w(\bar{D})$.*

Below we give the more general theorem 2, which also gives a similar characterization of generators of $\mathscr{M}_w(\bar{D})$ when $w \in \partial D$. When $n = 2$, Kerzman and Nagel [8] have shown that $z_1 - w_1$ and $z_2 - w_2$ generate $\mathscr{M}_w(\bar{D})$ when $w \in D$, as well as similar results for algebras with Hölder norms. I want to thank Dr. Kerzman for sending me a copy of his thesis [7], where these results are stated.

The main tool in the proof is the following result, which is proved in [11]:

LEMMA 1. *Suppose* $u \in C^{\infty}_{(0,q)}(D)$ *is bounded, with* $\bar{\partial} u = 0, q \geqq 1$. *Then there exists a* $v \in C^{\infty}_{(0,q-1)}(D)$ *with* $\bar{\partial} v = u$, *such that* v *has a continuous extension to* $\bar{D}$.

A closely related result is given in Lieb [10], while a stronger result for (0, 1)-forms, involving Hölder estimates, is given in Kerzman [7].

It is convenient to prove first a more general result. If U is open in $\bar{D}$, let $H(U)$ denote functions in $C(U)$ that are holomorphic in $D \cap U$. When $w \in \bar{D}$, we define $H_w = \varinjlim_{U \ni w} H(U)$, so H_w is the space of germs at w of continuous functions on $\bar{D}$ that are holomorphic in D. It is easy to see that H is the sheaf of A-holomorphic functions in the sense of [2].

PROPOSITION 1. *Let* D *be as above,* $w \in \bar{D}$, *and suppose* $f_1, \cdots, f_N$ *have* w *as their only common zero. We let* I *denote the ideal in* A *generated by* $f_1, \cdots, f_N$, *and* I_w *the ideal in* H_w *generated by* $f_{1w}, \cdots, f_{Nw}$. *If* $f \in A$ *and* $f_w \in I_w$, *then* $f \in I$.

Proof. By assumption, we may write $f = \sum_{i=1}^{N} g_i \cdot f_i$ on a neighbourhood U of w in $\bar{D}$, with $g_1, \cdots, g_N \in H(U)$. We want to write $f = \sum_{i=1}^{N} h_i \cdot f_i$, with $h_1, \cdots, h_N \in A$, and shall first solve the problem differentiably. As the sets $N_i = \{z \in \bar{D} \backslash \{w\} : f_i(z) = 0\}$, $i = 1, \cdots, N$, are closed in $\boldsymbol{C}^n \backslash \{w\}$, it is well known how to construct $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_N$ with $\tilde{\varphi}_i = 0$ on a neighbourhood of N_i, $i = 1, \cdots, N$, that form a C^{∞} partition of unity on $\boldsymbol{C}^n \backslash \{w\}$. Choose $\varphi_0 \in C_0^{\infty}(U')$, where $U' \cap \bar{D} = U$, with $\varphi_0 = 1$ on a neighbourhood U_1 of w, and define $\varphi_i = (1 - \varphi_0) \cdot \tilde{\varphi}_i$, $i = 1, \cdots, N$.

If we define

$$g_i' = \varphi_0 \cdot g_i + \frac{\varphi_i \cdot f}{f_i}, \text{ clearly } \sum_{i=1}^{N} g_i' \cdot f_i = f \text{ on } \bar{D}.$$

The $g_i's \in C^{\infty}(D) \cap C(\bar{D})$, and are holomorphic in $U_1 \cap D$.

We want to use Lemma 1 to modify the $g_i's$ to get h_i's in A. To handle the combinatorial difficulties, we apply the homological argument of [6].

NOTATION. $L_r = \{u \in C^{\infty}_{(0,r)}(D),\ u \text{ and } \bar{\partial} u \text{ have bounded coefficients}\}$, while $L_r^s = L_r \otimes_C \bigwedge^s \boldsymbol{C}^N$, $0 \leqq r, s$.

If we choose a basis $e_1, \cdots, e_N$ in $\boldsymbol{C}^N$, the elements in L_r^s may be written uniquely as $\sum_{|I|=s} u_I \otimes e^I$, where $u_I \in L_r$, $e^I = e_{i_1} \wedge \cdots \wedge e_{i_s}$, and we sum over strictly increasing sequences $I = (i_1, \cdots, i_s)$. We define $\bar{\partial}$ on L_r^s by $\bar{\partial}(u \otimes \omega) = (\bar{\partial} u) \otimes \omega$ and linearity. Clearly

$\bar{\partial} L_r^s \subset L_{r+1}^s$, and lemma 1 gives:

LEMMA 1′. *If* $k \in L_r^s$ *and* $\bar{\partial} k = 0$, $r \geqq 1$, *there exists a* $k' \in L_{r-1}^s$, such that $\bar{\partial} k' = k$, *and* k' *has a continuous extension to* $\bar{D}$.

The product determined by $(u \otimes \omega)\cdot(u' \otimes \omega') = (u \wedge u') \otimes (\omega \wedge \omega')$ is clearly a bilinear map $L_r^s \times L_{r'}^{s'} \to L_{r+r'}^{s+s'}$.

Let $e_1^*, \cdots, e_N^*$ be the reciprocal basis to $e_1, \cdots, e_N$, so $\langle e_i^*, e_j \rangle = \delta_{ij}$. We define $P_f: L_r^s \to L_r^{s-1}$ by

$$P_f(d \otimes \omega) = \sum_{i=1}^{N} (f_i \cdot u) \otimes (e_i^* \rfloor \omega), \text{ and linearity.}$$

(For the definition of $\rfloor$, se [12] Ch. 1.)
$P_f: L_r^1 \to L_r^0$ maps $\sum_{i=1}^N u_i \otimes e_i$ to $\sum_{i=1}^N f_i \cdot u_i$; in particular, $P_f g' = f$, when $g' = \sum_{i=1}^N g_i' \otimes u_i$.

A simple computation gives $P_f^2 = 0$, while the derivation property of $\rfloor$ gives

$$\text{(i)} \qquad P_f(k \cdot k') = (P_f k)\cdot k' + (-1)^s k \cdot P_f k'$$

when $k \in L_r^s$.

Let $M_r^s = \{k \in L_r^s : k|_{U_1} = 0\}$.

LEMMA 2. *The complex* $0 \leftarrow M_r^0 \xrightarrow{P_f} M_r^1 \xrightarrow{P_f} \cdots \xrightarrow{P_f} M_r^N \leftarrow 0$ *is exact.*

Proof. Let $\varphi \in C^\infty(C^N)$ be zero near w and one outside U_1. We put $k_0 = \sum_{i=1}^N (\varphi \cdot \tilde{\varphi}_i)/f_i \otimes e_i$. Clearly $k_0 \in L_0^1$, and $P_f k_0 \in L_0^0$ is identically one in $D \backslash U_1$. If $k \in M_r^s$ and $P_f k = 0$, $k_0 \cdot k \in M_r^{s+1}$, and by (*i*), $P_f(k_0 \cdot k) = (P_f k_0)\cdot k = k$.

As $f_1, \cdots, f_N$ are holomorphic in D, P_f and $\bar{\partial}$ commute.

LEMMA 3. *If* $k \in M_r^s$ *and* $P_f k = \bar{\partial} k = 0$, *there exists a* $k' \in L_r^{s+1}$, *with* $P_f k' = k$ *and* $\bar{\partial} k' = 0$.

This is trivially true when $r > n$, and the proof goes by downward induction on r. Suppose the lemma is valid for $r+1$. By Lemma 2, there exists a $k_1 \in M_r^{s+1}$ with $P_f k_1 = k$. Clearly $\bar{\partial} M_r^{s+1} \subset M_{r+1}^{s+1}$, while $P_f \bar{\partial} k_1 = \bar{\partial} P_f k_1 = 0$. Using the induction hypothesis, we can find $k_2 \in L_{r+1}^{s+2}$ with $P_f k_2 = \bar{\partial} k_1$ and $\bar{\partial} k_2 = 0$. By Lemma 1′, $k_2 = \bar{\partial} k_3$, with $k_3 \in L_r^{s+2}$. If we put $k' = k_1 - P_f k_3$, we get $k' \in L_r^{s+1}$, with $\bar{\partial} k' = \bar{\partial} k_1 - P_f \bar{\partial} k_3 = 0$, and $P_f k' = P_f k_1 - P_f^2 k_3 = k$. This completes the induction step.

Proof of Proposition 1. As the g_i's are holomorphic in $U_1 \cap D$, $\bar\partial g' \in M_1^1$. Applying Lemma 1′ and Lemma 3, we find a $k \in L_0^2$, with $\bar\partial P_f k = P_f \bar\partial k = \bar\partial g'$, such that k is continuous on $\bar D$. If $h = g' - P_f k$, $\bar\partial h = 0$. Writing $h = \sum_{i=1}^N h_i \otimes e_i$, this means that $h_1, \cdots, h_N \in A$, and $\sum_{i=1}^N h_i \cdot f_i = f$.

THEOREM 2. *Let $w \in \bar D$, and let M_w denote the unique maximal ideal of H_w. The family $(f_i)_{i \in I}$ in A generates $\mathscr{M}_w(\bar D)$ if and only if*

(1) *$(f_{i_w})_{i \in I}$ generates M_w, and*

(2) *w is the only common zero of functions f_i in $\bar D$*

Proof. I. The sufficiency of (1) and (2): If $f \in \mathscr{M}_w(\bar D)$, we have $f_w \in M_w$, and by (1) f_w belongs to some ideal $[f_{i_1,w}, \cdots, f_{i_M,w}]$. As $(z_1 - w_1)_w, \cdots, f(z_n - w_n)_w$ belong to M_w, the functions $z_i - w_i$; $i = 1, \cdots, n$, may be expressed as linear combinations of functions $f_{i_{M+1}}, \cdots, f_{i_P}$ in the family on some open neighbourhood V of w in $\bar D$. Then $f_{i_{M+1}}, \cdots, f_{i_P}$ have w as their only common zero in V. By condition (2) and the compactness of $\bar D \backslash V$, there exist $f_{i_{P+1}}, \cdots, f_{i_N}$ in the family with no common zeroes outside V. Now proposition 1 implies that $f \in [f_{i_1}, \cdots, f_{i_N}]$.

II. The necessity of (1) and (2): If $(f_i)_{i \in I}$ generate $\mathscr{M}_w(\bar D)$, condition (2) follows from the fact that A separates points in $\bar D$. Condition (1) follows from

PROPOSITION 2. *The germs at w of elements in $\mathscr{M}_w(\bar D)$ generate M_w.*

The following proof of Proposition 2 was kindly communicated to me by Dr. R. M. Range, and replaces a more complicated argument of my own:

When $w \in D$, $z_1 - w_1, \cdots, z_n - w_n$ generate $\mathscr{M}_w = M_w$. Thus we may assume $w \in \partial D$, and consider an $f \in H(U \cap \bar D)$ with $f(w) = 0$, where U is some neighbourhood of w in C^n. We choose $\varphi \in C_0^\infty(U)$ such that $\varphi \equiv 1$ on a smaller neighbourhood V of w. As D is strictly pseudoconvex, we may extend it inside V to a strictly pseudoconvex domain D' containing w. As $\bar\partial(\varphi \cdot f)$ vanishes on $V \cap D$, it may be extended by zero to a smooth, bounded, $\bar\partial$-closed $(0, 1)$-form ω on D'. By Lemma 1, the equation $\bar\partial g = \omega$ has a solution in $C^\infty(D') \cap C(\bar D')$, and we may assume $g(w) = 0$. As g is holomorphic in $D' \cap V$, we may write it near w as $g = \sum_{i=1}^n g_i(z_i - w_i)$, with $g_1, \cdots, g_n$ holomorphic. Thus $f_w = (\varphi \cdot f - g)_w + \sum_{i=1}^n g_{i_w}(z_i - w_i)_w$, and $\varphi \cdot f - g|_{\bar D} \in \mathscr{M}_w(\bar D)$.

When $w \in D$ and I is finite, Theorem 2 reduces to theorem 1. If $w \in \partial D$, it follows from Gleason's result that $\mathscr{M}_w(\bar{D})$ is not finitely generated. If M_w were finitely generated, it would by Proposition 2 be generated by finitely many elements of A, which implies by the argument of I that $\mathscr{M}_w(\bar{D})$ must be finitely generated. Thus M_w is not finitely generated when $w \in \partial D$. (This may also be proved in a more direct fashion).

Note. The Corollary to Theorem 1 has also been proved by G. M. Henkin in Bull. Acad. Polon. Sci., 24 (1971) 37–42, and by I. Lieb in Math. Ann., 190 (1970–71) 6–44, which contains a detailed version of [10].

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