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ZERO DIVISORS IN DIFFERENTIAL RINGS

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Let R be a commutative ordinary differential ring with 1. Let A be a commutative differential R-algebra satisfying the ascending chain condition on radical differential ideals. Let M be a differentially finitely generated R-module. We obtain the following results on the zero divisors of A and M in R. (i) If R satisfies the ascending chain condition on radical differential ideals and if A has zero nilradical, then the assassinator of A in R is finite and consists of differential ideals; it is contained in the support of A in R, and the minimal members of each set comprise exactly the minimal prime ideals which contain the annihilator of A in R; (ii) If $R \subseteqq A$ and I is a radical differential ideal of A, then we obtain the assassinator of A/I in R from the assassinator of A/I in A by intersecting with R; (iii) If R is noetherian, then the set of zero divisors of M in R is a unique union of prime differential ideals of R, each of which is maximal among annihilators in R of nonzero elements of M; (iv) If I is the annihilator or power annihilator of M in R, then any prime ideal of R minimal over I is the annihilator of a nonzero element of M. In the above, (iii) and (iv) require an additional hypothesis to be made explicit later.

These results (except (ii)) are well known for finite modules over noetherian rings.

2. Preliminaries. In what follows, all rings are commutative and all modules are unitary. R will always be a differential ring with 1, with fixed derivation denoted by "$'$". By a *differential module* M over R, one means an R-module M together with an additive map from M to M, again denoted by "$'$", which satisfies $(rm)' = r'm + rm'$ for each $r \in R$ and $m \in M$. If $x \in M$, the successive derivatives of x will be denoted by $x', x'', \cdots, x^{(n)}, \cdots$. By a *differential algebra* A over R, one means a differential module A which is a ring and for which the module derivation is a ring derivation. By an ideal of A, we always mean an algebra ideal.

Let M be any R-module and $T \subseteqq M$ a subset. We denote the zero divisors of T in R by $\mathscr{Z}_R(T)$ and the annihilator of T in R by $\mathscr{A}_R(T)$. The *assassinator* of M in R, written $\mathrm{Ass}_R\, M$, is the set of prime ideals of R which are the annihilators of nonzero elements of M. The *support* of M in R, written $\mathrm{Supp}_R\, M$, is the set of prime ideals P of R such that $M_P \neq 0$.

Now let R be a differential ring and M a differential R-module.

Denote by $[T]/R$ the smallest differential submodule of M containing T. We call M *d-finitely generated* if there exists $n \geqq 0$ and $x_1, \cdots, x_n$ in M such that $M = [x_1, \cdots, x_n]/R$.

Let $S \subseteqq R$ be a multiplicatively closed set with $0 \notin S$. Then the derivations on R and M extend by the usual quotient formula to make M_s into a differential R_s-module. (See [2; Lemma 1].)

Assume, in addition, that M is a differential R-algebra. Denote by $\{T\}/M$ the smallest radical differential ideal containing T. The following fact is a trivial consequence of [5; Lemma 1.3]. Let Rad $M = 0$ (i.e., M has zero nilradical), and let T be a subset of either R or M. Then $\mathscr{A}_R(T)$ and $\mathscr{A}_M(T)$ are radical differential ideals.

3. The assassinator.

We begin by stating the first main theorem.

THEOREM 1. *Let R be a differential ring and A a differential R-algebra. Let R and A satisfy the ascending chain condition on radical differential ideals, and let* Rad $A = 0$. *Then* $\mathrm{Ass}_R\, A$ *is finite, consists of differential prime ideals, and is contained in* $\mathrm{Supp}_R\, A$. *The minimal members of each of these sets are the same and coincide with the prime ideals of R minimal over $\mathscr{A}_R(A)$.*

Before proving Theorem 1, we need a series of lemmas.

LEMMA 1. *Let R be a differential ring satisfying the ascending chain condition on radical differential ideals. Let A be a nonzero differential R-algebra with* Rad $A = 0$. *Then* $\mathrm{Ass}_R\, A \neq \varnothing$.

Proof. For any nonzero $a \in A$, $\mathscr{A}_R(a)$ is a proper, radical differential ideal of R. By hypothesis, there are ideals of R maximal among annihilators of nonzero elements of A. That these ideals are prime is well known [4; Theorem 6].

LEMMA 2. *Let R be a differential ring, and let M be a differential R-module. Let T be a subset of M, and suppose that $\mathscr{A}_R(T)$ is a differential ideal. Then*:

(i) $\mathscr{A}_R(T) = \mathscr{A}_R([T]/R)$;

(ii) *if M is a differential R-algebra, then* $\mathscr{A}_R(T) = \mathscr{A}_R([T]/M)$; *if, in addition,* Rad $M = 0$, *then* $\mathscr{A}_R(T) = \mathscr{A}_R(\{T\}/M)$.

Proof. Let $y \in \mathscr{A}_R(T)$. Then, since $xy' + x'y = 0$ for any $x \in T$, and $y' \in \mathscr{A}_R(T)$, we see that $x'y = 0$. Hence, $x''y + x'y' = 0$. The above argument applied to y' instead of to y would have resulted in $x'y' = 0$. Hence $x''y = 0$. Continuing in this way, we see that

$x^{(k)}y = 0$ for each nonnegative integer k. Now since an arbitrary element of $[T]/R$ has the form $\sum_{i,j} a_{i,j}x_j^{(i)}$, and an arbitrary element of $[x]/M$ (when M is an R-algebra) has the form $\sum_{i,j} b_{ij}x_i^{(i)} + \sum_{i,j} c_{ij}x_j^{(i)}$, for $b_{ij} \in M$ and a_{ij}, $c_{ij} \in R$, for every i and j, we see that $\mathscr{A}_R(T) \subseteqq \mathscr{A}_R([T]/R)$ and $\mathscr{A}_R(T) \subseteqq \mathscr{A}_R([T]/M)$. Since the opposite inclusions are clear, we have equality.

Now assume that M is an R-algebra and that Rad $M = 0$. By the above, we will be through once we show that $\mathscr{A}_R([T]/M) \subseteqq \mathscr{A}_R(\{T\}/M)$. Now $\mathscr{A}_M(\mathscr{A}_R([T]/M))$ is a radical differential ideal of M. Since it contains T, it contains $\{T\}/M$; i.e., $\{T\}/M$ annihilates $\mathscr{A}_R([T]/M)$; therefore, $\mathscr{A}_R([T]/M)$ annihilates $\{T\}/M$. This completes the proof.

LEMMA 3. *Let R be a differential ring, and let A be a differential R-algebra satisfying the ascending chain condition on radical differential ideals, and such that* Rad $A = 0$. *Let P be a prime ideal of R containing $\mathscr{A}_R(A)$. Then $P \in$* Supp_R A.

Proof. Since A satisfies the ascending chain condition on radical differential ideals, there must be $a_1, \cdots, a_r$ in A such that $A = \{a_1, \cdots, a_r\}/A$. Suppose that $A_P = 0$. Then there are $s_i \in R - P$ such that $s_i a_i = 0$ for each i. Let $s = \prod_{i=1}^{r} s_i$. Then $sa_i = 0$ for each i. Since Rad $A = 0$, $\mathscr{Z}_A(s)$ is a radical differential ideal of A containing each a_i, and so must equal A. But then $sA = 0$; i.e., $s \in \mathscr{A}_R(A)$, which contradicts $s \notin P$. This completes the proof.

LEMMA 4. *Let R be a differential ring satisfying the ascending chain condition on radical differential ideals, and let A be a differential R-algebra with* Rad $A = 0$. *Then* $\mathrm{Ass}_R\, A \subseteqq \mathrm{Supp}_R\, A$, *and each member of* $\mathrm{Supp}_R\, A$ *contains a member of* $\mathrm{Ass}_R\, A$. *In particular, both sets have the same minimal elements.*

Proof. That $\mathrm{Ass}_R\, A \subseteqq \mathrm{Supp}_R\, A$ is just [1; §1, °3, Prop. 7(i)]. Now let $Q \in \mathrm{Supp}_R\, A$. Then $A_Q \neq 0$ as an R_Q-algebra. By Lemma 1, $\mathrm{Ass}_{R_Q}(A_Q) \neq \emptyset$. Let $P_1 \in \mathrm{Ass}_{R_Q}(A_Q)$ with $P_1 = \mathscr{Z}_{R_Q}(a/1)$. Since Rad $(A_Q) = 0$, P_1 is a differential ideal. Let $P = \{r \in 1 \mid r/1 \in P_1\}$. Then P is a prime differential ideal of R and $P \subseteqq Q$. We claim that $P \in \mathrm{Ass}_R\, A$. By hypothesis, $P = \{p_1, \cdots, p_n\}/R$ for some $p_1 \cdots, p_n \in R$. Since $p_i a/1 = 0$, there are $s_i \in R - P$ such that $p_i s_i a = 0$ for each i. Hence, if $s = \prod_{i=1}^{n} s_i$, $p_i \in \mathscr{Z}_R(sa)$ for each i. Since $\mathscr{Z}_R(sa)$ is a radical differential ideal of R, $P \subseteqq \mathscr{Z}_R(sa)$. On the other hand, if $x \in \mathscr{Z}_R(sa)$, then $xa/1 = 0$; i.e., $x/1 \in P_1$; i.e., $x \in P$. Hence $P = \mathscr{Z}_R(sa) \in \mathrm{Ass}_R\, A$, and we are done.

LEMMA 5. *Let R be a differential ring, and let A be a differential R-algebra satisfying the ascending chain condition on radical differential ideals. Assume that* Rad $A = 0$. *Then A has a normal series*

$$A = A_0 \supseteqq A_1 \supseteqq \cdots \supseteqq A_n = 0$$

where

(i) *A_i is a radical differential ideal of A for each i;*

(ii) *$\mathscr{A}_R(A_{i-1}/A_i) = \mathrm{Ass}_R\,(A_{i-1}/A_i)$ for each i, and both consist of a single prime differential ideal P_i of R.*

Proof. Let $B \neq A$ be a radical differential ideal of A. Then A/B is a differential R-algebra satisfying the ascending chain condition on radical differential ideals, and $\mathrm{Rad}\,(A/B) = 0$. Since $A/B \neq 0$, we are guaranteed by Lemma 1 that there exists in $\mathrm{Ass}_R\,(A/B)$ a differential prime ideal $P = \mathscr{Z}_R(x)(x \in A/B$ and nonzero) which is maximal among the annihilator of nonzero elements of A/B. Let $B_1 = \varphi^{-1}(\{x\}/(A/B))$ where φ is the canonical homomorphism of A onto A/B. Then B_1 is a radical differential ideal of A, $B \subsetneqq B_1$ and $B_1/B \cong \{x\}/(A/B)$ so that $\mathscr{A}_R(B_1/B) = P$ by Lemma 2. Now suppose that $Q \in \mathrm{Ass}_R(B_1/B)$. Then $Q = \mathscr{Z}_R(b_1)$ for some $b_1 \in B_1/B$. Since $Pb_1 = 0$, $P \subseteqq Q$; hence, by the maximality of P, $P = Q$ and $\mathrm{Ass}_R\,(B_1/B)$ consists of the single prime P.

Starting with $B = 0$ and using the above method, we construct an increasing chain of radical differential ideals of A satisfying the conclusions of the lemma. By hypothesis, this chain must stop; i.e., at some stage, $B_1 = $ A, and we are done.

Proof of Theorem 1. We follow the notation of Lemma 5. By [1; § 1, °1, Prop. 3],

$$\mathrm{Ass}_R\,A \subseteqq \cup_{i=1}^{n} \mathrm{Ass}_R\,(A_{i-1}/A_i) = \{P_1, \cdots, P_n\}$$

so that $\mathrm{Ass}_R\,A$ is finite and consists of differential ideals. By Lemma 4, $\mathrm{Ass}_R\,A \subseteqq \mathrm{Supp}_R\,A$, and each has the same minimal elements. (In fact, since $P_i \in \mathrm{Supp}_R(A_{i-1}/A_i)$ by Lemma 4 and since $0 \neq (A_{i-1}/A_i)_{P_i} = (A_{i-1})_{P_i}/(A_i)_{P_i}$ each $P_i \in \mathrm{Supp}_R\,A_{i-1} \subseteqq \mathrm{Supp}_R\,A$.) That these minimal elements coincide with the prime ideals of R minimal over $\mathscr{A}_R(A)$ follows from the following two facts: The minimal elements of $\mathrm{Ass}_R\,A$, and so of $\mathrm{Supp}_R\,A$, contain $\mathscr{A}_R(A)$; the primes minimal over $\mathscr{A}_R(A)$ are members of $\mathrm{Supp}_R\,A$ by Lemma 3. This completes the proof.

COROLLARY. *Let the hypotheses be as in Theorem* 1. *Then* $\mathrm{Supp}_R\,A$ *consists of exactly the prime ideals of R which contain $\mathscr{A}_R(A)$.*

We remark that if R contains the rational numbers and satisfies the ascending chain condition on radical differential ideals, then any quotient by a differential ideal of the differential polynomial ring over R in a finite number of differential indeterminates also satisfies the ascending chain condition on radical differential ideals.

If we assume that $R \subseteqq A$, we get the following result with no chain condition assumptions on R.

THEOREM 2. *Let R be a differential ring contained in the differential R-algebra A. Assume that A satisfies the ascending chain condition on radical differential ideals. Let I be a radical differential ideal of A. Then:* (i) *I can be written uniquely as $I = \cap_{i=1}^{n} P_i$ where the P_i are prime differential ideals of A;* (ii) *if $Q_i = P_i \cap R$, then*

$$\mathrm{Ass}_A(A/I) = \{P_1, \cdots, P_n\} \text{ and } \mathrm{Ass}_R(A/I) = \{Q_1, \cdots, Q_n\}.$$

Proof. We note that (i) is well known and proved more directly in [5; Theorem 7.5]. Now A/I, viewed as an A-algebra, satisfies the hypotheses of Lemma 1 and Theorem 1. Let $P_1, \cdots, P_n$ be the unique elements of $\mathrm{Ass}_A(A/I)$ minimal over $\mathscr{A}_A(A/I)$. Since $1 \in A$, $\mathscr{A}_A(A/I) = I$, and since I is a radical ideal, $I = \cap_{i=1}^{n} P_i$. This proves (i).

Since the P_i are minimal over $\mathscr{A}_A(A/I)$, they are minimal members of $\mathrm{Ass}_A(A/I)$ by Theorem 1. On the other hand, let $P = \mathscr{Z}_A(a_1) \in \mathrm{Ass}_A(A/I)$, with $a_1 \in A/I$. Let $a \in A$ be mapped to a_1. Then $a \notin P_j$ for some $j = 1, \cdots, n$. But $Pa \subseteqq I \subseteqq P_j$, so that $P \subseteqq P_j$; i.e., $P = P_j$. Hence $\mathrm{Ass}_A(A/I) = \{P_1, \cdots, P_n\}$.

Now let $P_i = \mathscr{Z}_A(a_i)$, $a_i \in A/I$ for each i. Then $Q_i = P_i \cap R$ must be $\mathscr{Z}_R(a_i)$ for each i; i.e., $Q_i \in \mathrm{Ass}_R(A/I)$.

To complete the proof, we must show that any $Q \in \mathrm{Ass}_R(A/I)$ is one of the Q_i. Localize A and R at Q. Then A_Q is an R_Q-algebra satisfying the hypotheses of the theorem and I_Q is a radical differential ideal of A_Q. Further, I_Q is a proper ideal of A_Q for, since $I \cap R \subseteqq Q$, we see that $(I \cap R)_Q = I_Q \cap R_Q \subseteqq Q_Q$; i.e., $R_Q \not\subseteq I_Q$. Since each P_i is prime, $I_Q = (\cap_{i=1}^{n} P_i)_Q = \cap_{i=1}^{r}(P_i)_Q$ where we have assumed that $P_1, \cdots, P_r$ are exactly those among $P_1, \cdots, P_n$ such that $(P_i)_Q \neq A_Q$. Note that $r > 0$ by Lemma 1 since $A_Q/I_Q \neq 0$. By the initial argument in this part of the theorem, $\mathrm{Ass}_{R_Q}(A_Q/I_Q) = \{(P_1)_Q, \cdots, (P_r)_Q\}$. Since $Q_Q \subseteqq \mathscr{Z}_{R_Q}(A_Q/I_Q)$, $Q_Q \subseteqq (P_i)_Q \cap R_Q = (Q_i)_Q$ for some i. Since Q_Q is maximal, $Q_Q = (Q_i)_Q$; i.e., $Q = Q_i$, and the proof is complete.

4. The case for modules. The situation for modules is less complete. However, under the restriction given below, we can gain some information about $\mathrm{Ass}_R(M)$ when M is a d-finitely generated R-module.

We say that the differential R-module M satisfies the property (♯) if ideals of R maximal among the annihilators of nonzero elements of M are differential ideals. We say that M satisfies the property (♯♯) if M/N satisfies the property (♯) for every differential submodule N of M.

THEOREM 3. *Let R be a noetherian differential ring and M a nonzero, d-finitely generated R-module which satisfies the property (♯♯). Then* $\mathrm{Ass}_R M$ *is finite.*

Proof. The assassinator of nonzero modules over noetherian rings is never empty. Using the condition (♯♯) and Lemma 2(i), we modify the proof of Lemma 5 to prove an analogue of Lemma 5 in which the A_i are replaced by differential R-modules. The result now follows as in the first part of Theorem 1.

Further progress in this direction is limited by the fact that prime ideals of R containing $\mathscr{A}_R(M)$ need not be in $\mathrm{Supp}_R M$. The correct modification is given in Lemma 7. (For example, let $R = Z$, the integers, with the trivial derivation. Let M be generated over $Z/2Z$ by 1 and the set $\{x/2^n\}$ for $n = 0, 1, 2, \cdots$, and have derivation defined by $(x/2^n)' = x/2^{n+1}$. Then $M = [1, x]/Z$. Now $\mathscr{A}_Z(M) = 0$; but if $P = 3Z, M_P = 0$.)

The following discussion indicates what is still true if we assume only the condition (♯). We shall need the result [2; Th. 1]:[1]

THEOREM A. *Let R be a noetherian differential ring, and let M be a d-finitely generated R-module. Then M satisfies the ascending chain condition on differential submodules.*

We can now prove

THEOREM 4. *Let R be a noetherian differential ring and M a d-finitely generated R-module which satisfies the property (♯). Then $\mathscr{Z}_R(M)$ is expressible uniquely as the union of a finite number of differential prime ideals, each of which is maximal among the annihilators of nonzero elements of M.*

Proof. Each nonzero $x \in M$ has an annihilator ideal, and $\mathscr{Z}_R(M)$ is clearly their union. Each such annihilator is contained in a maximal one which is prime, and differential by assumption. Let $\{P_\lambda\}_{\lambda \in \Lambda}$ be the set of these maximal annihilators, and let $P_\lambda = \mathscr{Z}_R(x_\lambda)$, $x_\lambda \in M$

[1] This theorem, in different language, is originally due to J. Johnson, *Differential dimension polynomials and a fundamental theorem on differential modules*, Amer. J. Math., **91** (1969), 239.

for each $\lambda \in \Lambda$. The differential submodule N of M generated by the x_λ's is d-finitely generated by Theorem A. Let $N = [x_1, \cdots, x_n]/R$, with the $x_1, \cdots, x_n$ chosen from among the x_λ's. Then, for any λ, $x_\lambda = \sum_{i,j} r_{ij} x_i^{(j)}$, with $r_{ij} \in R$ for each i and j, and only a finite number of values for j appearing. Since, by Lemma 2, $P_i = \mathscr{A}_R([x_i])$ for each $i = 1, 2, \cdots, n$, this implies that $P_\lambda \supseteqq \cap_{i=1}^n P_i$. This implies, by maximality, that P_λ is one of the P_i's. Hence, $\mathscr{Z}_R(M) = \cup_{i=1}^n P_i$.

To show uniqueness, we remark that if Q were a member of another such union, then $Q \subseteqq \cup_{i=1}^n P_i$ implies that Q equals one of the P_i's [4; Th. 8]. This proves the theorem.

For any R-module M, define $\mathscr{P}\mathscr{A}_R(M)$, the *power annihilator* of M in R, to be the set of r in R such that for every $m \in M$, there is a positive integer n with $r^n m = 0$. Then $\mathscr{P}\mathscr{A}_R(M)$ is an ideal which contains both $\mathscr{A}_R(M)$ and its radical. (If M is finitely generated, it equals this radical.)

LEMMA 6. *Let M be a differential R-module. Let $a \in M$ and $r \in R$, and suppose that $ra = 0$. Then, for every nonnegative integer n, we have $r^{n+1}a^{(n)} = 0$.*

Proof. We proceed by induction, the case $n = 0$ being satisfied by hypothesis.

If $r^n a^{(n-1)} = 0$, then $r^n a^{(n)} + nr^{n-1}r' a^{(n-1)} = 0$.

On multiplying through by r, we have the result.

LEMMA 7. *Let R be a differential ring M a d-finitely generated R-module. Let $P \subsetneqq R$ be a prime ideal containing $\mathscr{P}\mathscr{A}_R(M)$. Then $M_P \neq 0$.*

Proof. Let $M = [m_1, \cdots, m_r]/R$, and assume that $M_P = 0$. Then there is an $s \in R - P$ such that $sm_i = 0$ for each i. By Lemma 6, $s^k m_i^{(k-1)} = 0$ for each i and k, and so, for every $m \in M$, there is a positive integer t with $s^t m = 0$; i.e., $s \in \mathscr{P}\mathscr{A}_R(M)$. This contradicts $s \notin P$.

LEMMA 8. *Let M be any R-module. Let $I = \mathscr{A}_R(M)$ (resp., $I = \mathscr{P}\mathscr{A}_R(M)$), and let P be a prime ideal of R containing I. Assume that $M_P \neq 0$. Then $I_P \subseteqq \mathscr{A}_{R_P}(M_P) \subseteqq P_P$ (resp., $I_P \subseteqq \mathscr{P}\mathscr{A}_{R_P}(M_P) \subseteqq P_P$).*

Proof. The first inclusion is clear in both cases. We prove the second inclusion, $\mathscr{P}\mathscr{A}_{R_P}(M_P) \subseteqq P_P$. Let $x/t \in \mathscr{P}\mathscr{A}_{R_P}(M_P)$ with $x \in R$ and $t \in R - P$, and let $m \in M$ be such $m/1 \neq 0$. If $(x/t)^r m/1 = 0$, then there is an $s \in R - P$ with $sx^r m = 0$. If $x \notin P$, then $sx^r \in R - P$, so that $sx^n m = 0$ implies that $m/1 = 0$, a contradiction. Hence, $x \in P$, and we are done.

If M is any R-module, it is well known that any prime ideal of R minimal over $\mathscr{A}_R(M)$ is contained in $\mathscr{Z}_R(M)$ (See [4; Th. 84]). A minor variant of the proof in the reference proves.

LEMMA 9. *Let M be any R-module. Let P be a prime ideal minimal over $\mathscr{P}\mathscr{A}_R(M)$. Then $P \subseteqq \mathscr{Z}_R(M)$.*

THEOREM 5. *Let R be a noetherian differential ring and M a d-finitely generated R-module. Let $I = \mathscr{A}_R(M)$ (resp., $\mathscr{P}\mathscr{A}_R(M)$), and let P be a minimal prime ideal over I. Assume that $M_P \neq 0$ (note Lemma 7 in this regard) and that M_P satisfies the property (♯). Then P is a differential ideal and $P \in \mathrm{Ass}_R M$.*

Proof. M_P is a nonzero, d-finitely generated module over R_P. Since P is minimal over I, Lemma 8 implies that P_P is minimal over $\mathscr{A}_{R_P}(M_P)$ (resp., $\mathscr{P}\mathscr{A}_{R_P}(M_P)$). By Lemma 9 and the remark preceding it, $P_P \subseteqq \mathscr{Z}_{R_P}(M_P)$. It follows from Theorem 4 and the maximality of P_P that there is an $x \in M$ such that $P_P = \mathscr{Z}_{R_P}(x/1)$. Further, P_P is a differential R_P-ideal. Since $P = \{r \in R \mid r/1 \in P_P\}$, P is a differential ideal also. Since P_P is finitely generated, this implies the existence of an $s \in R - P$ such that $sPx = 0$. But then $P = \mathscr{Z}_R(sx)$. For if $ysx = 0$, for some $y \in R$, then $(y/1)(x/1) = 0$; i.e., $y/1 \in P_P$. It follows that $y \in P$, and we are done.

EXAMPLE. Let S be a noetherian ring containing the rational numbers and equipped with the trivial derivation. Let R be the ring of formal power series over S in the indeterminate z, equipped with the derivation defined by $z' = z$. Since every prime ideal of R is of the form PR or $PR + zR$, where P is a prime ideal of S, R satisfies the condition (♯♯) for any R-module. Let x be an indeterminate, and let $M_1 = R[x^{-1}]$, viewed as a differential R-module by the derivation $(x)' = r$ for some unit $r \in S$. Since $x^{-(n+1)} = (x^{-1})^{(n)}$ times a unit of S, M_1 is d-finitely generated over R by 1 and x^{-1}. Let M be any quotient module of M_1 by a differential submodule. Then M and R satisfy the hypotheses of Theorems 3, 4, and 5. Notice that if M_1 is considered as a ring, Rad M_1 need not be zero.

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