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DIRECTED GRAPHS AS UNIONS OF PARTIAL ORDERS

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The index of an irreflexive binary relation R is the smallest cardinal number $\sigma(R)$ such that R equals the union of $\sigma(R)$ partial orders. With $s(n)$ the largest index for an R defined on n points, it is shown that $s(n)/\log_2 n \to 1$ as $n \to \infty$. The index function is examined for symmetric R's and almost transitive R's, and a characterization for $\sigma(R) \leqq 2$ is presented. It is shown also that

$$\inf \{n: s(n) > 3\} \leqq 13 ,$$

but the exact value of $\inf \{n: s(n) > 3\}$ is presently unknown.

1. Introduction. A binary relation on a set X is a subset of ordered pairs xy in $X \times X$. A directed graph (hereafter *digraph*[1]) $G = (X, R)$ is a nonempty set X and an irreflexive ($xx \notin R$) binary relation R on X. If $\phi \subset Y \subseteqq X$ then $G \mid Y$ is the digraph obtained from $G = (X, R)$ by deleting all points in X-Y.

A *partial order* P on X is an irreflexive and transitive ($xy \in P$ & $yz \in P \Rightarrow xz \in P$) binary relation on X. A digraph $G = (X, R)$ is *resolved* by a set of partial orders on X if and only if R equals the *union* of the partial orders in the set. Since $\{xy\}$ is a partial order when $xy \in R$, every G is resolved by some set of partial orders.

The *index*[2] of a digraph $G = (X, R)$ is the smallest cardinal number $\sigma(R)$ such that R is resolved by $\sigma(R)$ partial orders on X. Clearly $\sigma(R) = 1$ if and only if R is a partial order. $\sigma(\{ab, ba\}) = 2$, and $\sigma(R) = 3$ for the cyclic triangle $R = \{ab, bc, ca\}$. The smallest X that we know of that admits an R with $\sigma(R) = 4$ has 13 points. (See Figure 1.) In connection with a later characterization of $\sigma \leqq 2$ we present an R with $\sigma(R) = 2$ where R cannot be the union of two *disjoint* partial orders.

Our definition of $\sigma(R)$ is motivated by Dushnik and Miller's definition [2] of the *dimension of a partial order* P on X as the smallest cardinal number $D(P)$ such that P equals the *intersection* of $D(P)$ linear orders on X. *A linear order* L on X is a complete ($x \neq y \Rightarrow xy \in L$ or $yx \in L$) partial order, and a *chain* in X is a linear

[1] We shall sometimes refer to a binary relation as a digraph, omitting explicit mention of the set on which the relation is defined.

[2] It is tempting to use "dimension" instead of "index," but since the former term is used for a number of other concepts in the theory of binary relations we favor the latter here. It would be proper to write $\sigma(G)$ instead of $\sigma(R)$, but since $\sigma(R) = \sigma(R')$ if R is isomorphic to R' the specific omission of X will cause no problems.

order on a subset of X. A number of facts about $D(P)$ are summarized in [1], which gives other references.

This paper examines the index function σ for digraphs. The next section focuses on large values for $\sigma(R)$. Our first theorem, based on a theorem in Folkman [4], shows that $\sigma(R)$ can be arbitrarily large for both *symmetric* ($xy \in R \Rightarrow yx \in R$) and asymmetric ($xy \in R \Rightarrow yx \notin R$) digraphs. The second theorem examines the behavior of σ in the following way. Let

$$s(n) = \sup\{\sigma(R) \colon R \text{ is an irreflexive binary relation on } n \text{ points}\}\ ,$$

the largest σ for a digraph with n points. When u is a real-valued function on $\{1, 2, \cdots\}$ and $u(n)$ remains bounded as n gets large, we write $u = 0(1)$ according to popular convention. Theorem 2 states that

$$\log_2 n - \frac{1}{2}\log_2\log_2 n + 0(1) \geqq s(n) \geqq \log_2 n - \frac{3}{2}\log_2\log_2 n - 0(1)\ .$$

This gives another proof that σ can be arbitrarily large, and shows that $s(n)/\log_2(n)$ approaches 1 as n gets large.

The rest of the paper is mostly concerned with small values of σ. Section 3 presents an (X, R) with $|X| = 13$ and $\sigma(R) = 4$. We do not presently know the smallest X that admits an R with $\sigma(R) = 4$.

Symmetric digraphs (X, S) are examined in § 4, where we give a necessary and sufficient condition for $\sigma(S) \leqq 2$. Suppose that P is a partial order on X and

$$S = \{xy\colon xy \in X \times X \ \&\ x \neq y \ \&\ xy \notin P \ \&\ yx \notin P\}\ .$$

Then S is a symmetric digraph. We note that when S is defined in this way, then $D(P) \leqq 2$ if and only if $\sigma(S) \leqq 2$, and

$$D(P) \leqq n \Rightarrow \sigma(S) \leqq 2(n-1)\ .$$

The question of whether $\sigma(S) \leqq n \Rightarrow D(P) \leqq f(n)$ for some function f is presently open.

A binary relation R is almost transitive[3] if and only if $(ab \in R\ \&\ bc \in R\ \&\ a \neq c) \Rightarrow ac \in R$. Section 5 proves that $\sigma(R) \leqq 2$ when R is an almost transitive digraph.

Section 6 then gives a general characterization of $\sigma(R) \leqq 2$ that is stated in terms of a partition of the subset of R whose elements

[3] Harary, Norman and Cartwright [7, p. 7] call this transitivity, but we use the modifier to distinguish it from the more common use of "transitivity" in which a, b and c do not have to be distinct.

are involved in nontransitive adjacent pairs such as $xy,\ yz \in R$ & $xz \notin R$.

2. Digraphs with large indices.

THEOREM 1. *If n is a positive integer then there are asymmetric and symmetric digraphs whose indices exceed n.*

Our proof is based on a specialization of Theorem 2 in Folkman [4]. A *graph* (X, E) is a nonempty set X and a set E of unordered pairs $\{x, y\}$ with $x, y \in X$ and $x \neq y$. A *triangle of* (X, E) is a set $\{\{a, b\}, \{b, c\}, \{a, c\}\} \subseteqq E$. A *partition* of X is a set of mutually disjoint subsets of X whose union equals X.

LEMMA 1 (Folkman). *Let m be a positive integer. Then there is a graph (X, E) that includes no triangles, and every partition $\{C_1, \cdots, C_k\}$ of X with $k \leqq m$ contains a C_i such that $a, b \in C_i$ for some $\{a, b\} \in E$.*

Proof of Theorem 1. Let (X, E) be such a graph for $m = 2^n$. Let (X, R) be any digraph for which $xy \in R$ or $yx \in R$ if and only if $\{x, y\} \in E$. Suppose that R is the union of partial orders $P_1, \cdots, P_n$ on X. Since E has no triangles, any subset of a P_i is a partial order and hence we can assume $P_i \cap P_j = \varnothing$ when $i \neq j$. Letting $A(x) = \{i$: for some $y \in X, xy \in P_i\}$, partition X so that x and y are in the same element of the partition if and only if $A(x) = A(y)$. The number of elements in the partition does not exceed 2^n. Thus, by Lemma 1, the partition contains an element Y with $x, y \in Y$ and $\{x, y\} \in E$. Then $A(x) = A(y)$. Since $xy \in R$ or $yx \in R$, take $xy \in P_j$ for definiteness with $j \in A(x)$. Since $j \in A(y)$ also, there is a $z \in X$ such that $yz \in P_j$. Transitivity then implies that $xz \in P_j$ and hence that E includes a triangle, which contradicts our initial hypothesis. Therefore $\sigma(R) > n$. By the definition of R it can be taken to be either asymmetric or symmetric (or neither).

Henceforth in this section all logarithms are to base 2 unless indicated otherwise. $[r] =$ (largest integer $\leqq r$) and $\{r\} =$ (smallest integer $\geqq r$).

THEOREM 2. $\log n - 1/2 \log\log n + 0(1) \geqq s(n) \geqq \log n - 3/2 \log\log n - 0(1)$.

We show first the upper bound, using two preparatory lemmas.

LEMMA 2. *In any digraph* $G = (H, R)$ *with* $|H| = m$ *there exists* $D \subseteqq H$ *such that* $|D| \geqq \{\log_4 m\} = \{1/2 \log m)$ *and* $\sigma(G|D) \leqq 2$.

Proof. We use induction on m, the lemma being obvious for small values of m. Fix $x \in H$. Split $H^* = H - \{x\}$ into four parts:

$$T_1 = \{y \in H^*: xy \notin R \ \&\ yx \notin R\} \qquad S_1 = \varnothing$$

$$T_2 = \{y \in H^*: xy \in R \ \&\ yx \notin R\} \qquad S_2 = \{x\} \times D_2$$

$$T_3 = \{y \in H^*: xy \notin R \ \&\ yx \in R\} \qquad S_3 = D_3 \times \{x\}$$

$$T_4 = \{y \in H^*: xy \in R \ \&\ yx \in R\} \qquad S'_4 = \{x\} \times D_4\,, \quad S''_4 = D_4 \times \{x\}\,.$$

Some $|T_i| \geqq \{(m-1)/4\}$. By induction find $D_i \subseteqq T_i$ with

$$|D_i| \geqq \{\log_4 |T_i|\} \geqq \{\log_4 \{(m-1)/4\}\} = \{\log_4 m\} - 1$$

and $G|D_i = P_1 \cup P_2$. Then set $D = D_i \cup \{x\}$. $G|D = (P_1 \cup S_i) \cup (P_2 \cup S_i)$ except for $i = 4$ when $G|D = (P_1 \cup S'_4) \cup (P_2 \cup S''_4)$.

LEMMA 3. *In any digraph* $G = (X, R)$ *with* $|X| = n$ *there is a partition* $\{D_1, \cdots, D_t\}$ *of* X *such that* $t < 3n/\log n$ *and* $\sigma(G|D_i) \leqq 2$ *for each* i.

Proof. Given G, by Lemma 2 find D_1 such that

$$|D_1| = x_1 \geqq \{\log_4 n\}\,.$$

By induction find D_i such that

$$|D_i| = x_i \geqq \left\{\log_4\left(n - \sum_{j=1}^{i-1} x_j\right)\right\}\,.$$

From elementary calculus we can show $\sum_{i=1}^{t} x_i \geqq n$ for

$$t \leqq (2+\varepsilon)n/\log n\,.$$

We now show the upper bound for Theorem 2. Let $G = (X, R)$ with $|X| = n$. Take $D_1, \cdots, D_t$ as in Lemma 3. Let $\{A_i^*, B_i^*\}$ be a partition of $\{1, \cdots, t\}$ for $i = 1, \cdots, s$ such that for all $1 \leqq j \neq k \leqq t$ there exists i, $1 \leqq i \leqq s$, such that $j \in A_i^*$ & $k \in B_i^*$. By Spencer [12] we may take

$$s = \log t + 1/2 \log\log t + 0(1) \leqq \log n - 1/2 \log\log n + 0(1)\,.$$

$\{A_i^*, B_i^*\}$ induces a partition $\{A_i, B_i\}$ of X with

$$A_i = \bigcup_{j \in A_i^*} D_j\,, \qquad B_i = \bigcup_{j \in B_i^*} D_j\,.$$

Then set

$$P_i = \{xy\colon x \in A_i \ \&\ y \in B_i \ \&\ xy \in R\} \quad \text{for } i = 1, \cdots, s\,.$$

Since $\sigma(G \mid D_i) \leqq 2$, $G \mid D_i = P_i' \cup P_i''$. Set

$$P' = \bigcup_{i=1}^{s} P_i',\ P'' = \bigcup_{i=1}^{s} P_i''\,.$$

Then $R = P' \cup P'' \cup P_1 \cup \cdots \cup P_s$, giving the upper bound of Theorem 2.

We turn to the lower bound of the theorem, again using two preliminary lemmas. A complete asymmetric digraph is a *tournament*.[4] We shall show that a "random" tournament $T = (X, R)$ with $|X| = n$ has $\sigma(T) \geqq \log n - 3/2 \log\log n - 0(1)$. Intuitively speaking, we show that all $P \subseteqq T$ are essentially bipartite.

Let T^n be the set of tournaments with $X = \{1, 2, \cdots, n\}$. We say that $T = (X, R) \in T^n$ has *property* α if and only if there are A, $B \subseteqq X$ with $|A| = |B| \geqq 3 \log n$ and $A \times B \subseteqq R$. T has *property* β if and only if there is an $A \subseteqq X$ and a linear order L on A such that $|A| \geqq (\log n)^2$ and

$$(*) \qquad |R \cap L| \leqq \frac{1}{3}\binom{|A|}{2}\,.$$

LEMMA 4. *For n sufficiently large there exists $T \in T^n$ satisfying neither property α nor property β.*

Proof. If $T \in T^n$ has property α, there are $A, B \subseteqq X$ with $|A| = |B| = [3 \log n]$ and $A \times B \subseteqq R$. Set $t = [3 \log n]$. For fixed A and B, 2^{-t^2} is the proportion of $T \in T^n$ that satisfy this condition. There are less than n^{2t} choices of A and B, so less than $n^{2t}2^{-t^2}$ of the $T \in T^n$ satisfy α. $n^{2t}2^{-t^2} \rightarrow 0$ as $n \rightarrow \infty$.

If $T \in T^n$ has property β, there exists $A \subseteqq X$ and L on A such that $|A| = [(\log n)^2]$ and (*) holds. There are less than $n^{(\log n)^2}$ choices of A and then $[(\log n)^2]!$ choices of L. Given A and L, the proportion of $T \in T^n$ satisfying (*) is the probability of at most $\binom{t}{2}/3$ heads in $\binom{t}{2}$ flips of a fair coin where $t = |A| \sim (\log n)^2$. This probability is approximately $p^{-\binom{t}{2}}$ where $p = 3^{1/3}\ (3/2)^{2/3} > 1$. Thus the proportion of $T \in T^n$ satisfying β is less than

$$n^{(\log n)^2}\,[(\log n)^2]!\ p^{-\binom{t}{2}},\ \text{which} \longrightarrow 0 \text{ as } n \longrightarrow \infty\,.$$

Thus for n sufficiently large some $T \in T^n$ can satisfy neither α nor β.

[4] See Moon [**9**] for extensive discussion of tournaments. See also [**3**, **10**, **11**] for resulted to the present paper.

LEMMA 5. *If* $T_1, \cdots, T_n \subseteqq \{1, \cdots, s\}$ *then there are* $n/\binom{s}{s/2}$ T_i *which are mutually comparable.*[5]

Proof. We use a technique due to Lubell [8]. There are $s!$ maximal chains of subsets of $\{1, \cdots, s\}$ under the ordering of $\subset$. If $|T_i| = a$ then T_i is in $a!\ (s-a)! \geqq (s/2)!^2 = s!/\binom{s}{s/2}$ maximal chains. Thus some maximal chain must contain $n[s!/\binom{s}{s/2}]/s!$ T_i.

In the following proof of the lower bound of Theorem 2 we use the fact that $1/\binom{s}{s/2} \sim \sqrt{\pi/2}\ \sqrt{s}\ 2^{-s}$.

Let $G = (X, R)$ be a tournament that satisfies neither α nor β (Lemma 4). Suppose that $R = P_1 \cup \cdots \cup P_s$. Define

$$\begin{aligned} W_i &= \{x \in X: |\{y \in X: xy \in P_i\}| > 3 \log n\} \\ L_i &= \{x \in X: |\{y \in X: yx \in P_i\}| > 3 \log n\} \\ R_i &= X - W_i - L_i \end{aligned}$$

for $1 \leqq i \leqq s$. (We split X into winners, losers, and the rest.) By Lemma 4, $W_i \cap L_i = \varnothing$. For $x \in X$ set

$$T_x = \{i: x \in W_i \cup R_i\} \subseteqq \{1, \cdots, s\}\ .$$

By Lemma 5 find $V \subseteqq X$ such that $|V| \geqq n\sqrt{\pi/2}\sqrt{s}\ 2^{-s}$ and $T_x \subseteqq T_y$ or $T_y \subseteqq T_x$ whenever $x, y \in V$. Induce a linear order L on V by setting $xy \in L$ if $T_x \subset T_y$: when $T_x = T_y$, L is defined in any fixed manner.

Now assume $s < \log n - 3/2 \log\log n - 7$. Then $|V| \geqq 2^7\sqrt{\pi/2}$ $(\log n)^2$. Set

$$Z_i = L \cap P_i \qquad 1 \leqq i \leqq s\ .$$

Given $xy \in Z_i$, $T_x \subseteqq T_y$ so that we cannot have $x \in W_i$ & $y \in L_i$. And since $W_i \cap L_i = \varnothing$ we cannot have $x \in L_i$ & $y \in W_i$. Therefore

$$Z_i = \{xy \in Z_i: x \text{ or } y \in R_i\} \cup \{xy \in Z_i: x, y \in W_i\} \cup \{xy \in Z_i: x, y \in L_i\}\ .$$

There are at most $6 \log n\ |V|$, $3 \log n\ |V|$ and $3 \log n\ |V|$ ordered pairs in the first, second and third parts respectively of this decomposition of Z_i. Thus $|Z_i| \leqq 12 \log n\ |V|$. Since G does not have property β it follows that

$$\frac{1}{3}\binom{|V|}{2} \leqq |R \cap L| \leqq \sum_{i=1}^{n} |Z_i| \leqq 12 (\log n)^2 |V|$$

and hence that $|V| \leqq 72 (\log n)^2 + 1$. Since this contradicts $|V| \geqq 2^7$ $\sqrt{\pi/2}(\log n)^2$ it must be true that $s \geqq \log n - 3/2 \log\log n - 0(1)$.

[5] T_i and T_j are mutually comparable if and only if $T_i \subseteq T_j$ or $T_j \subseteq T_i$.

This completes the proof of Theorem 2.

If a sufficiently good bound could be placed on

$$\{xy \in P_i\colon\ x \text{ or } y \in R_i \text{ or } x, y \in W_i \text{ or } x, y \in L_i\}$$

then one could prove $s(n) = \log n - 1/2 \log \log n + o\,(\log \log n)$. One might even show that $s(n) = \log n - 1/2 \log \log n + 0(1)$.

3. A digraph with $\sigma = 4$ and $|X| = 13$. Although the theorems of the preceding section show that there are digraphs with large indices, they are of little use in attempting to discover the smallest X that admits an R for which $\sigma(R) = n$. Figure 1 shows the smallest X that we know of for which $\sigma(R) = 4$.

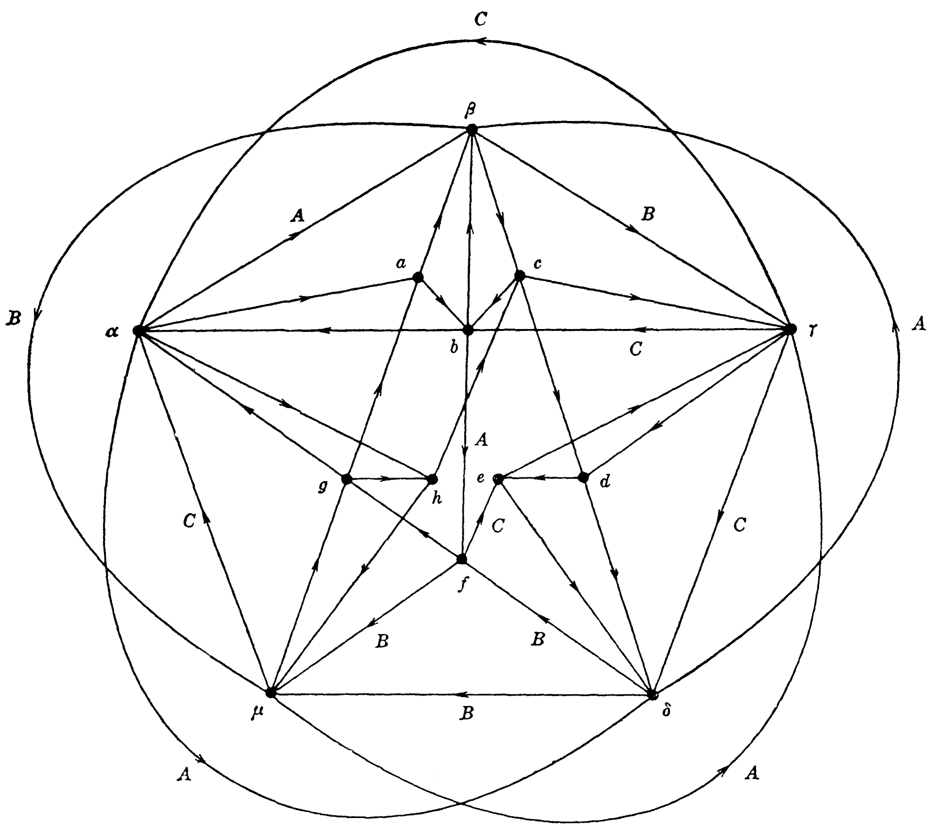

FIGURE 1

Assume that $\sigma(R) = 3$ for Figure 1, with A, B and C three partial orders whose union equals R. Then one of A, B and C must contain exactly one of $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\mu$ and $\mu\alpha$ and the other two must each contain exactly two of these ordered pairs in alternating fashion.

Suppose for example that $\alpha\beta \in A$, $\beta\gamma \in B$, $\gamma\delta \in C$, $\delta\mu \in B$, $\mu\alpha \in C$. Then $\gamma\alpha$, $\delta\beta$, $\mu\gamma$, $\alpha\delta$, and $\beta\mu$ must be respectively in C, A, A, A, and B. Then $\gamma b \in C$ and δf, $f\mu \in B$. Since $\gamma b \in C$ and $f\mu \in B$, $bf \in A$. Since $bf \in A$ and $\delta f \in B$, $fe \in C$. By the cyclic triangle $\{fe, e\delta, \delta f\}$, $e\delta$ must be in A. But since $\delta\beta \in A$ this implies $e\beta \in A$, which is false. A similar contradiction to $\sigma = 3$ is obtained when any alternative assignment is made for $\alpha\beta, \beta\gamma, \cdots, \mu\alpha$.

4. Indices of symmetric digraphs.

In this section we consider symmetric ($xy \in S \Rightarrow yx \in S$) digraphs (X, S). For any binary relation R, $R^* = \{xy: yx \in R\}$, the *converse* or *dual of* R.

A graph (X, E) is a *comparability graph* if and only if there is a partial order P on X such that $\{x, y\} \in E$ if and only if $xy \in P \cup P^*$. Ghouila-Houri [5] and Gilmore and Hoffman [6] provide characterizations of comparability graphs. When (X, S) is a symmetric digraph, $(X, E(S))$ will denote the graph in which $\{x, y\} \in E(S)$ if and only if $xy \in S$.

THEOREM 3. *Suppose that (X, S) is a symmetric digraph. Then $\sigma(S) \leqq 2$ if and only if $(X, E(S))$ is a comparability graph.*

Proof. If $(X, E(S))$ is a comparability graph then $S = P \cup P^*$ for a partial order P, and thus $\sigma(S) \leqq 2$. Conversely, if $S = P_1 \cup P_2$ with P_1 and P_2 partial orders, then $P_2 = P_1^*$.

In [1] it is shown that if (X, P) is a transitive digraph (so that P is a partial order) and if $S = \{xy: x \neq y \ \& \ xy \notin P \cup P^*\}$ then $D(P) \leqq 2$ if and only if $(X, E(S))$ is a comparability graph. Hence, as a corollary to Theorem 3 we have $D(P) \leqq 2$ if and only if $\sigma(S) \leqq 2$. Our next theorem extends this in one direction.

THEOREM 4. *Suppose that P on X is a partial order and let $S = \{xy: x \neq y \ \& \ xy \notin P \cup P^*\}$. Then $D(P) \leqq n \Rightarrow \sigma(S) \leqq 2(n-1)$ for $n > 1$.*

Proof. The theorem is true for $n = 2$. Using induction, assume it's true for all $n < m$ and suppose $D(P) = m$ with $P = \bigcap_1^m L_i$ where each L_i is a linear order. Let $P' = \bigcap_2^m L_i$ and

$$S' = \{xy: x \neq y \ \& \ xy \notin P' \cup (P')^*\} .$$

Since $D(P') \leqq m - 1$, the induction hypothesis gives $\sigma(S') \leqq 2(m-2)$. Clearly $S' \subseteqq S$ and $S - S' = (P' \cap L_1^*) \cup ((P')^* \cap L_1)$. Since $P' \cap L_1^*$ is a partial order (the intersection of two partial orders) and $(P')^* \cap L_1$ is a partial order, $\sigma(S) \leqq \sigma(S') + 2 \leqq 2(m-2) + 2 = 2(m-1)$.

5. Almost transitive digraphs. The proof of the next theorem has several similarities to Szpilrajn's proof [13] of the theorem that any partial order P on X can be extended to a linear order L with $P \subseteqq L$. We recall that R is almost transitive if and only if $(ab \in R$ & $bc \in R$ & $a \neq c) \Rightarrow ac \in R$.

THEOREM 5. *$\sigma(R) \leqq 2$ if (X, R) is an almost transitive digraph.*

Proof. Assume that (X, R) is an almost transitive digraph. Let $A = \{ab\colon ab \in R \ \& \ ba \notin R\}$, the asymmetric part of R. Let $A^+ = \{ab\colon ab \in A$ or $\{aa_1, a_1a_2, \cdots, a_nb\} \subseteqq A$ for distinct $a_1, \cdots, a_n$ in X that are different from a and $b\}$, the almost transitive closure of A. Clearly $A^+ \subseteqq R$ and A^+ is almost transitive.

To show that A^+ is a partial order it suffices to show that it is asymmetric. To the contrary suppose that $xy \in A^+$ and $yx \in A^+$. Then from the definition of A^+ and almost transitivity for R it follows easily that there is a $c \in X$ for which $cx \in A$ and $xc \in R$, which contradicts the definition of A. Hence A^+ is a partial order.

Let $\mathscr{P} = \{P\colon P$ is a partial order on X & $A^+ \subseteqq P \subseteqq R\}$. It follows easily from Zorn's lemma that there is a $P^* \in \mathscr{P}$ such that $P^* \subset P$ for no $P \in \mathscr{P}$. Letting P^* be maximal in this sense we now prove that

$$ab, ba \in R \Rightarrow ab \in P^* \text{ or } ba \in P^* .$$

To the contrary suppose that each of ab and ba is in R and neither is in P^*. Then let

$$W = \{xy\colon x \neq y \ \& \ (xa \in P^* \text{ or } x = a) \ \& \ (by \in P^* \text{ or } y = b)\} ,$$

and let $V = P^* \cup W$, so that $P^* \subset V$. We show that V is a partial order (clearly $A^+ \subseteqq V \subseteqq R$), thus contradicting the maximality of P^*. V is irreflexive since P^* and W are irreflexive. For transitivity take $xy, yz \in V$. If both xy and yz are in P^* then $xz \in P^*$ by the transitivity of P^*.

Suppose next that $xy \in P^*$ and $yz \in W$. The latter gives $(ya \in P^*$ or $y = a)$, from which $xa \in P^*$ follows, and it gives also $(bz \in P^*$ or $z = b)$, from which $xz \in V$ follows unless $x = z$. But if $x = z$ we have $xa \in P^*$ and $(bx \in P^*$ or $x = b)$, which give $ba \in P^*$, contradicting the hypothesis that $ba \notin P^*$. Hence $xy \in P^*$ & $yz \in W \Rightarrow xz \in V$. Similarly, $xy \in W$ & $yz \in P^* \Rightarrow xz \in V$.

The final case for transitivity is $xy, yz \in W$. Then $(xa \in P^*$ or $x = a)$ and $(bz \in P^*$ or $z = b)$ so that $xz \in W$ unless $x = z$. But if $x = z$ then $[(xa \in P^*$ or $x = a)$ & $(bx \in P^*$ *or* $x = b)] \Rightarrow (ba \in P^*$ or $b = a)$, which is false. Hence V is a partial order, a contradiction to the maximality of P^*, and therefore

$$ab,\ ba \in R \Longrightarrow ab \in P^* \text{ or } ba \in P^* .$$

Finally, let $Q = R - P^*$ so that $R = P^* \cup Q$. Q is irreflexive since R is irreflexive. Suppose that $xy, yz \in Q$. Then, since both xy and yz are in R but not A, yx and zy are in R and must be in P^* by the preceding analysis. Therefore $zx \in P^*$ and $z \neq x$. Then, by almost transitivity of R, $xz \in R$ and thus $xz \in Q$ since P^* is asymmetric.

Thus $R = P^* \cup Q$, the union of two partial orders.

6. A partition characterization for $\sigma \leqq 2$. Given a digraph (X, R) let K be the set of all ordered pairs of pairs in R that deny transitivity, so that

$$xyKyz \text{ if and only if } xy \in R \ \&\ yz \in R \ \&\ xz \notin R ,$$

and let V be the subset of R involved in these intransitivities so that

$$V = \{xy\colon xyKyz \text{ or } zxKxy \text{ for some } z \in X\} .$$

Suppose that $\sigma(R) \leqq 2$. If $xyKyz$ then xy and yz must be in different resolving partial orders, so that the digraph (V, K) must be bipartite or 2-colorable. Moreover, if xy and yz are in V and in the same resolving partial order and if $xz \in V$ also, then transitivity requires that xz be in this partial order. These two necessary conditions for $\sigma(R) \leqq 2$ are reflected in A1 and A2 of Theorem 6. Their insufficiency for $\sigma(R) \leqq 2$ is noted later. (Note that $\sigma(R) = 1$ if and only if $V = \varnothing$.)

THEOREM 6. *Suppose that (X, R) is a digraph and $V \neq \varnothing$. Then $\sigma(R) = 2$ if and only if V can be partitioned into V_1 and V_2 so that*

A1. *$xyKyz \Rightarrow xy$ and yz are in different V_i ,*

A2. *$xy, yz \in V_i$ & $xz \in V \Rightarrow xz \in V_i$,*

A3. *$xy \in R - V \Rightarrow$ (1) and (2) do not hold simultaneously:*

(1) $(yz \in V_2 \ \&\ xz \in V_1)$ *or* $(zx \in V_2 \ \&\ zy \in V_1)$, *for some* $z \in X$,

(2) $(yw \in V_1 \ \&\ xw \in V_2)$ *or* $(wx \in V_1 \ \&\ wy \in V_2)$, *for some* $w \in X$.

If $R = P_1 \cup P_2$ then $V_i = P_i \cap V$ for $i = 1, 2$ are easily seen to satisfy A1 through A3, and $V_1 \cap V_2 = \varnothing$.

Before proving sufficiency we show that A1 and A2 are not sufficient for $\sigma = 2$. All directed edges in the 13-point asymmetric

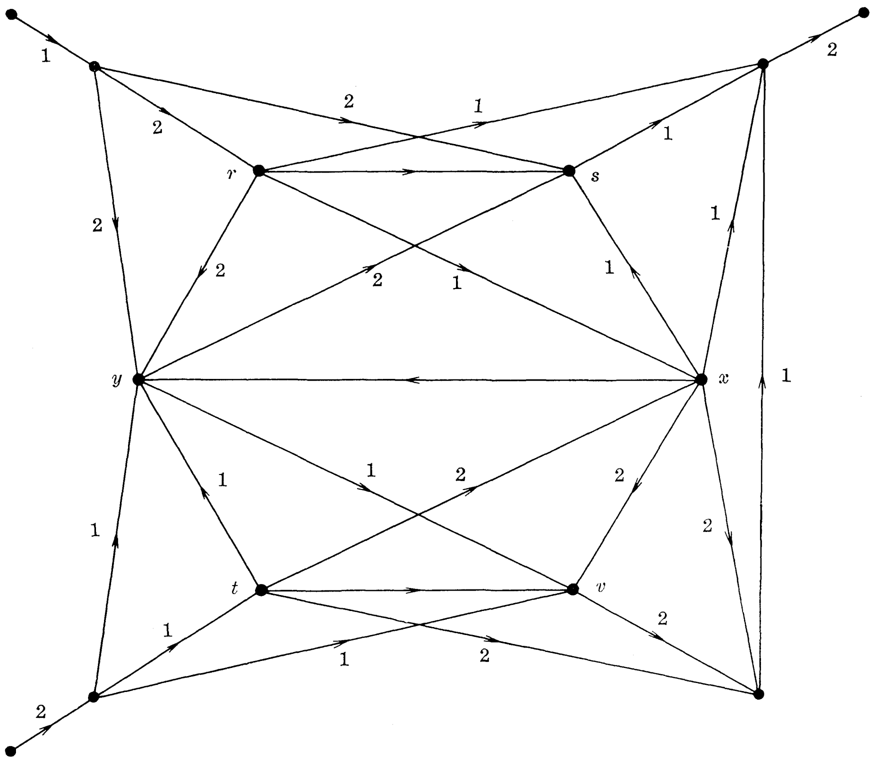

FIGURE 2

digraph of Figure 2 are in V except for xy, rs and tv, and A1 and A2 hold. Labels 1 and 2 for P_1 and P_2 are assigned to the edges in V in the only way consistent with A1 and A2, beginning with P_1 in the upper left corner. For $\sigma(R) = 2$ we require rs and tv in both P_1 and P_2, but xy violates A3 and cannot be assigned either

$$P_1\,[rx \in P_1 \;\&\; ry \notin P_1] \text{ or } P_2\,[tx \in P_2 \;\&\; ty \notin P_2]\;.$$

By deleting the edge xy from Figure 2 we obtain an R with $\sigma(R) = 2$ where R is not the union of two *disjoint* partial orders.

Sufficiency Proof for Theorem 6. With $V \neq \emptyset$ let A1, A2 and A3 hold. For $i = 1, 2$ let

$$S_i = \{xy\colon xy \in R - V \;\&\; \text{(i) holds}\}\;.$$

Let $R^0 = R - V - S_1 - S_2$ and for $i = 1, 2$ define P_i by

$$P_i = V_i \cup S_i \cup R^0\;.$$

Since $P_i \subseteqq R$, it is irreflexive. We now prove that P_1 is transitive. The proof for P_2 is similar.

Assume that $xy, yz \in P_1$. Then $xz \in R$, for if both xy and yz are in V_1 then $xz \in R$ by A1, and if one of xy and yz is in $S_1 \cup R^0$ then $xz \in R$ by the definitions. Thus $xz \in P_1$ unless $xz \in V_2 \cup S_2$. $xz \in V_2$ is contradicted in all cases:

1. $xy, yz \in V_1 \Rightarrow xz \notin V_2$, by A2;
2. $xy \in V_1$ & $yz \in S_1 \Rightarrow xz \notin V_2$, by A3;
3. $xy \in V_1$ & $yz \in R^0 \Rightarrow xz \notin V_2$, by A3;
4. $xy, yz \in S_1 \cup R^0$. Then $ax \in R \Rightarrow ay \in R \Rightarrow az \in R$ and $za \in R \Rightarrow ya \in R \Rightarrow xa \in R$. Hence neither $axKxz$ nor $xzKza$ can hold.

It remains to show that $xz \notin S_2$. Assume $xz \in S_2$ to the contrary and for definiteness take $zw \in V_1$ and $xw \in V_2$ (Figure 3). We note first

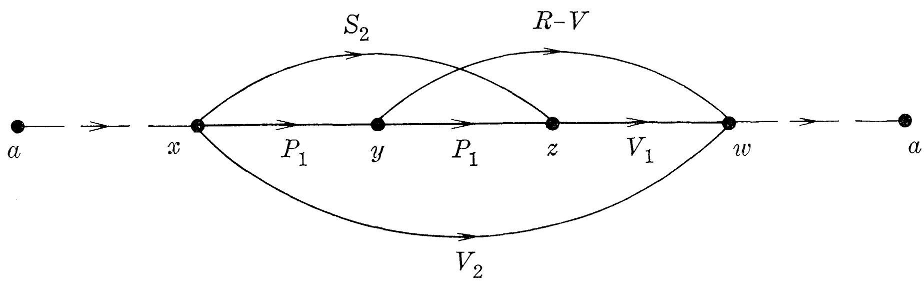

FIGURE 3

that $yw \notin V_2$, for $yw \in V_2 \Rightarrow yz \in S_2$. Moreover, $yw \notin V_1$, for $yw \in V_1$ & $xy \in V_1$ contradict A2, and $yw \in V_1$ & $xy \in S_1 \cup R^0$ contradict the definition of S_2 along with A3. Hence $yw \in R - V$. Now if $ax \in V_1$ then $ay \in R$ and hence (since $yw \in R - V$) $aw \in R$; and if $wa \in V_1$ then $za \in R$ and hence (since $xz \in R - V$) $xa \in R$. Since $xw \in V_2$ requires either $axKxw$ with $ax \in V_1$ or $xwKwa$ with $wa \in V_1$, and since $ax \in V_1$ contradicts $axKxw$ (since $aw \in R$) and $wa \in V_1$ contradicts $xwKwa$ (since $xa \in R$), the proof is complete.

REFERENCES

1. K. A. Baker, P. C. Fishburn, and F. S. Roberts, *Partial Orders of Dimension 2, Interval Orders, and Interval Graphs*, RAND Paper P-4376, The RAND Corporation, 1970.
2. B. Dushnik and E. W. Miller, *Partially Ordered Sets*, Amer. J. Math., **63** (1941), 600-610.
3. P. Erdös and L. Moser, *On the Representation of Directed Graphs as Unions of Orderings*, Math. Inst. Hung. Acad. Sci., **9** (1964), 125-132.
4. J. Folkman, *Graphs with Monochromatic Complete Subgraphs in Every Edge Coloring*, SIAM J. Appl. Math., **18** (1970), 19-24.
5. A. Ghouila-Houri, *Caractérisation des graphes nonorientés dont on peut orienter les arêtes de manière à obtenir le graphe d'une relation d'ordre*, C. R. Acad. Sci. Paris **254** (1962), 1370-1371.

6. P. C. Gilmore and A. J. Hoffman, *A Characterization of Comparability Graphs and of Interval Graphs*, Canad. J. Math., **16** (1964), 539-548.
7. F. Harary, R. Z. Norman, and D. Cartwright, *Structural Models: An Introduction to the Theory of Directed Graphs*, Wiley, New York, 1965.
8. D. Lubell, *A Short Proof of Sperner's Lemma*, J. Combinatorial Theory, **1** (1966), 299.
9. J. W. Moon, *Topics on Tournaments*, Holt, New York, 1969.
10. K. B. Reid and E. T. Parker, *Disproof of a conjecture of Erdös and Moser on tournaments*, J. Combinatorial Theory, **9** (1970), 225-238..
11. J. Spencer, *Notes on Combinatorial Mathematics: Transitive Subtournaments*, RAND Memorandum RM-6156-PR, The RAND Corporation, 1969.
12. ———, *Minimal Completely Separating Systems*, J. Combinatorial Theory, **8** (1970), 446-447.
13. E. Szpilrajn, *Sur extension de l'ordre partiel.* Fundamenta Math., **16** (1930), 386-389.

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