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AN EXTENSION OF SOME RESULTS OF TAKESAKI IN THE REDUCTION THEORY OF VON NEUMANN ALGEBRAS

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Briefly, the results in this paper are that both for measurable fields of von Neumann algebras and for families of measurable fields of operators, pointwise isomorphism implies isomorphism.

In the special case when half the measurable fields considered are constant, these results were established by Takesaki. If the Borel space on which the fields are defined is standard, the results can be established by classical means; in the case considered by Takesaki they are due to von Neumann.

For the results of the present paper, two new tools seem to be needed. The first is a measurable choice theorem of Aumann which generalizes the classical one. This has already been applied to reduction theory by Flensted-Jensen. The second is a criterion for a von Neumann algebra containing the diagonal operators to be decomposable: it should consist of decomposable operators. This answers a question of Dixmier.

We shall use the terminology of reduction theory developed in [2], Chapitre II.

2. LEMMA (Aumann). *Let T be a Borel space and let X be a standard Borel space. Let G be a Borel subset of $T \times Y$ such that the projection of G onto T is all of T. Let there be given a finite measure on T. Then there exists a measurable map $g: T \to X$ such that $(t, g(t)) \in G$ for almost all $t \in T$.*

Proof. See [1]. The proof is by reduction to the case that T is standard.

3. THEOREM. *Let T be a Borel space, and suppose given a finite measure on T and a measurable field of Hilbert spaces on T with direct integral H. Let A and B be decomposable von Neumann algebras in H. If for each $t \in T$ there is a spatial isomorphism of $A(t)$ onto $B(t)$ then there exists a decomposable spatial isomorphism of A onto B. This statement also holds with the word "spatial" removed.*

Proof. The proof of the first assertion is the same as the proof of Lemma 2 on page 179 of [2], with the exception that 2 above is used instead of the more well known measurable choice theorem for standard measures.

The second assertion is reduced to the first by tensoring with the scalars on a separable infinite dimensional Hilbert space, just as in Theorem 3 of [4].

4. LEMMA. *Let T be a Borel space, and suppose given a finite measure on T and a measurable field of Hilbert spaces on T with direct integral H. Then a von Neumann algebra in H containing the diagonal operators is decomposable if it consists of decomposable operators.*

This answers affirmatively the question on page 174 of [2].

Proof. We may suppose that the field of Hilbert spaces is constant. By [2], page 178, Corollaire, it then follows that the algebra of all decomposable operators is spatially isomorphic to $Z \otimes B$ with Z a commutative algebra and B the algebra of all operators on a separable Hilbert space. Since the algebra of diagonal operators is countably decomposable (the measure of T is finite), so is Z; therefore both Z and B and hence also $Z \otimes B$ have a countable separating set of vectors.

Let $\xi_1, \xi_2, \cdots$ be a countable separating set of vectors for the algebra of all decomposable operators, such that $\Sigma \|\xi_i\|^2 < \infty$. Then for any operator x we have $(\Sigma \|x\xi_i\|^2)^{1/2} < \infty$. On the algebra of decomposable operators this expression defines a norm, which on bounded sets determines the strong topology. We shall denote this norm by N.

Let A be a von Neumann algebra containing the algebra of diagonal operators, and consisting of decomposable operators. To show that A is decomposable, we must show that A is countably generated over the algebra of diagonal operators ([2], page 174, Théorème 2). Writing as before the algebra of decomposable operators as $Z \otimes B$ with Z commutative and B a type I_n factor, n countable, let $x_1, x_2, \cdots$ be a sequence strongly dense in the unit ball of $1 \otimes B$. For each $k = 1, 2, \cdots$ let y_k be an element of A which is at minimal distance from x_k with respect to the norm N of the preceding paragraph (such y_k exists because bounded weakly closed sets of A are weakly compact, and N is weakly lower semicontinuous). Then $y_1, y_2, \cdots$ generate A over the algebra of diagonal operators. For if e is a diagonal projection then for each $k = 1, 2, \cdots$ the distance from ey_k to ex_k with respect to N is minimal. Hence, if $e_1, \cdots, e_p$ are diagonal projections with sum 1, and if $k_1, \cdots, k_p = 1, 2, \cdots$ then the distance of $e_1y_{k_1} + \cdots + e_py_{k_p}$ to $e_1x_{k_1} + \cdots + e_px_{k_p}$ with respect to N is minimal. The assertion follows, because the operators $e_1x_{k_1} + \cdots + e_px_{k_p}$ as above are strongly dense in the unit ball of decomposable operators, and the strong topology on this unit ball is metrized by N.

5. THEOREM. *Let T be a Borel space, and suppose given a finite measure on T and a measurable field of Hilbert spaces on T with direct integral H. Let (x_i) and (y_i) be families of decomposable operators in H. If for each $t \in T$ the families $(x_i(t))$ and $(y_i(t))$ are simultaneously unitarily equivalent, then (x_i) and (y_i) are simultaneously unitarily equivalent, with the equivalence implemented by a decomposable unitary operator.*

Proof. Suppose first that the families (x_i) and (y_i) are countable. Then the conclusion may be deduced as in A 82, page 348 of [3], using again 2 instead of the classical measurable choice theorem.

If the families (x_i) and (y_i) are not countable, by 4 it is still true that the von Neumann algebra A generated by the x_i and the diagonal operators is decomposable. It follows that there exists a countable family $w_1, w_2, \cdots$ in A generating A over the diagonal operators. We may suppose that the x_i form a sub involutive algebra, containing the diagonal operators. Then the x_i are strongly dense in A, and the x_i of norm $\leqq 1$ are strongly dense in the unit ball of A. As shown in the proof of 4, the strong topology on the unit ball of A is metrizable. We may suppose that $w_1, w_2, \cdots$ lie in the unit ball of A. Then there exists a countable subfamily of (x_i) which generates A over the diagonal operators (namely, the union of sequences converging strongly to each w_k, $k = 1, 2, \cdots$). By the first paragraph of the proof this countable subfamily of (x_i) is simultaneously unitarily equivalent to the corresponding subfamily of (y_i), by a decomposable unitary operator, say v.

We claim that $vx_iv^* = y_i$ for every i. The subfamily of (x_i) such that $vx_iv^* = y_i$ contains the diagonal operators and also a set (the above countable subfamily) which generates A over the diagonal operators. It therefore contains a sub involutive algebra dense in A. By metrizability of the strong topology on bounded sets, this subfamily is closed under strong limits (use Proposition 4, page 160 of [2]). It follows that $vx_iv^* = y_i$ for all i.

6. REMARKS. Once 5 has been reduced by use of 4 to the case that the families are countable, the proof can also be finished by a variant of the method of Takesaki, in [4] (in which not just a measur-

able choice but a Borel choice is made).

On the other hand, although Takesaki was able to prove his special case of 3 by a Borel choice argument, the author does not see how to extend this approach and was forced to be content in the proof of 3 with making a measurable choice.

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