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INTEGRAL REPRESENTATION OF EXCESSIVE FUNCTIONS OF A MARKOV PROCESS

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Let X_t be a standard Markov process on a locally compact separable metric space E having a Radon reference measure. Let $\mathscr{S}$ denote the set of locally integrable excessive functions of X_t and $ex\mathscr{S}$ the set of elements lying on the extremal rays of $\mathscr{S}$. Then if $u \in ex\mathscr{S}$ is not harmonic, it is shown that there is an $x \in E$ such that $P_V u = u$ for all neighborhoods V of x where P_V is the hitting operator of V. A regularity condition is introduced which guarantees that two functions in $\mathscr{S}$ having the above property at x are proportional. A subset $\hat{E} \subset E$ and a metric topology on $\hat{E}$ are defined which allows one to represent each potential $p \in \mathscr{S}$ in the form $p(x) = \int u(x, y)\nu(dy)$ for some finite Borel measure $\nu \geqq 0$ on $\hat{E}$. Here the function $u: E \times \hat{E} \rightarrow [0, \infty]$ is measurable with respect to the product Borel field and has the property that for each $y \in \hat{E}$ the function $x \rightarrow u(x, y)$ is an extremal excessive function. In the course of this study a dual potential operator is introduced and some of its properties are investigated.

In § 2 we introduce the notation and assumptions which will be assumed to hold throughout the paper. Section 3 begins our study of $ex\mathscr{S}$ and using a result of Meyer [7] we show that to each function $u \in ex\mathscr{S}$ which is not harmonic we can associate a point $x \in E$ such that $P_V u = u$ for all open neighborhoods V of x. Here P_V is the hitting operator associated with V. We then say that u has support at x in analogy to the property introduced in axiomatic potential theory by Hervé [4]. We then discuss the axiom of proportionality, i.e., when is it true that if $u_1, u_2 \in ex\mathscr{S}$ have support at x, it follows that $u_1 = \alpha u_2$ for some $\alpha \geqq 0$. Some conditions are given which guarantee this property.

In § 4 we begin the discussion of representation of elements of $\mathscr{S}$. A uniform integrability condition on $\mathscr{S}$ is imposed and we define a suitable compact, convex set $\mathscr{K}$ in $\mathscr{S}$. Using the Choquet theorem and the characterization of $ex\mathscr{S}$ established in § 3, we define a subset $\hat{E} \subset E$ and a metric topology on $\hat{E}$ which allows us to represent each potential $p \in \mathscr{K}$ in the form $p(x) = \int u(x, y)\nu(dy)$ for some Borel measure $\nu \geqq 0$ on $\hat{E}$. Here $u: E \times \hat{E} \rightarrow [0, \infty]$ is a function measurable with respect to the product Borel field on $E \times \hat{E}$ and having the property that the function $x \rightarrow u(x, y)$ is an extremal excessive function for each $y \in \hat{E}$.

In § 5 the dual operator $\hat{U}$ is introduced, defined for a continious function on E with compact support by $\hat{U}f(y) = \int f(x)u(x, y)dx$. Some properties of $\hat{U}$ are investigated, and the integral representation is then extended to all potentials $p \in \mathscr{S}$.

2. Preliminaries and notation. The primary reference for the material in this paper will be Blumenthal and Getoor [2], and most of the notation will be taken from that book. Let therefore E be a locally compact separable metric space, and write $E_\Delta = E \cup \{\Delta\}$ where Δ is the point at infinity if E is not compact and an isolated point otherwise. We denote by $\mathscr{B}(E)$ and $\mathscr{B}(E_\Delta)$ the Borel sets of E and E_Δ respectively. Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a standard process with state space $(E, \mathscr{B}(E))$. Thus $X_t: \Omega \to E_\Delta$ for each t, $0 \leqq t \leqq \infty$, such that $X_s(\omega) = \Delta$ for all s $\geqq t$ if $X_t(\omega) = \Delta$. The path functions $t \to X_t(\omega)$, $\omega \in \Omega$, are right continuous on $[0, \infty)$ and have left-hand limits on $[0, \zeta)$ almost surely. Here $\zeta = \inf\{t: X_t = \Delta\}$ is the lifetime of X. The shift operators $\theta_t: \Omega \to \Omega$ are defined by $X_t \circ \theta_h = X_{t+h}$. For each $x \in E_\Delta$, P^x is a probability measure on the σ-algebra $\mathscr{F}$ such that $x \to P^x(\Lambda)$ is $\mathscr{B}(E_\Delta)$ measurable for each $\Lambda \in \mathscr{F}$ and $P^x(X_0 = x) = 1$. The reader is referred to [2] for the definitions of $\{\mathscr{F}_t\}$ and $\mathscr{F}$. Finally, X is assumed to be strong Markov and quasi-left continuous on $[0, \zeta)$.

If F is any topological space, we write $B(F)$ for the real-valued Borel measurable functions on F, and $bB(F)$ for the bounded elements of $B(F)$. If F is locally compact Hausdorff, $C_K(F)$ will denote the real-valued continuous functions on F with compact support. If L is any space of functions, L^+ will denote the nonnegative elements of L. If $f \in B(E)$ we extend f to E_Δ by setting $f(\Delta) = 0$.

We denote by P_t^α, $\alpha \geqq 0$, the α-transition operator so that $P_t^\alpha f(x) = e^{-\alpha t}E^x[f(X_t)]$ for $f \in bB(E)$. Set $P_t = P_t^0$. Our notation for the resolvent of the semi-group is $U^\alpha f(x) = \int_0^\infty P_t^\alpha f(x)dt = E^x\int_0^\infty e^{-\alpha t}f(X_t)dt$, and we put $U = U^0$, the potential operator. Recall that for $\alpha > 0$, $U^\alpha: bB(E) \to bB(E)$ is a bounded linear operator on the Banach space $bB(E)$ with the supremum norm, and $\| U^\alpha \| \leqq \alpha^{-1}$. If B is Borel, then $P_B^\alpha f(x) = E^x[e^{-\alpha T_B}f(X_{T_B}); T_B < \zeta]$ defines the α-hitting operators. Here $T_B = \inf\{t > 0; X_t \in B\}$ is the hitting time of B. Recall that if $B \in \mathscr{B}(E)$, then $B^r = \{x: P^x[T_B = 0] = 1\}$ is the set of regular points for B, and $B \cup B^r$ is the closure of B in the fine topology. Also if $D \in \mathscr{B}(E)$ and $D = D^r$ then for each $x \in E$ there is a decreasing sequence $\{G_n\}$ of open sets containing D such that $T_{G_n} \uparrow T_D$ a.s., P^x on $\{T_D < \infty\}$. A Borel set D for which $D = D^r$ is called finely perfect.

We let $\mathscr{S}^\alpha$ denote the α-excessive functions of X and set $\mathscr{S} =$

$\mathscr{S}^0$. Thus a nonnegative Borel function f is in $\mathscr{S}^\alpha$ if $P_t^\alpha f \leqq f$ for all $t \geqq 0$ and $P_t^\alpha f(x) \uparrow f(x)$ as $t \downarrow 0$ for all $x \in E$. Recall that the fine topology is the coarsest topology on E relative to which each $f \in \mathscr{S}^\alpha$ is continuous, $\alpha > 0$. Let $u \in \mathscr{S}$. Unless otherwise qualified, the statement $u = 0$ will mean that u is the zero function. Similarly, $u \neq 0$ will mean that u is not identically zero.

One basic assumption which will be assumed to hold throughout is the existence of a (Radon) reference measure. This is a Radon measure dx having the property that a set $B \in \mathscr{B}(E)$ is of potential zero, i.e., $U(x, B) = 0$ for all $x \in E$, if and only if $\int_B dx = 0$. This condition is satisfied if the elements in $\mathscr{S}^\alpha$ are lower semi-continuous for some $\alpha > 0$. If $f, g \in \mathscr{S}^\alpha$ and $f = g$ a.e., dx, then f and g are identical. Also, under this assumption each $f \in \mathscr{S}^\alpha$ is Borel measurable. An important situation where a reference measure exists is when there is a dual Markov process $\hat{X}_t$ as in Chapter VI of [2]. Here the resolvent kernel is of the form $U^\alpha f(x) = \int u^\alpha(x, y) f(y) \xi(dy)$ where $u^\alpha: E \times E \to \bar{R}^+$ is $\mathscr{B}(E) \times \mathscr{B}(E)$ measurable, $\xi(dy)$ is a Radon measure on E, and the function $x \to u^\alpha(x, y)$ is α-excessive for each $y \in E$, $\alpha \geqq 0$. Moreover, the resolvent of the dual process $\hat{X}_t$ is given by $\hat{U}^\alpha f(y) = \int u^\alpha(x, y) f(x) \xi(dx)$, and for each $x \in E$, the function $y \to u^\alpha(x, y)$ is α-excessive for $\hat{X}_t$. One can then define, analogous to X_t, a cofine topology for $\hat{X}_t$, and it turns out that the notion of semi-polar is equivalent in these two topologies. If D is Borel, then ${}^rD \backslash D^r$ is semi-polar, where ${}^rD$ denotes the set of points cofinely regular for D.

We make finally the following assumption on U: If f is a bounded Borel measurable function on E with compact support, then the function $x \to Uf(x)$ is finite. This condition is always satisfied by the operator U^α for $\alpha > 0$ and in fact the assumption is mainly a convenience that simplifies the notation. The reader can easily convince himself that all of the following results are true when stated in terms of α-potentials for $\alpha > 0$. Under this assumption each excessive function is the limit of an increasing sequence $\{Uf_n\}$ of finite potentials where each $f_n \geqq 0$ is in $B(E)$.

We fix once and for all a reference measure dx and, changing our notation slightly, we agree to denote by $\mathscr{S}$ the set of all excessive functions of X which are locally integrable with respect to dx. Now $\mathscr{S}$ is a convex, proper, pointed cone of functions on E and we denote by $ex\mathscr{S}$ the set of extreme rays of $\mathscr{S}$: $u \in ex\mathscr{S}$ if and only if for any representation of u in the form $u = u_1 + u_2$ with $u_1, u_2 \in \mathscr{S}$ it follows that $u_1 = \alpha u_2$ for some $\alpha \geqq 0$. We will draw heavily upon the following result found in Meyer [7, p. 59]:

THEOREM 2.1. *Let $\{u_n\}$ be a sequence of excessive functions. Then* *there is a subsequence* $\{u_{n'}\}$ *and an excessive function* u *such that* $u_{n'} \to u$ a.e., dx.

From now on all "almost everywhere (a.e.)" statements will be in reference to the measure dx.

3. Characterization of $ex\mathscr{S}$**.** We now want to give a characterization of the extremal rays of $\mathscr{S}$. For this we make the

DEFINITION 3.1. An excessive function $u \in \mathscr{S}$ is said to have support at $x \in E$ if for any open neighborhood V of x, $P_V u = u$. Also, u is said to be harmonic if $P_{K^c} u = u$ for all compact subsets $K \subset E$.

REMARK 3.2. If $u \in \mathscr{S}$ has a support at x, then u is harmonic in $E \backslash \{x\}$. In this connection, see Bauer [1, Chap. V].

We now prove

THEOREM 3.3. *Let* $u \in ex\mathscr{S}$. *Then if* u *is not harmonic,* u *has support at some* $x \in E$.

For the proof, we will need a series of lemmas.

LEMMA 3.4. *Let* $\{u_1^n\}$ *and* $\{u_2^n\}$ *be sequences of excessive functions in* $\mathscr{S}$ *such that* $u_1^n + u_2^n \to u$ *for some* $u \in \mathscr{S}$. *Then if* $u_1^n \to u_1$ *a.e., and* $u_2^n \to u_2$ *a.e., for* $u_1, u_2 \in \mathscr{S}$, *we have* $u_1^n \to u_1$ *and* $u_2^n \to u_2$ *on* $\{u < \infty\}$.

Proof. Of course $u = u_1 + u_2$ since they agree almost everywhere, hence everywhere. The important fact here is that if $v_n, v \in \mathscr{S}$ and $v_n \to v$ a.e., then $v \leqq \liminf v_n$ [*Proof*: We have by Fatou's lemma $\alpha U^\alpha(x, \liminf v_n) \leqq \liminf \alpha U^\alpha(x, v_n) \leqq \liminf v_n(x)$ for any $\alpha > 0$, so $\liminf v_n$ is super-median. If $\bar{v}$ is the excessive regularization of $\liminf v_n$, then $\bar{v} \leqq \liminf v_n$. But $\bar{v} = \liminf v_n$ a.e., and therefore $\bar{v} = v$ a.e., hence $\bar{v} = v$ everywhere so that $v \leqq \liminf v_n$]. Now if $u_1^n + u_2^n \to u = u_1 + u_2$, then on $\{u < \infty\}$, $A \equiv \{\limsup u_1^n > u_1\} \subset \{\liminf u_2^n < u_2\}$ since $x \in A$ and $u(x) < \infty$ implies there is a subsequence $\{n'\}$ such that $u_1^{n'}(x) \to \alpha > u_1(x)$ and hence $u_2^{n'}(x) \to \beta < u_2(x)$. Therefore $\liminf u_2^n(x) \leqq \liminf u_2^{n'}(x) < u_2(x)$. But $\{\liminf u_2^n < u_2\} = \phi$ by the above remark. Thus $A = \phi$ and for any $x \in E$ with $u(x) < \infty$ we have $\limsup u_1^n(x) \leqq u_1(x) \leqq \liminf u_1^n(x)$; therefore $u_1^n \to u_1$ and hence $u_2^n \to u_2$ on $\{u < \infty\}$.

LEMMA 3.5. *Suppose* $\{u_n\} \subset \mathscr{S}$ *and* $u_n \to \beta u$ *on* $\{u < \infty\}$ *with* $\beta > 0$ *and* $u_n \leqq u \in \mathscr{S}$ *for all* n. *Let* B *be Borel. Then if* $P_B u_n = u_n$ *for all* n, *we have* $P_B u = u$.

Proof. Since $u \in \mathcal{S}$ we always have $P_B u \leqq u$. To show $P_B u \geqq u$, consider a point $x \in E$ where $u(x) < \infty$. Then the measure $P_B(x, .)$ puts no mass on $\{u = \infty\}$. Since $u_n \leqq u$ for all n, the dominated convergence theorem implies $P_B(x, u_n) \to P_B(x, \beta u) = \beta P_B u(x)$. But $P_B(x, u_n) = u_n(x) \to \beta u(x)$ and since $\beta > 0$, $P_B u(x) = u(x)$. Hence $P_B u = u$ on $\{u < \infty\}$ and since $\{u = \infty\}$ has dx-measure zero, $P_B u = u$ everywhere.

LEMMA 3.6. *Suppose $u \in ex\mathcal{S}$ is not harmonic. Then there is a compact $K \subset E$ and a sequence $\{f_n\} \subset B^+(E)$ of Borel functions vanishing outside of K such that $Uf_n \leqq u$ for all n and $Uf_n \to u$ as $n \to \infty$ on $\{u < \infty\}$.*

Proof. Since $u \in ex\mathcal{S}$, there is a sequence $\{g_n\}$ of nonnegative Borel functions with $Ug_n \uparrow u$. Assume the conclusion is not true, and let $K \subset E$ be an arbitrary compact. Then $1 = I_K + I_{K^c}$ and hence $Ug_n = UI_K g_n + UI_{K^c} g_n \uparrow u$. Here I_B denotes the indicator function of B, for any $B \in \mathscr{B}(E)$. By Theorem (2.1) and Lemma (3.4) and the fact that $Ug_n \leqq u$ for all n, we can find a subsequence $\{n'\}$ and excessive functions $u_1, u_2 \in \mathcal{S}$ such that $UI_K g_{n'} \to u_1$ and $UI_{K^c} g_{n'} \to u_2$ on $\{u < \infty\}$ with $u = u_1 + u_2$. Since $u \in ex\mathcal{S}$, $u_2 = \beta u$ for some $\beta \geqq 0$. Now $\beta \neq 0$ since otherwise $UI_K g_{n'} \to u$ and $I_k g_{n'} = 0$ on K^c for all n'. Thus $UI_{K^c} g_{n'} \to \beta u$ on $\{u < \infty\}$ and $\beta > 0$. But for any $x \in E$,

$$P_{K^c} UI_{K^c} g_{n'}(x) = E^x \int_{T_{K^c}}^{\infty} I_{K^c}(X_t) g_{n'}(X_t) dt = E^x \int_0^{\infty} I_{K^c}(X_t) g_{n'}(X_t) dt$$
$$= UI_{K^c} g_{n'}(x) \ .$$

Hence Lemma (3.5) implies that $P_{K^c} u = u$. But K was an arbitrary compact and u is therefore harmonic, giving a contradiction.

Proof of Theorem (3.3). Suppose $u \in ex\mathcal{S}$ is not harmonic. Then by Lemma (3.6) we can find a compact $K \subset E$ and a sequence $\{f_n\} \subset B^+(E)$ with each f_n vanishing outside of K and $Uf_n \to u$ on $\{u < \infty\}$, $Uf_n \leqq u$ for all n. We define recursively a decreasing sequence $\{B_j\}$ of nonempty Borel sets such that diameter $(B_j) \downarrow 0$ and such that for each $j > 0$ there is an $\alpha_j > 0$ and subsequence $\{n'\} \subset \{n\}$ with $UI_{B_j} f_{n'} \to \alpha_j u$ on $\{u < \infty\}$. Set $B_1 = K$ and assume B_j has been defined with a corresponding $\alpha_j > 0$ and subsequence $\{n'\} \subset \{n\}$. Since $\bar{B}_j \subset K$ is compact, we can find a finite Borel partition $\{C_i\}$ of B_j such that diameter $(C_i) < 1/j$ diameter (B_j) for each i. Then $I_{B_j} = \sum_i I_{C_i}$ and hence $UI_{B_j} f_{n'} = \sum_i UI_{C_i} f_{n'} \to \alpha_j u$. By Theorem (2.1) and Lemma (3.4), there is an i_0, a subsequence $\{n''\} \subset \{n'\}$, and excessive functions $u_1, u_2 \in \mathcal{S}$ with $u_1 \neq 0$ such that $UI_{C_{i_0}} f_{n''} \to u_1$ and $\sum_{i \neq i_0} UI_{C_i} f_{n''} \to u_2$ on $\{u < \infty\}$. Since $\alpha u = u_1 + u_2 \in exS$, $u_1 = \beta \alpha_j u$ for some $\beta > 0$. Let $B_{j+1} = C_{i_0}$ and $\alpha_{j+1} = \beta \alpha_j > 0$. Then diameter $(B_{j+1}) \leqq 1/j$ diameter (B_j)

and $UI_{B_{j+1}}f_{n''} \to \alpha_{j+1}u$ on $\{u < \infty\}$, thus completing the definition of the sequence $\{B_j\}$.

Consider now the decreasing sequence $\{\bar{B}_j\}$ of nonempty compact subsets of E, and let $x \in \bigcap_j \bar{B}_j$. Let V be any neighborhood of x. Since diameter $(\bar{B}_j) \downarrow 0$, there is some j_0 with $V \supset \bar{B}_{j_0} \supset B_{j_0}$, and hence $T_V \leqq T_{B_{j_0}}$ a.s. Now there is a subsequence $\{n'\} \subset \{n\}$ and an $\alpha_{j_0} > 0$ such that $UI_{B_{j_0}} f_{n'} \to \alpha_{j_0} u$ on $\{u < \infty\}$, $UI_{B_{j_0}} f_{n'} \leqq u$ for all n'. But for each $x \in E$

$$P_V UI_{B_{j_0}} f_{n'}(x) = E^x \int_{T_V}^{\infty} I_{B_{j_0}}(X_t) f_{n'}(X_t) dt = UI_{B_{j_0}} f_{n'}(x)$$

since $T_V \leqq T_{B_{j_0}}$ a.s., Lemma (3.5) implies that $P_V u = u$ and the proof is complete.

We list here a property of $ex\mathscr{S}$.

PROPOSITION 3.7. (i) *If $u \in S$ has support at x, there is a sequence $\{x_n\}$ with $x_n \to x$ and $u\{x_n\} \uparrow \|u\| = \{\sup u(y) \colon y \in E\}$.*

(ii) *If u is harmonic and E is not compact, there is a sequence $\{x_n\}$ such that $x_n \to \Delta$ and $u(x_n) \uparrow \|u\|$.*

Proof. (i) Suppose not. Then there is a neighborhood V of x and a constant $M < \|u\|$ such that $u(x) \leqq M$ for all $x \in V$. Let G be a neighborhood of x with $\bar{G} \subset V$. Then $u(X_{T_G}) \leqq M$ a.s., on $\{T_G < \infty\}$ since $X_{T_G} \in G \subset G^r \subset \bar{G} \subset V$ a.s., on $\{T_G < \infty\}$. But $u(y) = P_G u(y) = E^y[u(X_{T_G}); T_G < \infty]$ and hence $u(y) \leqq M$ for all $y \in E$, a contradiction.

(ii) Same proof as in (i) using neighborhoods of infinity.

Recall that a point $x \in E$ is polar if $P^y[T_x < \infty] = 0$ for all $y \in E$ where T_x is the hitting time of $\{x\}$. It follows from (3.5) of [2, Chap. II] that if $u \in \mathscr{S}$, then $\{u = \infty\}$ is polar. As a converse to this result, we prove

PROPOSITION 3.8. *Assume $U^\alpha \colon C_K(E) \to C(E)$ for some $\alpha \geqq 0$. Then if x is polar and $0 \neq u \in ex\mathscr{S}$ has support at x, $\|u\| = \infty$.*

Proof. Suppose x is polar and let $0 \neq u \in ex\mathscr{S}$ have support at x with $\|u\| = M < \infty$. Let $\{G_n\}$ be a decreasing sequence of open sets containing x with $\bigcap_n \bar{G}_n = \{x\}$. Let $y \in E$ be distinct from x. Then $T_{G_n} \uparrow \infty$ a.s., P^y and $u(y) = P_{G_n} u(y) = E^y[u(X_{T_{G_n}})] \leqq MP^y[T_{G_n} < \infty]$. By (4.24) of [2, Chap. II], $X_{T_{G_n}} \to \Delta$ a.s., P^y as $n \to \infty$. Since $X_{T_{G_n}} \in \bar{G}_n$ on $\{T_{G_n} < \infty\}$ a.s., it follows that $T_{G_n} = \infty$ a.s. P^y for large n. Hence $P^y[T_{G_n} < \infty] \downarrow 0$ as $n \to \infty$ and therefore $u(y) = 0$. Since $y \neq x$ was arbitrary, $u(y) = 0$ for all $y \neq x$ and hence $u = 0$ as dx does not charge the polar set $\{x\}$. This contradicts the fact that $u \neq 0$, thus completing the proof.

We now investigate the following uniqueness problem: When is it true that if $u_1, u_2 \in ex\mathscr{S}$ have support at x, then $u_1 = \alpha u_2$ for some $\alpha \geqq 0$? For this we make the following

DEFINITION 3.9. (i) If u has support at $x \in E$, then u is said to be regular at x if $P_D u = u$ for all finely perfect sets $D = D^r$ containing x of the form $D = G^r$ where G is finely open.

(ii) A family $\mathscr{U} \subset ex\mathscr{S}$ of excessive functions is said to be regular if any $u \in \mathscr{U}$ which has support at x is regular at x.

PROPOSITION 3.10. *Suppose $u \in \mathscr{S}$ has support at $x \in E$ and has the following property: For every decreasing sequence $\{G_n\}$ of open sets containing x with $\lim_n T_{G_n} = T$ a.s., we have $P_{G_n} u \rightarrow P_T u$. Then u is regular at x.*

Proof. Let D be a finely perfect set containing x, and let $y \in E$ be arbitrary. Then there is a decreasing sequence $\{G_n\}$ of open sets containing D such that $T_{G_n} \uparrow T_D$ a.s. P^y on $\{T_D < \infty\}$; hence $P_{G_n} u(y) \rightarrow P_D u(y)$. But each G_n is a neighborhood of x, therefore $P_{G_n} u(y) = u(y)$ for all n, and it follows that $u(y) = P_D u(y)$. Since y was arbitrary, $P_D u = u$.

REMARK 3.11. If $u \in \mathscr{S}$ has support at x and is regular at x, then $P_V u = u$ for all finely open V containing x.

We now prove the main result concerning regularity.

THEOREM 3.12. *Suppose $\mathscr{U} \subset ex\mathscr{S}$ is regular, and let $x \in E$. Then up to a nonnegative multiplicative constant, there is at most one $u \in \mathscr{U}$ having support at x. Moreover, if $u \in \mathscr{S}$ has support at x and is regular at x, then $u \in ex\mathscr{S}$.*

Proof. We first show that if $u_1, u_2 \in \mathscr{S}$ have support at x and are regular at x, then $u_1 \leqq u_2$ or $u_2 \leqq u_1$. Indeed, set $D_1 = \{u_1 < u_2\}^r$ and $D_2 = D_1^{cr} \subset \{u_2 \leqq u_1\}$. Now D_1 and D_2 are finely perfect and since $E = D_1 \cup D_2$, x must be regular for one of these sets. Assume that $x \in \{u_1 < u_2\}^r = D_1$ (the other case is treated similarly). Since u_1 and u_2 are finely continuous, $u_1 = P_{D_1} u_1 \leqq P_{D_1} u_2 = u_2$, i.e., $u_1 \leqq u_2$. Let now $\beta = \sup\{\alpha \geqq 0 \colon \alpha u_1 \leqq u_2\} \geqq 1$. We claim that if $\beta = \infty$ then $u_1 = 0$. For in this case $u_2 = \infty$ on $\{u_1 > 0\}$. But $u_2 \in \mathscr{S}$ and hence $\int_{\{u_1 > 0\}} dx = 0$, for otherwise there would exist a compact $K \subset \{u_1 > 0\}$ such that $\int_K dx > 0$ which would imply that $\int_K u_2 dx = \infty$. Thus $u_1 = 0$ a.e., hence $u_1 = 0$ everywhere. Assume therefore that $\beta < \infty$. Then $\beta u_1 \leqq u_2$.

On the other hand, if $\varepsilon > 0$, there is an $x \in E$ such that $u_2(x) < (\beta + \varepsilon)u_1(x)$. But $(\beta + \varepsilon)u_1$ and u_2 also have support at x and are regular at x, implying that $u_2 \leqq (\beta + \varepsilon)u_1$. Since $\varepsilon > 0$ was arbitrary, $u_2 \leqq \beta u_1$ and therefore $\beta u_1 = u_2$, proving the first part of the theorem.

To prove the second part, assume that $u \in \mathscr{S}$ has support at x and is regular at x. Then if $u = u_1 + u_2$ with $u_1, u_2 \in \mathscr{S}$, we have $u = P_D u = P_D u_1 + P_D u_2 = u_1 + u_2$ for all finely perfect D containing x. But $P_D u_i \leqq u_i$ $(i = 1, 2)$ and hence $P_D u_i = u_i$. Thus u_1 and u_2 have support at x and are regular at x. The preceding proof implies that $u_1 = \alpha u_2$ for some $\alpha \geqq 0$ and therefore $u \in ex\mathscr{S}$.

Suppose $ex\mathscr{S}$ has the following property: If $u \in ex\mathscr{S}$ has support at x, then u is locally bounded and continuous on E. Using Proposition (3.10), it is easy to see that $ex\mathscr{S}$ is regular. We show that in certain cases a form of continuity is actually necessary for regularity to hold.

PROPOSITION 3.13. *Assume X is a Hunt process. Let x_0 be regular for $\{x_0\}$ and suppose $u \in ex\mathscr{S}$ has support at x_0 and $u(x_0) \neq 0$. Then u is the unique* (up to a nonnegative multiplicative constant) *element in $ex\mathscr{S}$ having support at x_0 if and only if $u(x) \leqq u(x_0) < \infty$ for all $x \in E$.*

Proof. Since x_0 is not polar and $u(x_0) \neq 0$, it follows that the excessive function $P_{x_0}u(x) = E^x[u(x_{T_{x_0}})] \leqq u(x)$ is not identically zero, has support at x_0 and is regular there, and is therefore in $ex\mathscr{S}$ from Theorem (3.12). If $u(x_0) = \infty$, then $E^x[u(x_{T_{x_0}})]$ could only take the values 0 and ∞ since $X_{T_{x_0}} = x_0$ a.s., on $\{T_{x_0} < \infty\}$. But then $P_{x_0}u = 0$ a.e. since $P_{x_0}u \in \mathscr{S}$, and hence $P_{x_0}u = 0$, a contradiction. Now the uniqueness assumption on u implies that $u = \alpha P_{x_0}u$ for some $\alpha \geqq 0$ and since $0 < P_{x_0}u(x_0) = u(x_0) < \infty$ it follows that $\alpha = 1$ and therefore $u(x) = P_{x_0}u(x) = E^x[u(X_{T_{x_0}})] \leqq u(x_0) < \infty$ for all $x \in E$.

Conversely, assume $u(x) \leqq u(x_0) < \infty$ for all $x \in E$. Let $\{G_n\}$ be a decreasing sequence of open sets containing x_0 such that $\bigcap_n \bar{G}_n = \{x_0\}$. Then $T_{G_n} \uparrow T_{x_0}$ a.s. Since X is a Hunt process, $X_{T_{G_n}} \to X_{T_{x_0}} = x_0$ and $\lim_n u(X_{T_{G_n}}) \geqq u(X_{T_{x_0}}) = u(x_0)$ on $\{T_{x_0} < \infty\}$. But $u(x) \leqq u(x_0)$ for all $x \in E$ and hence $\lim_n u(X_{T_{G_n}}) = u(x_0)$ on $\{T_{x_0} < \infty\}$. The bounded convergence theorem now implies that $u(x) = P_{G_n}u(x) = E^x[u(X_{T_{G_n}})] \to E^x[u(X_{T_{x_0}})] = P_{T_{x_0}}u(x)$ for each $x \in E$ and the proof is complete.

The property of regularity is not shared by all standard processes (consider translation to the right on the line), and we now seek other conditions which guarantee the uniqueness property announced in Theorem (3.12). First let us state this property explicitly.

(A) Let $x \in E$ be arbitrary. If $u_1, u_2 \in ex\mathscr{S}$ have support at x, then $u_1 = \alpha u_2$ for some $\alpha \geqq 0$.

This property was first studied by Hervé [4] in axiomatic potential and is known as the axiom of proportionality. We introduce now a property that will guarantee (A) in a large number of cases.

(B) Suppose $u \in ex\mathscr{S}$ has support at x, and let D be finely perfect set containing x. Then $P_D u$ has support at x.

Note that the property includes the case $P_D u = 0$. We will state explicitly when (B) is assumed to hold.

Let $T = \inf\{t: X_t \neq X_0\}$. A point $x \in E$ is called an instantaneous point if $P^x[T = 0] = 1$. It is easy to see that if dx does not charge singletons, then the points of E are instantaneous.

THEOREM 3.14. *Assume* (B) *and that dx does not charge singletons. Let $u \in ex\mathscr{S}$ have support at x_0 and suppose that either x_0 is polar or $u(x_0) = 0$. Then if $D = D^r$ contains x_0, we have $P_D u = u$ or $P_D u = 0$.*

Proof. Let $v = u - P_D u \geqq 0$. Then (B) implies $P_V v = v$ for all open neighborhoods V of x_0. It follows that if $B \subset E$ is any Borel set such that x_0 is in the interior of B^c, then $P_{B^c} v = v$. Let $E' = E \backslash \{x_0\}$ and consider the standard process $\tilde{X}_t$ defined by $\tilde{X}_t = X_t$ if $t < T_{x_0}$ and $\tilde{X}_t = \Delta$ if $t \geqq T_{x_0}$. Then $\tilde{X}_t$ has state space E' and transition function $\tilde{P}_t f(x) = E^x[f(X_t); t < T_{x_0}]$. Let d be a metric on E compatible with the topology and suppose $x \in E'$. Then there is a closed ball $B \subset E'$ with center x such that x_0 is in the interior of B^c. Thus if $y \in E'$, $\tilde{P}_{B^c} v(y) = E^y[v(X_{T_{B^c}}); T_{B^c} < T_{x_0}] \leqq E^y[v(X_{T_{B^c}})] = v(y)$. Since v is nonnegative and finely continuous, it follows from [2, Chap. II, (5.9)] that v is excessive for $\tilde{X}_t$. Therefore if we denote by $\{\tilde{U}^\alpha\}$ the resolvent operators for $\tilde{X}_t$, we have

$$\alpha \tilde{U}^\alpha v(x) = \alpha E^x \int_0^{T_{x_0}} e^{-\alpha t} v(X_t) dt \leqq v(x)$$

for all $x \in E'$. Now if $x \neq x_0$,

$$\begin{aligned} \alpha U^\alpha v(x) &= \alpha \tilde{U}^\alpha v(x) + \alpha E^x \int_{T_{x_0}}^\infty e^{-\alpha t} f(X_t) dt \\ &\leqq v(x) + E^x[e^{-\alpha T_{x_0}} \alpha U^\alpha v(X_{T_{x_0}})]. \end{aligned}$$

If x_0 is polar, the third term in the inequality is zero. If $u(x_0) = 0$, then $P_D u(x_0) = 0$ and $\alpha U^\alpha u(x_0) \leqq u(x_0) = 0$; similarly $\alpha U^\alpha P_D u(x_0) = 0$. It follows that $\alpha U^\alpha v(x_0) = 0$ and therefore $E^x[e^{-\alpha T_{x_0}} \alpha U^\alpha v(X_{T_{x_0}})] = 0$ since $X_{T_{x_0}} = x_0$ a.s., on $\{T_{x_0} < \infty\}$. Thus in both cases $\alpha U^\alpha v(x) \leqq v(x)$ for all $x \neq x_0$. We now define a function $\tilde{v}$ by $\tilde{v}(x) = v(x)$ if $x \neq x_0$, $\tilde{v}(x_0) = \infty$. Sin e x_0 has dx-measure zero, $\{x_0\}$ has zero measure with respect to the measures $\alpha U^\alpha(x, .)$, $x \in E$. It follows that $\alpha U^\alpha v(x) = \alpha U^\alpha \tilde{v}(x) \leqq \tilde{v}(x)$ for all $x \in E$ and therefore $\lim_{\alpha \to \infty} \alpha U^\alpha \tilde{v}(x) = v(x)$ is in $\mathscr{S}$. Thus we have a decomposition of u in the form $u = v + P_D u$

where v and $P_D u$ are in $\mathscr{S}$. Since $u \in ex\mathscr{S}$, $P_D u = u$ for some $\alpha \geqq 0$. If $\alpha = 0$ or $P_D u = 0$, then $u = 0$. We claim that if $P_D u \neq 0$, then $D \cap \{0 < u < \infty\} \neq \phi$. For if otherwise, $D = D \cap \{u = 0\} \cup D \cap \{u = \infty\}$, a disjoint union. But $\{u = \infty\}$ is polar, hence $T_D = T_{D \cap \{u=0\}}$ a.s., and therefore $X_{T_D} \in \{u = 0\}$ a.s., on $T_D < \infty$. Hence $P_D u(x) = E^x[u(X_{T_D})] = 0$ for all $x \in E$, a contradiction. Thus if $\alpha > 0$ and $P_D u \neq 0$, there is a point $x \in D$ with $0 < u(x) < \infty$ and hence $\alpha P_D u(x) = \alpha u(x) = u(x)$ implying that $\alpha = 1$, or $P_D u = u$.

COROLLARY 3.15. *Assume* (B) *and suppose points are polar and that* $\mathscr{S}$ *has the following property: if* $u \in ex\mathscr{S}$ *has support at* x *and* $u \neq 0$, *then* $u(x) \neq 0$. *Then* $ex\mathscr{S}$ *is regular.*

Proof. If points are polar, then dx certainly does not charge singletons. If $0 \neq u \in ex\mathscr{S}$ has support at x and $D = D^r$ contains x, then $P_D u = u$ or $P_D u = 0$ by Theorem (3.14). But $P_D u(x) = u(x) \neq 0$ and therefore $P_D u = u$, proving that $ex\mathscr{S}$ is a regular.

According to Theorem (3.3), to each $u \in ex\mathscr{S}$ which is not harmonic we can associate a point $x \in E$ such that u has support at x. We want to consider the case where to each $u \in ex\mathscr{S}$ which is not harmonic, there is a unique point x at which u has its support. In axiomatic potential theory this property holds by virtue of the sheaf properties of the harmonic functions in that theory. Here, however, we do not have the property that if G_1 and G_2 are open and u is harmonic in G_1 and G_2, then u is harmonic in $G_1 \cup G_2$. For a Hunt process this property holds if u is locally bounded (cf. Meyer [7]).

For the moment we content ourselves with the following results.

PROPOSITION 3.18. *Assume* $\mathscr{U} \subset ex\mathscr{S}$ *is regular. If* $u \in \mathscr{U}$ *has support at* x_1 *and* x_2, *then* $u(x_1) = u(x_2)$.

Proof. Suppose $u(x_1) < \delta < u(x_2)$. Then $V = \{u < \delta\}$ is finely open and contains x_1. Now $u(X_{T_V}) \leqq \delta$ a.s., on $\{T_V < \infty\}$ since u is finely continuous; hence $u(x_2) = P_V u(x_2) = E^{x_2}[u(X_{T_V}); T_V < \infty] \leqq \delta$, a contradiction.

DEFINITION 3.19. $\mathscr{U} \subseteqq ex\mathscr{S}$ is separating if to each $u \in \mathscr{U}$ there is a unique $x \in E$ such that u has support at x.

From Proposition (3.7), it follows that if $\mathscr{U} \subseteqq ex\mathscr{S}$ contains no harmonic functions and each $u \in \mathscr{U}$ has the property that its supremum is approached in any neighborhood of one and only one point in E, then $\mathscr{U}$ is separating. The following proposition justifies the terminology.

PROPOSITION 3.20. *Assume $\mathscr{U} \subseteqq ex\mathscr{S}$ is regular and contains no harmonic functions.*

(i) *Suppose $\mathscr{U}$ has the property that if $u \in \mathscr{U}$ has support at x, then $0 < u(x) < \infty$. Then $\mathscr{U}$ is separating if $\mathscr{S}$ separates points.*

(ii) *Suppose $\mathscr{U}$ has the following property: If $u \in \mathscr{U}$ has support at x and if $y \neq x$, there is a function $v \in \mathscr{S}$ and a Borel set $D = D^r$ containing x such that $v \geqq u$ on D and $v(y) < v(x)$. Then $\mathscr{U}$ is separating.*

Proof. (i) It suffices to consider the case where $u \in \mathscr{U}$ has support at two distinct points x and y. By Proposition (2.16), $u(x) = u(y) = \beta > 0$. Let $v \in \mathscr{S}$ satisfy $v(x) > v(y)$. Then there is an $\alpha > 0$ such that $\alpha v(x) > \beta > \alpha v(y)$. Now $V = \{\alpha v > u\}$ is finely open and contains x. Therefore $\alpha v > P_V \alpha v \geqq P_V u = u$, i.e., $u \leqq \alpha v$. But $\alpha v(y) < u(y)$, a contradiction.

(ii) Suppose u has support at x and y, $x \neq y$ and let v and D be as in the hypothesis. We have from Hunt's theorem [2, p. 141], $u = P_D u = \inf \{s \in \mathscr{S}: s \geqq u \text{ on } D\}$. Thus, from $v \geqq u$ on D it follows that $u \leqq v$ and hence $u(x) \leqq v(x) < v(y) = u(y)$. But $u(x) = u(y)$ by Proposition (2.16), a contradiction.

4. Representation of excessive functions. In this section we prove a representation theorem, in integral form, for a certain class of potentials of the standard process X. In the next section we extend this representation to all potentials in $\mathscr{S}$. Recall that $\mathscr{S}$ denotes the set of all excessive functions that are locally integrable with respect to the reference measure dx. We now topologize $\mathscr{S}$ as a subset of $M^+(E)$, the nonnegative Radon measures on E: to each $u \in \mathscr{S}$ we associate the measure $u(x)dx$. This topology on $\mathscr{S}$ is locally convex and it is given by the family of semi-norms $\{p_f: f \in C_K(E)\}$ defined by $p_f(u) = \int fu\,dx$. Thus a sequence $\{u_n\} \subset \mathscr{S}$ converges to $u \in \mathscr{S}$ if and only if $\int fu_n dx \to \int fu\,dx$ for all $f \in C_K(E)$. Moreover, because of the hypotheses on the state space E, $\mathscr{S}$ is metrizable (Cf. Choquet [3]).

A cap of $\mathscr{S}$ is a compact subset of $\mathscr{S}$ of the form $\{h \leqq 1\}$ where h is a map of $\mathscr{S}$ into $[0, \infty]$, linear in the sense that $h(0) = 0$, $h(u + v) = h(u) + h(v)$ for $u, v \in \mathscr{S}$, and $h(\alpha u) = \alpha h(u)$ for $u \in \mathscr{S}$, $\alpha \in R^+ = [0, \infty)$. In order to guarantee the existence of a sufficient number of caps of $\mathscr{S}$, we will make a special assumption. Recall that a sequence $\{\nu_n\}$ of nonnegative Radon measures on E is bounded if the sequence $\{\nu_n(f)\}$ is bounded for each $f \in C_K^+(E)$. Our special assumption, which holds in the situation discussed in [7, Chap. II], is as follows:

(4.1) Suppose $\{u_n\} \subset \mathscr{S}$ is a bounded sequence in $M^+(E)$ and $u_n \rightarrow u$ a.e., for some $u \in \mathscr{S}$. Then there is a subsequence $\{u_{n'}\} \subset \{u_n\}$ such that $u_{n'} \rightarrow u$ in $\mathscr{S}$.

It follows that $\mathscr{S}$ is a closed subset of $M^+(E)$, for if $\{u_n\}$ is a sequence of excessive functions in $\mathscr{S}$ and $u_n \rightarrow \nu$ in $M^+(E)$ for some $\nu \in M^+(E)$, then by Theorem (2.1) we can find a subsequence $\{u_{n'}\} \subset \{u_n\}$ and an excessive function u such that $u_{n'} \rightarrow u$ a.e. But for each $f \in C_K(E)$ we have by Fatou's lemma $\int fudx = \int f \liminf u_{n'}dx \leqq \liminf \int fu_{n'}dx = \int fd\nu(x)$ so that $u \in \mathscr{S}$. By (4.1) there is a subsequence $\{u_{n''}\} \subset \{u_{n'}\}$ such that $u_{n''} \rightarrow u$ in $\mathscr{S}$ and therefore $\int fudx = \int fd\nu(x)$ for all $f \in C_K(E)$, implying that $d\nu(x) = u(x)dx$. Note that (4.1) is satisfied if $\mathscr{S}$ has the following property: If $\{u_n\} \subset \mathscr{S}$ and $u_n \rightarrow u$ a.e. for some $u \in \mathscr{S}$, then for each compact $K \subset E$, there is a subsequence $\{u_{n'}\} \subset \{u_n\}$ which is uniformly integrable over K.

Now (4.1) implies that $\mathscr{S}$ is well-capped, i.e., $\mathscr{S}$ is the union of its caps (Meyer [6, Chap. XI]). Thus Choquet's representation theorem applies (cf. [3]). Let $\mathscr{S}'$ denote the continuous linear forms on $\mathscr{S}$. Then if $v \in \mathscr{S}$, there is a nonnegative Radon measure ν carried by $ex\mathscr{S}$ such that for $l \in \mathscr{S}'$, $l(v) = \int_{ex\mathscr{S}} l(u)\nu(du)$.

Let now $\{K_n\}$ be an increasing sequence of compact subsets of E with $K_n \subseteq K_{n+1}$ and $E = \bigcup_n K_n$. Let $\{f_n\}$ be a sequence of nonnegative continuous functions with compact support such that for each $n, f_n(x) = 1$ for all $x \in K_n$. Choose numbers $\alpha_n > 0$ such that $\sum_n \alpha_n \int f_n dx = 1$, and denote by $h\colon \mathscr{S} \rightarrow [0, \infty]$ the functional defined by $h(u) = \sum_n \alpha_n \int f_n u dx$. It is clear that $h(0) = 0$, $h(u + v) = h(u) + h(v)$ for $u, v \in \mathscr{S}$, and $h(\beta u) = \beta h(u)$ for $\beta \geqq 0$. If we let $\mathscr{K} = \{u\colon h(u) \leqq 1\} = \{u\colon \sum_n \alpha_n \int f_n u dx \leqq 1\}$, then (4.1) implies that $\mathscr{K}$ is a compact, convex set in $\mathscr{S}$. Therefore, if $\hat{\mathscr{S}}$ is the convex, proper cone generated by $\mathscr{K}$, $\hat{\mathscr{S}}$ will have compact base $\mathscr{K}$ and will be σ-compact. Note that $\hat{\mathscr{S}} = \{u \in \mathscr{S}\colon h(u) < \infty\}$ and that if $v \in \mathscr{S}$ is bounded, then $v \in \hat{\mathscr{S}}$. finally, we denote by $\mathscr{B}(\mathscr{K})$ the Borel sets of $\mathscr{K}$.

LEMMA 4.2. *Suppose $\{u_j\}$ is a sequence of excessive functions in $\mathscr{K}$ such that $u_j \rightarrow u$ in $\mathscr{K}$ for some $u \in \mathscr{K}$. Then for each integer $n > 0$ and $\alpha > 0$ we have $U^\alpha(x, u_j \wedge n) \xrightarrow{j} U^\alpha(x, u \wedge n)$ for all $x \in E$.*

Proof. Consider an integer $n > 0$ and $\alpha > 0$. We show first that $\int_B u_j \wedge n dx \rightarrow \int_B u \wedge n dx$ for all Borel sets $B \subset E$ having compact closure. Assume this is not the case so that there is an $\varepsilon > 0$ and a subsequence $\{j'\} \subset \{j\}$ with $\left|\int_B u_{j'} \wedge n dx - \int_B u \wedge n dx\right| \geqq \varepsilon$ for some Borel set B with compact closure and for all j'. By Theorem (2.1) and (4.1) we can find a subsequence $\{j''\} \subset \{j'\}$ and an excessive function u such that $u_{j''} \rightarrow u$ a.e. as $j'' \rightarrow \infty$ and that $\int f u_{j''} dx \rightarrow \int f \tilde{u} dx$ for all $f \in C_K(E)$. It follows that $\int f \tilde{u} dx = \int f u dx$ for all $f \in C_K(E)$ and therefore $u = \tilde{u}$ a.e., hence everywhere. Thus $u_{j''} \rightarrow u$ a.e., and $\int_B u_{j''} \wedge n dx \rightarrow \int_B u \wedge n dx$, giving the desired contradiction.

Fix $x \in E$. Then the Borel measure $B \rightarrow U^\alpha(x, B)$ is absolutely continuous with respect to dx, and $U^\alpha(x, E) = U^\alpha 1(x) \leqq 1/\alpha < \infty$. Since $\int_B u_j \wedge n dx \xrightarrow{j} \int_B u \wedge n dx$ for all $B \in \mathscr{B}(E)$ with compact closure, it follows that $U^\alpha(x, u_j \wedge n) \rightarrow U^\alpha(x, u \wedge n)$ as $j \rightarrow \infty$ and the proof is complete.

THEOREM 4.3. *The map* $\Phi: E \times \mathscr{K} \rightarrow \bar{R}^+ = [0, \infty]$ *defined by* $\Phi(x, u) = u(x)$ *is* $\mathscr{B}(E) \times \mathscr{B}(\mathscr{K})$ *measurable.*

Proof. It is sufficient to show that for each $\alpha > 0$, the map Φ^α: $E \times \mathscr{K} \rightarrow \bar{R}^+$ defined by $\Phi^\alpha(x, u) = U^\alpha(x, u) = U^\alpha u(x)$ is $\mathscr{B}(E) \times \mathscr{B}(\mathscr{K})$ measurable since for each $x \in E$ and $u \in \mathscr{K}$, $\alpha\Phi^\alpha(x, u) = \alpha U^\alpha u(x) \uparrow u(x) = \Phi(x, u)$ as $\alpha \rightarrow \infty$. Let $\alpha > 0$, and for each integer $n > 0$ define the map Φ^α_n: $E \times \mathscr{K} \rightarrow \bar{R}^+$ by $\Phi^\alpha_n(x, u) = U^\alpha(x, u \wedge n)$. For fixed $u \in \hat{\mathscr{S}}$ the map $x \rightarrow \Phi^\alpha_n(x, u)$ is $\mathscr{B}(E)$ measurable, and Lemma (4.2) implies that for fixed $x \in E$ the map $u \rightarrow \Phi^\alpha_n(x, u)$ is continuous on $\mathscr{K}$. Since $\mathscr{K}$ is a compact metric space, if follows that Φ^α_n is $\mathscr{B}(E) \times \mathscr{B}(\mathscr{K})$ measurable. But $\Phi^\alpha_n(x, u) = U^\alpha(x, u \wedge n) \uparrow U^\alpha(x, u)$ as $n \rightarrow \infty$ and therefore Φ^α is $\mathscr{B}(E) \times \mathscr{B}(\mathscr{K})$ measurable, completing the proof of Theorem (4.3).

COROLLARY 4.4. *Let* $B \in \mathscr{B}(E)$. *Then the map* $P_B: E \times \mathscr{K} \rightarrow \bar{R}^+$ *defined by* $P_B(x, u) = \int P_B(x, dy) u(y) = P_B u(x)$ *is* $\mathscr{B}(E) \times \mathscr{B}(\mathscr{K})$ *measurable.*

Proof. Let $H = \{\varphi \in B(E \times \mathscr{K}): (x, u) \rightarrow \int P_B(x, dy)\varphi(y, u)$ is $\mathscr{B}(E) \times \mathscr{B}(\mathscr{K})$ measurable$\}$. Then H contains all functions of the form $\varphi_1(x)\varphi_2(u)$ where $\varphi_1 \in B(E)$ and $\varphi_2 \in B(\mathscr{K})$. Moreover, if $\{\varphi_n\}$ is an increasing sequence of functions in H with $\varphi = \lim \varphi_n$, then the monotone convergence theorem implies that φ is in H. Hence, by the monotone class theorem, $B(E \times \mathscr{K}) \subset$ H. Since the function $(x, u) \rightarrow u(x)$ is in $B(E \times \mathscr{K})$, the result follows.

COROLLARY 4.5. (i) *Suppose* $\nu \geqq 0$ *is a finite Borel measure* *on* $\hat{\mathscr{S}}$ *carried by* $\mathscr{K}$, *and* $v(x) = \int u(x)\nu(du)$. *Then* $v \in \hat{\mathscr{S}}$.

(ii) *Suppose* $\nu \geqq 0$ *is a finite Borel measure on* $\hat{\mathscr{S}}$ *carried by* $\mathscr{K}$ *and* v *is an excessive function such that* $l(v) = \int l(u)\nu(du)$ *for all* $l \in \mathscr{S}'$. *Then* $v \in \hat{\mathscr{S}}$ *and* $v(x) = \int u(x)\nu(du)$ *for all* $x \in E$.

Proof. (i) Note first that the integral makes sense by the joint measurability of the map $(x, u) \to u(x)$. We have by Fubini's theorem $\alpha U^\alpha(x, v) = \int \alpha U^\alpha(x, u)v(du) \leqq \int u(x)\nu(du) = v(x)$ since $\alpha U^\alpha(x, u) \leqq u(x)$ for all $u \in \mathscr{K}$, $\alpha \geqq 0$. Also, since $\alpha U^\alpha(x, u) \uparrow u(x)$ as $\alpha \to \infty$ for all $u \in \mathscr{K}$, the monotone convergence theorem implies that $\alpha U^\alpha(x, v) \uparrow v(x)$ as $\alpha \to \infty$ so that v is excessive. To see that $v \in \hat{\mathscr{S}}$, use Fubini's theorem to write

$$h(v) = \sum_n \alpha_n \int f_n(x) \int u(x)\nu(du)dx = \sum_n \alpha_n \int \left(\int f_n(x)u(x)dx \right) \nu(du)$$
$$= \int \left(\sum_n \alpha_n \int f_n(x)u(x)dx \right) \nu(du) = \int h(u)\nu(du) < \infty$$

since $h(u) \leqq 1$ for all $u \in \mathscr{K}$.

(ii) Since $p_f \in \mathscr{S}'$ for each $f \in C_K(E)$, we have

$$\int f(x)v(x)dx = \int \left(\int f(x)u(x)dx \right) \nu(du)$$

for all $f \in C_K(E)$. On the other hand, the function $\tilde{v}(x) = \int u(x)\nu(du)$ is in $\hat{\mathscr{S}}$ by (i), and for each $f \in C_K(E)$,

$$\int f(x)\tilde{v}(x)dx = \int f(x) \int u(x)\nu(du)dx = \int \left(\int f(x)u(x)dx \right) \nu(du)$$
$$= \int f(x)v(x)dx$$

and therefore $\tilde{v} = v$ a.e., and hence everywhere since $\tilde{v}$ and v are excessive.

Consider again our increasing sequence $\{K_j\}$ of compact subsets of E with $\mathring{K}_j \subset K_{j1}$ and $E = \bigcup_j K_j$. For each j, define $\Psi_j : E \times \mathscr{K} \to \bar{R}^+$ by $\Psi_j(x, u) = P_{K_j^c}(x, u)$, *a* $\mathscr{B}(E) \times \mathscr{B}(\mathscr{K})$ *measurable function, and set* $\Psi(x, u) = \lim_j \downarrow P_{K_j^c}(x, u)$. From Fubini's theorem, the map $u \to h(\Psi_j(., u)) = \sum_n \alpha_n \int f_n \Psi_j(x, u)dx$ is $\mathscr{B}(\mathscr{K})$ measurable, and therefore $h(\Psi(., u)) = \lim_j \downarrow h(\Psi_j(., u)$ is $\mathscr{B}(\mathscr{K})$ measurable. Therefore the set $\mathscr{P} = \{u \in \mathscr{K} : h(\Psi(., u)) = 0\}$ is a Borel subset of $\mathscr{K}$. It is clear that $u \in \mathscr{P}$ if and only if $u \in \mathscr{K}$ and $P_{K_j^c} u \downarrow 0$ a.e., as $j \to \infty$ for all increasing sequences $\{K_j\}$ of compacts such that $\mathring{K}_j \subset K_{j+1}$ and $E = \bigcup_j K_j$. Finally, we put $\hat{\mathscr{P}} = ex\mathscr{K} \cap \mathscr{P} \backslash \{0\}$ where $ex\mathscr{K}$ is the set of extreme points of the compact, convex set $\mathscr{K}$. Then $\hat{\mathscr{P}} \subset \{u \in$

$\mathscr{P}: h(u) = 1\}$. See Meyer [6, Chap. XI]. We make the following assumption on $\hat{\mathscr{P}}$, which is valid if $\hat{\mathscr{P}}$ is regular and separating:

(4.6) $\hat{\mathscr{P}}$ is separating and the proportionality axiom holds.

Note that $\hat{\mathscr{P}}$ contains no harmonic elements for if $u \in \hat{\mathscr{P}}$ is harmonic, then $u = P_{K_j^c} u \downarrow 0$ a.e., for a sequence $\{K_j\}$ of compacts with $\mathring{K}_j \subset K_{j+1}$ and $E = \bigcup_j K_j$. Thus $u = 0$ a.e., hence everywhere and $0 \notin \hat{\mathscr{P}}$. Therefore, according to Theorem (3.3) and the assumption (4.6), to each $u \in \hat{\mathscr{P}}$ we can associate a unique $y \in E$, the point at which u has its support. We indicate this relation by setting $u = u_y$. Consider now the map $\Gamma: \hat{\mathscr{P}} \to E$ defined by $\Gamma(u_y) = y$. Define $\hat{E} = \Gamma(\hat{\mathscr{P}}) \subset E$. Then Γ is one-one onto $\hat{E}$. Moreover, we can give $\hat{E}$ the topology which makes Γ a homeomorphism between $\hat{\mathscr{P}}$ and $\hat{E}$. It is easily seen that this topology is given by the metric $d: \hat{E} \times \hat{E} \to R^+$ defined by $d(x, y) = \rho(u_x, u_y)$ where ρ is the metric on $\mathscr{K}$. In other words, the topology on $\hat{E}$ is defined by the family of semi-norms $\{p_f: f \in C_K(E)\}$ given for $y \in \hat{E}$ by $p_f(y) = \int f u_y dx$.

Consider now the function $u: E \times \hat{E} \to \bar{R}^+$ defined by $u(x, y) = u_y(x)$. This function is $\mathscr{B}(E) \times \mathscr{B}(\hat{E})$ measurable since it is the restriction of the $\mathscr{B}(E) \times \mathscr{B}(\mathscr{K})$-measurable map $(x, u) \to u(x)$ to the set $E \times \hat{\mathscr{P}}$ and $\hat{\mathscr{P}}$ is Borel in $\mathscr{K}$. We come now to the main result of this development. Recall that an excessive function $p \in \mathscr{S}$ is called a potential if $P_{K_n^c} p \downarrow 0$ a.e., for all increasing sequences $\{K_n\}$ of compacts such that $\mathring{K}_n \subseteq K_{n+1}$ and $E = \bigcup_n K_n$.

THEOREM 4.7. *There is a subset $\hat{E} \subseteq E$ with a metric topology and a function $u: E \times \hat{E} \to \bar{R}^+$ which is $\mathscr{B}(E) \times \mathscr{B}(\hat{E})$ measurable and having the property that the function $x \to u(x, y)$ is an extremal excessive function for each $y \in \hat{E}$. Each potential $p \in \hat{\mathscr{S}}$ has a representation of the form*

$$p(x) = \int u(x, y)\nu(dy)$$

for some finite Borel measure $\nu \geqq 0$ on $\hat{E}$.

Proof. The only statement to prove is the last sentence of the theorem. If $p \in \hat{\mathscr{S}}$, then by Choquet's theorem there is a nonnegative Radon measure μ carried by $ex\mathscr{K}$ such that $l(p) = \int_{ex\mathscr{K}} l(u)\mu(du)$ for $l \in \mathscr{S}'$; therefore $p(x) = \int_{ex\mathscr{K}} u(x)\mu(du)$ by Corollary (3.4). Since $\hat{\mathscr{P}} \subset \mathscr{K}$ is Borel, $p(x) = \int_{\hat{\mathscr{P}}} u(x)\mu(du) + \int_{\mathscr{F}} u(x)\mu(du)$ where $\mathscr{F} = ex\mathscr{K} \backslash \hat{\mathscr{P}}$. Now $\int_{\mathscr{P}} u(x)\mu(du) = \int_{\Gamma(\hat{\mathscr{P}})} \Gamma^{-1}(y)(x)\mu \circ \Gamma^{-1}(dy) = \int_{\hat{E}} u(x, y)\nu(dy)$ where $\nu = \mu \circ \Gamma^{-1}$ is a Borel measure on $\hat{E}$.

It remains to show that $\int_{\mathscr{T}} u(x)\mu(du) = 0$. Let $\{K_n\}$ be an increasing sequence of compacts such that $\mathring{K}_n \subset K_{n+1}$ and $E = \bigcup_n K_n$. Then Fubini's theorem yields

$$P_{K_n^c}p(x) = \int_{\hat{\mathscr{T}}} P_{K_n^c}u(x)\mu(du) + \int_{\mathscr{T}} P_{K_n^c}u(x)\mu(du)\ .$$

Now $P_{K_n^c}p \downarrow 0$ a.e., and hence $\int_{\mathscr{T}} \lim \downarrow P_{K_n^c}u(x)\mu(du) = 0$ a.e., or $\int_{\mathscr{T}} \Psi(x, u)\mu(du) = 0$ a.e.

Using Fubini's theorem again, we can write

$$0 = h\Big(\int_{\mathscr{T}} \Psi(., u)\mu(du)\Big) = \int_{\mathscr{T}} h(\Psi(., u))\mu(du).$$

Thus $\mu\{u \in \mathscr{T}: h(\Psi(., u)) > 0;\} = \mu\{ex\mathscr{K}\backslash\hat{\mathscr{P}}\} = 0$ and therefore μ is carried by $\hat{\mathscr{P}}$, completing the proof of Theorem (4.7).

We are going to improve Theorem (4.7), but before this we consider a related notion which is of independent interest.

5. Dual operator and the representation theorem.

We introduce now a dual operator associated with the potential operator U.

DEFINITION 5.1. The linear operator $\hat{U}: C_K(E) \to C(\hat{E})$ is defined for $f \in C_K(E)$ by $\hat{U}f(y) = \int f(x)u(x, y)dx$ and is called the dual operator of U.

The fact that $\hat{U}f(y)$ is a continuous function on $\hat{E}$ follows from the observation that $\hat{U}f(y) = \int f(x)u(x, y)dx = \int f(x)u_y(x)\,dx = p_f(y)$ where p_f is a semi-norm defining the topology on $\hat{E}$.

We want to investigate some of the properties of $\hat{U}$. The results obtained here are analogus to the case where a dual process exists as in [2, Chap. VI] or [7, Chap. II]. Now Meyer [5] has shown that $\mathscr{S}$, and therefore $\hat{\mathscr{S}}$, is a lattice in its own order, i.e., the order defined for $u, v \in \mathscr{S}$ by $u < v$ if and only if there is an $s \in \mathscr{S}$ such that $v = u + s$. The Choquet-Meyer Uniqueness Theorem [3] then implies that each $u \in \hat{\mathscr{S}}$ is represented by a unique nonnegative Radon measure carried by $ex\mathscr{K}$.

If ν is a signed Borel measure on $\hat{E}$ having finite total variation, we denote by $U\nu(x)$ the function $x \to \int u(x, y)\nu(dy)$. If $\nu \geqq 0$ is finite, then $U\nu \in \hat{\mathscr{S}}$ from Corollary (4.5).

PROPOSITION 5.2. (i) *If ν is a signed Borel measure on $\hat{E}$ of* *finite total variation, and if* $U\nu = 0$ *a.e., then* $\nu = 0$.

(ii) *If* $K \subset \hat{E}$ *is compact, then the restrictions of the functions in image* (U) *to* K *is dense in* $C(K)$.

Proof. (i) If ν is a such a measure, write $\nu = \nu_1 - \nu_2$ where ν_1 and ν_2 are finite and nonnegative. Then $\int u(x, y)\nu_1(dy) = \int u(x, y)\nu_2(dy)$ a.e., or $U\nu_1 = U\nu_2$ a. e.. But each of these functions is in $\hat{\mathscr{S}}$, hence $U\nu_1 = U\nu_2$. The Choquet-Meyer uniqueness theorem then implies $\nu_1 = \nu_2$ and therefore $\nu = \nu_1 - \nu_2 = 0$.

(ii) Let $K \subset \hat{E}$ be compact. Let ν be a Radon measure on K and suppose that $\int_K \hat{U}f(y)\nu(dy) = 0$ for all continuous functions f with compact support. Then $0 = \int_K \hat{U}f(y)\nu(dy) = \int\left(\int u(x, y)f(x)dx\right)\nu(dy) = \int f(x)dx\int u(x, y)\nu(dy) = \int f(x)U\nu(x)dx$ for all $f \in C_K(E)$. But then $U\nu = 0$ a.e., and hence by (ii), $\nu = 0$. The result now follows from the Hahn-Banach Theorem.

We now make the following observations: The set $\hat{E} = \hat{\mathscr{P}} \subset \mathscr{K}$ is a subset of the compact set $\mathscr{K}$, and therefore $F = \hat{E}^a$, the closure of $\hat{E}$ in $\mathscr{K}$, is a compact subset of $\mathscr{K}$. Note that $0 \notin F$. We claim that if $f \in C_K(E)$, then the function $\hat{U}f$ extends uniquely to a continuous function on F which we continue to denote by $\hat{U}f$. This follows from the previously mentioned fact that $\hat{U}f(y) = p_f(u_y)$ and p_f is one of the semi-norms defining the topology on F. Note that if $u \in F\backslash\hat{E}$, then $\hat{U}f(u) = \int f(x)u(x)dx$. In the terminology of [7], F is a "Martin Compactification" of the space $\hat{E}$. Finally, recall that $M^+(F)$ denotes the nonnegative Radon measures on F, and that any finite nonnegative Borel measure ν on $\hat{E}$ can be regarded as an element $\tilde{\nu} \in M^+(F)$ by the formula $\tilde{\nu}(B) = \tilde{\nu}(B \cap \hat{E})$ for $B \in \mathscr{B}(F)$. We now generalize Theorem (4.7).

THEOREM 5.3. *There is a subset* $\hat{E} \subset E$, *a metric topology on* $\hat{E}$ *making* $\hat{E}$ *a dense subset of a compact metric space* F, *and a function* $u: E \times \hat{E} \to [0, \infty]$ *having the following properties: The function* u *is* $\mathscr{B}(E) \times \mathscr{B}(\hat{E})$ *measurable and for each* $y \in \hat{E}$, *the function* $x \to u(x, y)$ *is an extremal excessive function. Each potential* $p \in \mathscr{S}$ *has a representation of the form*

$$p(x) = \int_{\hat{E}} u(x, y)\nu(dy)$$

for some uniquely determined finite Borel measure $\nu \geqq 0$ *on* $\hat{E}$. *For any* $f \in C_K(E)$, $\hat{U}f$ *has a unique continuous extension to* F.

Proof. According to Theorem (4.7) and the preceding remarks, the only part of the theorem to prove is the representation for potentials $p \in \mathscr{S}$. We show that if $p \in \mathscr{S}$ is potential, then $p \in \hat{\mathscr{S}}$ and hence the representation holds from Theorem (4.7). But if $p \in \mathscr{S}$, then $p_n(x) = (p \wedge n)(x)$ is an element of $\hat{\mathscr{S}}$ and therefore $p_n(x) = \int u(x, y)\nu_n(dy)$ for some finite Borel measure $\nu_n \geqq 0$ on $\hat{E}$. Let $f \in C_K^+(E)$. Then

$$\begin{aligned}\int f(x)p_n(x)dx &= \int f(x)\Big(\int u(x, y)\nu_n(dy)\Big)dx \\ &= \int\Big(\int f(x)u(x, y)dx\Big)\nu_n(dy) = \int \hat{U}f(y)\nu_n(dy) \\ &= \int_F \hat{U}f(y)\tilde{\nu}_n(dy) \uparrow \int f(x)p(x)dx < \infty .\end{aligned}$$

Since F is compact and $0 \notin F$, we can find a finite number $\{f_i\}$ of function in $C_K^+(E)$ such that $\sum_i p_{f_i}(u) > 0$ for all $u \in F$. But $p_{f_i}(u) = \int f_i u dx = \hat{U}f_i(u)$ on F and therefore $\sum_i \hat{U}f_i > 0$ on F. But then

$$\int_F \sum_i \hat{U}f_i(y)\tilde{\nu}_n(dy) = \int \sum_1 f_i(x)p_n(x)dx \uparrow \int \sum_i f_i(x)p(x)dy < \infty$$

as $n \to \infty$. Hence $\tilde{\nu}_n(F) \leqq M < \infty$ for some finite $M > 0$, and $\{\tilde{\nu}_n\}$ is bounded set in $M^+(F)$ and hence is pre-compact in the vague topology. There exists therefore a finite Radon measure $\nu \in M^+(F)$ and a subsequence $\{n'\}$ such that $\tilde{\nu}_{n'}(g) \to \nu(g)$ for all $g \in C(F)$. Since $\hat{U}f \in C(F)$ for $f \in C_K^+(E)$, we have

$$\int \hat{U}f(y)\tilde{\nu}_{n'}(dy) = \int f(x)p_{n'}(x)dx \uparrow \int_F \hat{U}f(u)\nu(du) = \int f(x)p(x)dx.$$

Now $\hat{U}f(u) = \int f(x)u(x)dx$ for $u \in F$ and therefore

$$\int f(x)p(x)dx = \int\Big(\int f(x)u(x)dx\Big)\nu(du) = \int f(x)\Big(\int u(x)\nu(du)\Big)dx .$$

Here we use the joint measurability of the function $(x, u) \to u(x)$ and Fubini's theorem. Since this equation holds for all $f \in C_K^+(E)$, it follows that $p(x) = \int_F u(x)\nu(du)$ a.e., and hence everywhere since each function is excessive by Corollary (4.5). Since $\nu(F) < \infty$ and $F \subset \mathscr{K}$, the same Corollary implies that $p \in \hat{\mathscr{S}}$, thus completing the proof of Theorem (5.3).

Recall that an excessive function $v \in \mathscr{S}$ is said to be harmonic if $P_B v = v$ whenever B is the complement of a compact subset of E. Now according to [2, p. 272], each $u \in \mathscr{S}$ has a unique representation of the form $u = p + v$ where p is a potential and v is an harmonic excessive function: The reader can easily convince himself that the proof given in the cited reference is equally valid under our assumptions. If we let $\mathscr{R} = \{u \in ex\mathscr{S}: u \text{ is harmonic}\}$ and $P = \{u \in ex\mathscr{S}: u \text{ is a potential}\}$, then the following corollary is an immediate consequence of the above fact

COROLLARY 5.4. (i) *Each $u \in \mathscr{S}$ has a unique representation of the form $u(x) = \int u(x, y)\nu(dy) + v(x)$ where $\nu \geqq 0$ is a finite Borel measure on $\hat{E}$ and $v \in \mathscr{S}$ is harmonic.*

(ii) *$ex\mathscr{S} = P \cup \mathscr{R}$. Of course, $P \cap \mathscr{R} = \{0\}$.*

REMARK 5.5. In § 3 we introduced the assumption (4.6) and we now show how to obtain a representation as in Theorem (5.3) under the single assumption that to each $x \in E$ there is at most one $u \in ex\mathscr{S}$ having support at x. Define $\hat{E} = \{x \in E$: there is a $u \in \hat{\mathscr{P}}$ having support at $x\}$ and write $x \sim y$ if and only if there is $u \in \hat{\mathscr{P}}$ having support at x and y. It is easy to see that $\sim$ is an equivalence relation on $\hat{E}$ and we put $\tilde{E} = \hat{E}/\sim$, the set of equivalence classes of $\hat{E}$. We denote by $\tilde{x}$ the equivalence class containing x. If we define $\tilde{\Gamma}: \tilde{E} \to \hat{\mathscr{P}}$ by $\tilde{\Gamma}(\tilde{x})$ = the unique $u \in \hat{\mathscr{P}}$ having support at x, then $\tilde{\Gamma}$ is one-one onto $\hat{\mathscr{P}}$, and the metric d on E defined by $d(\tilde{x}, \tilde{y}) = \rho(\tilde{\Gamma}(\tilde{x}), \tilde{\Gamma}(\tilde{y}))$, where ρ is the metric on $\hat{\mathscr{P}}$, endows $\tilde{E}$ with a topology that makes $\tilde{\Gamma}$ a homeomorphism between $\mathscr{P}$ and $\tilde{E}$. Imitating the proof of Theorem (4.7) we obtain an analogous representation with the space $\hat{E}$ replaced by $\tilde{E}$. Of course $\tilde{E}$ is no longer a subset of E, but rather a set of equivalence classes of points of E. Note that $\hat{\mathscr{P}}$ is separating if and only if $x \sim y$ implies that $x = y$.

REMARK 5.6. Denote by $\hat{E}'$ the subset $\hat{E} \subset E$ equipped with the subspace topology, i.e., the topology induced by E. A natural question to ask is if there is any relation between $\hat{E}'$ and $\hat{E} = \hat{\mathscr{P}}$ as topological spaces. We show that is a dual process exists as in Chapter VI of [2], then the map $\Gamma': \hat{\mathscr{P}} \to E'$ defined by $\Gamma'(u_x) = x$ is a homeomorphism so that $\hat{E} = \hat{E}'$ as topological spaces. Now the dual process $\tilde{X}_t$ has a potential operator $\tilde{U}$ of the form $\tilde{U}f(y) = \int g(x, y)f(x)dx$, and it follows from [7, Chap. III, T7 and T10] that $g(x, y) = u(x, y)$ for $y \in \hat{E} = \hat{\mathscr{P}}$. In other words, $\hat{E} = \{y \in E: x \to g(x, y)$ is an extremal potential$\}$ and therefore $\hat{U}f(y) = \tilde{U}f(y)$ for all $y \in \hat{E}$ and $f \in C_K(E)$. If $u_{y_n} \to u_{y_0}$ in $\hat{\mathscr{P}}$, then $\hat{U}f(y_n) = \tilde{U}f(y_n) \to \tilde{U}f(y_0) = \hat{U}f(y_0)$ for each $f \in C_K(E)$. Now it is easy to see that the operator $\tilde{U}: C_K(E) \to C(E)$ has an image which separates points of E so that $y_n \to y_0$ in E, hence E'. Thus Γ' is continuous. On the other hand, if $y_n \to y_0$ in E' then $\hat{U}f(y_n) = \tilde{U}f(y_n) \to \tilde{U}f(y_0) = \hat{U}f(y_0)$ for all $f \in C_K(E)$ by the continuity of $\tilde{U}f$. Thus $u(x, y_n) \rightarrow u(x, y_0)$ in $\hat{\mathscr{P}}$ and Γ'^{-1} is continuous, proving that Γ' is a homeomorphism.

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