

A DUALITY BETWEEN TRANSPOTENCE ELEMENTS AND MASSEY PRODUCTS

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The purpose of this note is to show that if v is an element whose suspension is nonzero, and if u is dual to v , then the transpotence $\varphi_k(v)$ is defined and nonzero if and only if the k -Massey product $\langle u \rangle^k$ is defined and nonzero.

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1. Preliminaries.

1.1. *The Cobar Construction:* (Adams [1]). Let C be a simply connected *DGA* coalgebra over K with co-associative diagonal map where K is a commutative ring with unit. The Cobar Construction $\bar{F}(C)$ is the direct sum of the n -fold tensor products of the desuspension of $\bar{C} = \text{Ker}(\varepsilon)$ where $\varepsilon: C \rightarrow K$ is the augmentation. Suppose C has a differential $\{d_n: C_n \rightarrow C_{n-1}\}$. A typical element is a linear combination of elements of the form

$$x = s^{-1}(c_1) \otimes \cdots \otimes s^{-1}(c_n) = [c_1 | \cdots | c_n]$$

where x has bidegree $(-n, m)$ and $m = \sum_{i=1}^n \text{degree}(c_i)$. The differential in $\bar{F}(C)$ is defined on elements of bidegree $(-1, *)$ by

$$d[c] = [-dc] + \sum_i (-1)^{\text{deg } c_i'} [c_i' | c_i'']$$

where

$$\Delta(c) = c \otimes 1 + 1 \otimes c + \sum_i c_i' \otimes c_i''$$

$\Delta: C \rightarrow C \otimes C$ being the diagonal mapping of C . The differential is extended to all of $\bar{F}(C)$ by the requirement that $\bar{F}(C)$ be a *DGA*-algebra.

If C has a differential of degree $+1$ instead of -1 , we no longer ask that C be a simply connected but only connected, and the element $[c_1 | \cdots | c_n]$ is assigned bidegree (n, m) .

1.2. *The Bar Construction.* Let A be a connected associative *DGA* algebra over K . Let $\varepsilon: A \rightarrow K$ be the augmentation. Let $\bar{A} = \text{ker } \varepsilon$. Then the Bar Construction $\bar{B}(A)$ is the direct sum of the n -fold tensor products of the suspension of \bar{A} . Let

$$\{d_n: A_n \rightarrow A_{n-1}\}$$

be the differential in A . $\bar{B}(A)$ is bigraded by assigning the element $[a_1 | \cdots | a_n]$ degree (n, m) where $m = \sum_{i=1}^n \deg a_i$. $\bar{B}(A)$ has a differential $d = d_E + d_I$ where

$$d_E([a_1 | \cdots | a_n]) = \sum_{i=1}^{n-1} (-1)^{u(i)} [a_1 | \cdots | a_i a_{i+1} | \cdots | a_n]$$

$$d_I([a_1 | \cdots | a_n]) = \sum_{i=1}^n (-1)^{u(i-1)} [a_1 | \cdots | \partial a_i | \cdots | a_n]$$

where

$$u(i) = i + \sum_{k=1}^i \deg a_k .$$

We also mention that $[a | \cdots (k) \cdots | a]$ is $\gamma_k[a]$, the k th divided power of $[a]$.

If instead of the above the differential of A has degree $+1$, we put the bidegree of $[a_1 | \cdots | a_n]$ to be $(-n, m)$. In this case we will always assume A is simply connected.

1.3. The Suspension Map. In the case of the Bar Construction the suspension map $\sigma: H_*(A) \rightarrow H_*(\bar{B}(A))$ is represented by $a \rightarrow [a]$. In the case of the Cobar Construction, $\sigma: H_*(PA) \rightarrow H_*(\bar{F}(A))$ is represented by $a \rightarrow [a]$ where PA is the subcomplex of primitive chains.

DEFINITION 1. The Massey Product $\langle u \rangle^k$. (Kraines [6]).

Let A be a DGA algebra over K . Suppose a_1, \dots, a_{k-1} are given in A such that a_1 is a cycle (or cocycle) and that

$$\partial a_n = \sum_{r=1}^{n-1} (-1)^{\deg a_r} a_r a_{n-r} \text{ for } n = 2, \dots, k-1 .$$

Suppose u is represented by a_1 . Then the Massey Product $\langle u \rangle^k$ is represented by the cycle

$$\sum_{r=1}^{k-1} (-1)^{\deg a_r} a_r \cdot a_{k-r}$$

THEOREM 1. (Kraines, [6]). *The operation $\langle u \rangle^k$ depends only on the class $\{a_1\} \in H(A)$.*

DEFINITION 2. (Gitler, [5]). Suppose that A is an associative DGA algebra. Suppose $x \in H(A)$ is such that $v^k = 0$. The transpotence $\varphi_k(v) \in H(\bar{B}(A))_{I, m\sigma}$ is defined as follows: If $b \in A$ represents v then there exists $M \in A$ such that $\partial M = -b^k$. $\varphi_k(v)$ is represented by

$$(-1)^w [b^{k-1} | b] + [M] \text{ where } w = (1)^{\deg b^{k-1}} + 1 .$$

2. Main Theorem.

THEOREM 2. *Let C be a co-associative DGA coalgebra over K and let A be the dual associative DGA algebra over K . Suppose $H(A; K)$ and $H(\bar{B}(A); K)$ are free and of finite type over K . Let v in $H(A)$ and u in $H(\bar{F}(C); K)$ be such that the Kronecker index $\langle \sigma(v), u \rangle$ is 1. Then $\varphi_k(v)$ is defined and is not zero in $H(\bar{B}(A); K)$ if and only if $\langle u \rangle^k$ is defined and not zero in $H(\bar{F}(C); K)$. In this case*

$$\langle \varphi_k(v), \langle u \rangle^k \rangle = 1 .$$

In order to prove this theorem we shall consider the Eilenberg-Moore Spectral Sequences with

$$\begin{aligned} E^2 &= \text{Cotor}^{H(\bar{B}(A); K)}(K, K) \\ E^r &\Rightarrow E^0 H(\bar{F}(\bar{B}(A)); K) \approx H(A; K) \text{ as algebras, and dually,} \\ (E')^2 &= \text{Tor}^{H(\bar{F}(C); K)}(K, K) \\ (E')^r &\Rightarrow E^0 H(B(\bar{F}(C)); K) \approx H(C; K) \text{ as coalgebras.} \end{aligned}$$

We also note that the Kronecker Index $\langle , \rangle: C \otimes A \rightarrow K$ induces a pairing

$$\langle , \rangle: \bar{F}(C) \otimes \bar{B}(A) \rightarrow K$$

LEMMA 1. *Let $b \in A$ represent $v \in H(A)$. Suppose $v^k = 0$. Then*

$$d_k[\varphi_k(v)] = [\sigma b]^k \text{ in } E^k .$$

Proof. Let

$$V = \sum_{i=1}^{k-1} P(i) [[b^i | b]] ([[b]])^{k-i-1} \text{ where } P(i) = (-1)^{\text{deg } b^{i+1}}$$

and the outside bars refer to the Cobar Construction and the inside bars refer to the Bar Construction.

Taking ∂V gives a telescoping series and so

$$\partial V = [\sigma b]^k + (-1)^w [\sigma(b^k)]. \text{ Here } (-1)^w = P(k-1) .$$

In E^1 , V represents the class $(-1)^w [[b^{k-1} | b]] + [[M]] = [\varphi_k(v)]$.

The Lemma follows from the definition of a spectral sequence of a bi complex.

LEMMA 2. *Let $a \in \bar{F}(C)$ represent u . Then, by definition,*

$$\gamma_k[a] = [a | \cdots (k) \cdots | a] \in \bar{B}(\bar{F}(C)) .$$

If $\gamma_k[a]$ lives to E^{k-1} then $\langle u \rangle^k$ is defined and

$$d_k(\gamma_k[a]) = \langle u \rangle^k \text{ in } (E')^k .$$

Proof. We first make an observation: Suppose $\langle u \rangle^t$ is defined. Let (a_i) be a defining system for $\langle u \rangle^t$. Let

$$W = \sum_{r=2}^t \sum_{i_1 + \dots + i_r = t} [a_{i_1} | \dots | a_{i_r}] \in \bar{B}(\bar{F}(C)) .$$

Then

$$\partial W = \sum_{i=1}^{t-1} (-1)^{\deg a_{i+1}} [a_i a_{t-i}] .$$

Now to prove Lemma 2, we use induction on k . Suppose the lemma is true for $k-1$. Suppose $\gamma_k[a]$ lives to E_{k-1} . Since E is a spectral sequence of DGA coalgebras, and $d_{k-1}(\gamma_k[a]) = 0$, we have

$$\Delta d_{k-1} \gamma_k[a] = d_{k-1}^{\otimes} \Delta \gamma_k[a] = d_{k-1}^{\otimes} \sum_{i=0}^k \gamma_i[a] \otimes \gamma_{k-i}[a] = 0$$

where d^{\otimes} is the differential in $E' \otimes E'$. That is, in particular when $i = k-1$ in the above, we see

$$d_{k-1} \gamma_{k-1}[a] \otimes [a] = 0 \text{ so } d_{k-1} \gamma_{k-1}[a] = 0 .$$

Now by inductive hypothesis, $\langle u \rangle^{k-1}$ is defined so there is a defining system (a_1, \dots, a_{k-1}) for $\langle u \rangle^{k-1}$ and a cochain a_k such that

$$\delta a_k = \sum_{i=2}^{k-2} (-1)^{\deg a_{i-1}} a_{i-1} a_{k-i}$$

since $\langle u \rangle^{k-1} = d_{k-1} \gamma_{k-1}[a] = 0$.

The observation at the beginning of this lemma shows that

$$d_k \gamma_k[a] = \langle u \rangle^k .$$

We now give the proof of Theorem 2:

Assume $\varphi_k(v)$ is defined and nonzero. We are assuming $1 = \langle \sigma v, u \rangle$.

Hence

$$\begin{aligned} 1 = \langle \sigma v, u \rangle &= \langle \sigma b, a \rangle = \langle [\sigma b]^k, \gamma_k[a] \rangle = \langle d_k \varphi_k(v), \gamma_k[a] \rangle \\ &= \langle \varphi_k(v), d_k \gamma_k[a] \rangle = \langle \varphi_k(v), \langle u \rangle^k \rangle \end{aligned}$$

by the duality of the two spectral sequences and Lemma 2.

It remains to be shown that if $\langle u \rangle^k$ is defined and nonzero, then so is $\varphi_k(v)$. Consider the map

$$\begin{aligned} A &\rightarrow \bar{F}(\bar{B}(A)) \text{ defined by} \\ b &\rightarrow [[b]] . \end{aligned}$$

This map is homotopy multiplicative (in fact is a *SHM* map) and is an equivalence. Hence $[[b^k]]$ differs from $[\sigma b]^k$ by a boundary. But $[\sigma b]^k = [\sigma b | \dots | (k) \dots | \sigma b]$ is dual to $\gamma_k[a] = [a | \dots | (k) \dots | a]$ in $\bar{B}\bar{F}(C)$, and so $d_k \gamma_k[a] = \langle u \rangle^k$ is not zero in E^k (Lemma 2) and so does not

survive to E^∞ , i.e., represents 0 in E^∞ . The dual element $[\sigma b]^k$ represents 0 in E^∞ , i.e., $[[b^k]] \sim [\sigma b]^k \sim 0$. Therefore $b^k \sim 0$ and so $\varphi_k(v)$ is defined.

We wish to mention two applications:

A1: Let $K = Z_p$ and let X be a $K(\pi, n)$ space (p an odd prime). Let $C = C^*(X; Z_p)$ and $A = C_*(X; Z_p)$ be cochain and chain complexes for X of finite type. In the notation of Cartan, $A = A_*(\pi, n; Z_p)$ ([2]). Cartan proved that $\langle \varphi_p(v), \beta P^m(u) \rangle = \langle \sigma v, u \rangle$. Now by Theorem 2, if

$$\langle \sigma v, u \rangle = 1$$

then $\langle \varphi_p(v), \langle u \rangle^p \rangle = 1$. Hence $\langle \varphi_p(v), \beta P^m u + \langle u \rangle^p \rangle = 0$. By Lemma 18 ([5]), $\langle u \rangle^p = c \beta P^m u$. This gives an easy proof of the fact that $c = -1$. (Compare Theorem 19 [5]).

A2: Now let $x = CP^{k-1}$. Then in $H^*(CP^{k-1}; Z) = P(v)_{I(v^k)}$ we have $v^k = 0$. Then $\varphi_k(v)$ is defined in $H^*(\Omega CP^{k-1}; Z)$ and by the Theorem 2, so is $\langle u \rangle^k$ in $H_{3k-2}(\Omega CP^{k-1}; Z)$ where $u \in H_2(\Omega CP^{k-1}, Z)$ and $\langle \varphi(v), \langle u \rangle^k \rangle = 1$. This gives another proof of the results of Stasheff ([7]).

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