PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 1, 1971

ON REPRESENTING F^*-ALGEBRAS

R. M. BROOKS

The purpose of this paper is to obtain a concrete representation for F^*-algebras with identity: a Frechet algebra with involution for which there exists a determining sequence of B^*-seminorms. The main result is Theorem 3.4 which is described here. Let A be an F^*-algebra with identity. Let $\{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ be a complete family of irreducible Hilbert space representations of A. Let $H = \Sigma_\lambda \oplus H_\lambda$, define $E \subseteqq \Lambda$ to be equicontinuous provided $\sup_{\lambda \in E} ||\pi_\lambda(a)|| < \infty$ $(a \in A)$, and let $X = \{x \in H: \mathrm{Supp}\,(x)$ is equicontinuous$\}$. The linear space X is given the final topology τ_f determined by the family $\{H_E = [x \in H: \mathrm{Supp}\,(x) \subseteqq E]$: E equicontinuous$\}$ of subspaces of X. Let X_f be (X, τ_f) and let $\mathscr{L}^*(X_f)$ be all operators on X which have an adjoint relative to the inner product inherited from H such that both the operator and its adjoint are τ_f-continuous. This algebra will be endowed with the topology $\mathscr{T}_b$ of bounded convergence. Let $\mathscr{L}^+(X)$ be all operators which have adjoints. It has a natural topology $\mathscr{T}_+$ described in § 2. Define $\pi: A \to \mathscr{L}_a(X)$ by $\pi(a)\{x_\lambda\} = \{\pi_\lambda(a)x_\lambda\}$ for $a \in A$ and $x = \{x_\lambda\} \in X$. Then $\pi(A) \subseteqq \mathscr{L}^+(X) = \mathscr{L}^*(X_f)$, and (1) $\pi: A \to (\mathscr{L}^*(X_f), \mathscr{T}_b)$ is a topological *-isomorphism (into) and (2) $\pi: A \to (\mathscr{L}^+(X), \mathscr{T}_+)$ is a topological *-isomorphism (into).

In § 1 we recall some results about Fréchet *-algebras with identity, their positive functionals and Hilbert space representations, and set the notation for the remainder of the paper.

In § 2 we obtain the results about algebras of operators on certain inner product spaces necessary to prove the main representation theorem.

In § 4 we define the concept of an enveloping algebra $E(A)$ for a Fréchet *-algebra with identity, A, and show that $E(A)$ can be realized either as the inverse limit of the enveloping algebras of the Banach *-algebras in an inverse limit decomposition of A or as an algebra of operators naturally constructed from the irreducible Hilbert space representations of A. Also we show that $E(A)$ has the property that every Hilbert space representation of A factors through $E(A)$, but that there are representations of A in algebras $\mathscr{L}^+(X)$ which fail to factor through $E(A)$.

1. Preliminaries. A *Fréchet algebra* is a complete metrizable topological algebra whose topology is determined by a (countable) family of seminorms (submultiplicative, convex, symmetric functionals). We may assume that such a family $\{\|\cdot\|_n\}_{n=1}^\infty$ for A is *ascending*: $\|a\|_n \leqq \|a\|_{n+1}(a \in A, n \in N)$, and that $\|e\|_n = 1(n \in N)$ if A has an identity e. A *Fréchet *-algebra* is a Fréchet algebra with a continuous involution. If A is a Fréchet *-alebra with identity we can choose a sequence $\{\|\cdot\|_n\}_{n=1}^\infty$ of seminorms for A such that (1) $\{\|\cdot\|_n\}$ determines the topology of A, (2) $\{\|\cdot\|_n\}$ is ascending, (3) $\|e\|_n = 1(n \in N)$, and (4) $\|a^*\|_n = \|a\|_n(a \in A, n \in N)$. Such a sequence we shall call a **-sequence of seminorms for A*. An *F^*-algebra* is a Fréchet *-algebra, for which there is an ascending determining sequence $\{\|\cdot\|_n\}$ of seminorms for A each of which has the B^*-property: $\|a^*a\|_n = \|a\|_n^2(a \in A, n \in N)$. Such a sequence we shall call an *F^*-sequence of seminorms*. The usual constructions (see [5]) show that every Fréchet *-algebra (resp., F^*algebra) is an inverse limit of Banach *-algebra (resp., B^*-algebras).

Let $(A, \{\|\cdot\|_n\})$ be a Fréchet *-algebra with identity e. We denote by $P(A)$ the set of all positive functionals on A and by $K(A)$ those $f \in P(A)$ for which $f(e) = 1$. For each $n \in N$ we let $P_n(A)$ (resp., $K_n(A)$) be the set of all $f \in P(A)$ (resp., $K(A)$) such that $|f(a)| \leqq f(e)\|a\|_n(a \in A)$. If $\{A_n, \rho^n, N\}$ is the inverse system generated by $\{\|\cdot\|_n\}$ with $\rho_n: A \to A_n$ natural map of A onto the nth member A_n, then for each n the dense homomorphism ρ_n induces a one-to-one map ρ_n^* of $P(A_n)$ onto $P_n(A)$. $(K(A_n))$ onto $K_n(A))$. Moreover, ρ_n^* preserves indecomposability. A theorem of Do-Shing [2] states every positive functional on A is continuous so we have $P(A) = \cup_{n=1}^\infty P_n(A)$ and $K(A) = \cup_{n=1}^\infty K_n(A)$. Also, $K(A)$ is a weak*-closed, convex subset of A^* and is the closed convex hull of its extreme points $\operatorname{ext}(K(A))$ which is exactly $\cup_{n=1}^\infty \operatorname{ext}(K_n(A))$.

A *Hilbert space representation* of a Fréchet *-algebra A with identity is a *-homomorphism $\mu: A \to \mathfrak{B}(H)$ for a Hilbert space H. A consequence of Do-Shing's theorem (see Lemma 3.1 below) is that every such representation is continuous. Moreover, there is a one-to-one correspondence between the members of $K(A)$ and the equivalence classes of cyclic Hilbert space representations of A (with unit cyclic vectors). This correspondence matches elements of $K_n(A)$ with those representations which can be factored through A_n. Also, the indecomposable positive functionals on A correspond to classes of irreducible Hilbert space representations of A.

The **-radical*, $R^*(A)$, of A is the set $\{a \in A: f(a^*a) = 0\ (f \in P(A))\} = \{a \in A: f(a^*a) = 0\ (f \in \operatorname{ext}(K(A))\} = \cap \{\ker \pi: \pi$ is an irreducible Hilbert space representation of $A\}$. If A is an F^*-algebra with identity, then $R^*(A) = (0)$. Hence, if we let Λ be all equivalence classes of irreducible Hilbert space representations of A and for each $\lambda \in \Lambda$ we choose $\pi_\lambda \in \lambda$ with representation space H_λ, then $\{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ is a complete family of irreducible Hilbert space representations of A. A family constructed in this manner for a Fréchet *-algebra A will be called a *standard family of irreducible Hilbert space representations of* A. If $\{\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ is a standard family for A and we let $E_n = \{\lambda: \pi_\lambda$ factors through $A_n\}$, then for each $\lambda \in E_n$ there exists a unique irreducible representation σ_λ of A_n on H_λ so that $\sigma_\lambda \circ \rho_n = \pi_\lambda$. The family $\{\sigma_\lambda: \lambda \in E_n\}$ is a complete family of irreducible representations for A_n (in case A is F^*) and the direct sum $\sum_{\lambda \in E_n} \sigma_\lambda$ on $\sum_{\lambda \in E_n} \oplus H_\lambda$ is an isometry and *-isomorphism of A_n into $\mathfrak{B}(\sum_{\lambda \in E_n} \oplus H_\lambda)$.

We have included no proofs of the facts quoted above since those concerning Fréchet *-algebras are proved for the more general class of locally m-convex *-algebras in [1], and those relating to Banach *-algebras can be found in [6].

2. Certain operator algebras. In this section we obtain the results about special algebras of operators on direct sums and inductive limits of Hilbert spaces which we need in the proof of the main representation theorem in § 3. The concepts considered in the first part of this section are discussed in detail in G. Lassner's work [4].

We first establish our notation. If X is a complex vector space we denote by $\mathscr{L}_a(X)$ the algebra of all linear transformations on X. If X has a locally convex topology τ we denote by $\mathscr{L}(X)$, or by $\mathscr{L}(X_\tau)$ if there are several topologies on X in the discussion, the subalgebra of $\mathscr{L}_a(X)$ consisting of all τ-continuous operators. For a locally convex TVS (X, τ) we denote by $\mathscr{S}$ the family of all τ-bounded subsets of X (with an appropriate subscript on $\mathscr{S}$ if there are several topologies on X). The topology of bounded convergence $\mathscr{T}_b$ is the topology on $\mathscr{L}(X)$ with base at 0 $\{Nbd(M, U): M \in \mathscr{S}, U$ a τ-neighborhood of 0 in $X\}$, where $Nbd(M, U) = \{T \in \mathscr{L}(X): T(M) \subseteq U\}$.

DEFINITION. Let X be an inner product space with inner product $(\cdot, \cdot)$, and let H be the completion of X. $\mathscr{L}^+(X)$ is the subset of $\mathscr{L}_a(X)$ which consists of all $T \in \mathscr{L}_a(X)$ which have an adjoint in $\mathscr{L}_a(X)$: there exists $S \in \mathscr{L}_a(X)$ such that $(Tx, y) = (x, Sy) (x, y \in X)$.

LEMMA 2.1. (Lassner) *$\mathscr{L}^+(X)$ is a *-algebra with involution $T \rightarrow T^*$. Also,* (1) *each $T \in \mathscr{L}^+(X)$ is closed,* (2) *if $X = H$, then $\mathscr{L}^+(X) = \mathfrak{B}(H)$, and* (3) *if there is a closed operator in $\mathscr{L}^+(X)$, then $X = H$.*

Proof. This is merely a compilation of Lemmas 2.1 and 2.2 of [4].

DEFINITION. An *Op^*-algebra* on X is a *-subalgebra $\mathfrak{A}$ with identity I of $\mathscr{L}^+(X)$, (i.e., the identify of $\mathfrak{A}$ is the identity operator on X).

DEFINITION. Let $\mathfrak{A}$ be an Op^*-algebra on X. We define a locally convex topology $\tau_{\mathfrak{A}}$ by taking as a sub-basic family of seminorms $\{\|\cdot\|_T: T \in \mathfrak{A}\}$, where $\|x\|_T = \|Tx\|$ $(x \in X)$. This is the coarsest topology on X with respect to which each operator in $\mathfrak{A}$ is a continuous map into H.

Lassner shows [Lemma 3.1, 4] that each $T \in \mathfrak{A}$ is a continuous linear transformation on $(X, \tau_{\mathfrak{A}})$. Since $I \in \mathfrak{A}$ it follows that $\tau_{\mathfrak{A}}$ is finer than the norm topology of H restricted to X.

DEFINITION. Let $\mathfrak{A}$ be an Op^*-algebra on X. We define two topologies $\mathscr{T}_{\mathfrak{A}}$ and $\mathscr{T}^{\mathfrak{A}}$ on $\mathfrak{A}$ by:

(1) $\mathscr{T}_{\mathfrak{A}}$ is defined the family $\{\|\cdot\|_M: M \in \mathscr{S}_{\mathfrak{A}}\}$ of seminorms where, (a) $\mathscr{S}_{\mathfrak{A}}$ is the family of $\tau_{\mathfrak{A}}$-bounded subset, of X and (b) $\|T\|_M = \sup\{|(Tx, y)|: x, y \in M\}$.

(2) $\mathscr{T}^{\mathfrak{A}}$ is the restriction to $\mathfrak{A}$ of the topology $\mathscr{T}_b$ on $\mathscr{L}(X, \tau_{\mathfrak{A}})$.

LEMMA 2.2. (Lassner) *If $\mathfrak{A}$ is an Op^*-algebra on X, then,*

(1) $(\mathfrak{A}, \mathscr{T}^{\mathfrak{A}})$ *is a locally convex algebra* (separate continuity of multiplication), *but the involution is not in general continuous.*

(2) $(\mathfrak{A}, \mathscr{T}_{\mathfrak{A}})$ *is a locally convex algebra with continuous involution.*

(3) $\mathscr{T}_{\mathfrak{A}} \leqq \mathscr{T}^{\mathfrak{A}}$, *and* $\mathscr{T}_{\mathfrak{A}} = \mathscr{T}^{\mathfrak{A}}$ *if, and only if, the multiplication in* $(\mathfrak{A}, \mathscr{T}_{\mathfrak{A}})$ *is* (jointly) *continuous.*

Proof. This is a compilation of Theorems 4.1 and 4.2 and Example 5.1 of [4].

NOTATION. For the maximal Op^*-algebra $\mathscr{L}^+(X)$ on X we shall let τ_+ and $\mathscr{T}_+$ replace the clumsier notation $(\tau_{\mathscr{L}^+(X)}, \mathscr{T}_{\mathscr{L}^+(X)})$ of the definitions above.

We now specialize to a particular class of inner product spaces. Let $\{H_\rho: \beta \in B\}$ be a family of Hilbert spaces, let $H = \Sigma_\beta \oplus H_\beta$ and let $X = \Sigma_\beta H_\beta$ (the algebraic direct sum). For $\beta \in B$ we let $p_\beta: X \to H_\beta$ be the natural projection and let $q_\beta: H_\beta \to X$ be the natural injection. For $x \in X$ we define $\mathrm{Supp}(x) = \{\beta: p_\beta(x) \neq 0\}$.

The locally convex direct sum topology, τ_f, on X is the final topology determined by the family $\{q_\beta: \beta \in B\}$. We shall abbreviate (X, τ_f) by X_f.

LEMMA 2.3. *If $X = \Sigma_\beta H_\gamma$, then $\mathscr{L}^+(X)$ is isomorphic to the algebra of all B^2-matrices $(T_{\alpha\beta})_{\alpha,\beta \in B}$ such that*

(1) $T_{\alpha\beta} \in \mathscr{L}(H_\beta, H_\alpha)(\alpha, \beta \in B)$,
(2) *for each* $\alpha \in B$ *the set* $B_\alpha = \{\beta: T_{\alpha\beta} \neq 0\}$ *is finite, and*
(3) *for each* $\beta \in B$ *the set* $B^\beta = \{\alpha: T_{\alpha\beta} \neq 0\}$ *is finite.*

Proof. *If* $(T_{\alpha\beta})$ is a matrix satisfying (1)—(3) we define $T: X \rightarrow X$ by $(T\{X_\beta\})_\alpha = \sum_\beta T_{\alpha\beta}x_\beta$ for each α. Since Supp (x) is finite, $\sum_\beta T_{\alpha\beta}x_\beta$ converges in H_α for each $\alpha \in B$ and it is easily seen that the set of α for which $\Sigma_\beta T_{\alpha\beta}x_\beta$ is nonzero is contained in $\cup\{B^\beta: \beta \in \text{Supp}(x)\}$, a finite set. Thus, $T \in \mathscr{L}_a(X)$ and by considering the matrix $(T^*_{\alpha\beta})(T^*_{\alpha\beta}: H_\alpha \rightarrow H_\beta)$ we obtain an adjoint for T in $\mathscr{L}_a(X)$; hence, $T \in \mathscr{L}^+(X)$.

Fix $T \in \mathscr{L}^+(X)$. For $\alpha, \beta \in B$ define $T_{\alpha\beta} = p_\alpha T q_\beta: H_\beta \rightarrow H_\alpha$. Clearly, $T_{\alpha\beta}$ is a linear transformation. We show that it has an everywhere defined adjoint, hence is bounded. Set $S_{\beta\alpha} = p_\beta T^* q_\alpha: H_\alpha \rightarrow H_\beta$. Fix $x_\alpha \in H_\alpha$, $x_\beta \in H_\beta$, then;

$$\begin{aligned}
&(T_{\alpha\beta}x_\beta, x_\alpha) = (p_\alpha T q_\beta x_\beta, x_\alpha)\\
&= (p_\alpha T q_\beta x_\beta, p_\alpha q_\alpha x_\alpha)\\
&= \Sigma_\gamma(p_\gamma T q_\beta x_\beta, p_\gamma q_\alpha x_\alpha)\\
&= (T q_\beta x_\beta, q_\alpha x_\alpha) = (q_\beta x_\beta, T^* q_\alpha x_\alpha)\\
&= (x_\beta, S_{\beta\alpha}x_\alpha) \text{ (by reversing the steps above)} .
\end{aligned}$$

Fix $\beta \in B$. If B^β is not finite, then there exists a sequence $\{\alpha_j\}$ it B so that $T_{\alpha_j\beta} \neq 0$ (j $= 1, 2, \cdots$). For each $x_\beta \in H_\beta$ there exists $n(x_\beta)$ so that $T_{\alpha_j\beta}(x_\beta) = 0$ for $j > n(x_\beta)$. We choose sequences $\{n_j\}$ in $\boldsymbol{N}$ and $\{x_j\}$ in H_β by the following procedure. Let $n_1 = 1$ and choose $x_1 \in H_\beta$ so that $||x_1|| < 2^{-1}$ and $T_{n_1}x_1 \neq 0$ (hereafter T_j will be used for $T_{\alpha_j\beta}$). There exists $n_2 > n_1$ such that $T_j x_1 = 0$ for $j \geqq n_2$. Choose $x_2 \in H_\beta$ such that $T_{n_2}x_2 \neq 0$ and $||x_2|| < \min(2^{-2}, 2^{-2}||T_{n_1}x_1||/||T_{n_1}||)$. Continuing inductively we obtain sequences $\{n_j\}$ and $\{x_j\}$ so that:

(1) $1 = n_1 < n_2 < \cdots$.
(2) $||x_j|| < \min(2^{-j}, 2^{-j}||T_{n_1}x_1||/||T_{n_1}||, \cdots, 2^{-j}||T_{n_{j-1}}x_{j-1}||/||T_{n_{j-1}}||)$
(3) $T_{n_j}x_i \neq 0$
(4) $T_{n_i}x_j = 0$ $(i > j)$.

We let $x = \sum_{j=1}^\infty x_j \in H_\beta$. We claim that $T_{n_k}x \neq 0$ $(k \geqq 1)$. Fix $k \in \boldsymbol{N}$, then $T_{n_k}x = \sum_{j=1}^{k-1} T_{n_k}x_j + T_{n_k}x_k + \sum_{j=k+1}^\infty T_{n_k}x_j$. For $j \leqq k-1$ we have $T_{n_k}x_j = 0$ and for $j > k+1$ we have $||T_{n_k}x_j|| \leqq ||T_{n_k}|| \, ||x_j|| < 2^{-j}||T_{n_k}x_k||$. If $T_{n_k}x = 0$, then, $||T_{n_k}x_k|| = ||\sum_{j=k+1}^\infty T_{n_k}x_j|| \leqq \sum_{j=k+1}^\infty 2^{-j}$ $||T_{n_k}x_k|| < ||T_{n_k}x_k||$, a contradiction.

That B_α is finite for each $\alpha \in B$ follows by applying the same argument to T^*.

LEMMA 2.4. *Let* $\{X_\beta\}$ *be a family of Banach spaces and let* $X = \sum_\beta X_\beta$. *For* $c \in \boldsymbol{R}_+^B$ *define* $p_c: X \rightarrow \boldsymbol{R}_+$ *by* $p_c(x) = \sum_\beta c_\beta ||x_\beta||$. *Then*

$\{p_c: c \in \boldsymbol{R}_+^B\}$ defines the locally convex direct sum topology τ_f on X.

Proof. Clearly, $\{p_c\}$ defines a separated locally convex topology τ' on X. Since τ_f is the final topology generated by the injections q_β: $X_\beta \to X(\beta \in B)$, it suffices to show (1) if p is a τ_f-continuous seminorm on X, then there exists $c \in \boldsymbol{R}_+^B$ so that $p \leqq p_c$ (hence, $\tau_f \leqq \tau'$), and (2), for each $\beta \in B$ the map $q_\beta: X_\beta \to (X, \tau')$ is continuous.

(1) Fix a τ_f-continuous seminorm p on X. For each $\beta \in B$ the map $p \circ q_\beta: X_\beta \to \boldsymbol{R}$ is a continuous seminorm on X_β. Hence, there exists $c_\beta \in \boldsymbol{R}_+$ so that $p \circ q_\beta(x_\beta) \leqq c_\beta \|x_\beta\|$ $(x_\beta \in X_\beta)$. This defines the function $c \in \boldsymbol{R}_+^B$. If $x = \{x_\beta\} \in X$, then,

$$\begin{aligned} p(x) &= p(\textstyle\sum_\beta q_\beta(x_\beta)) \leqq \sum_\beta p \circ q_\beta(x_\beta) \\ &\leqq \textstyle\sum_\beta c_\beta \|x_\beta\| = p_c(x) . \end{aligned}$$

(2) Fix $\beta \in B, c \in \boldsymbol{R}_+^B$. Then $p_c(q_\beta x_\beta) = c_\beta \|x_\beta\|$ and $q_\beta: X_\beta \to (X, \tau')$ is continuous.

LEMMA 2.5. *Let $X = \sum_\beta H_\beta$. For each $c \in \boldsymbol{R}_+^B$ we define $\|\ \|_c: X \to \boldsymbol{R}$ by $\|x\|_c = [\sum_\beta c_\beta^2 \|x_\beta\|^2]^{1/2}$ for $x = \{x_\beta\} \in X$. If B is countable then τ_f is defined by the seminorms $\{\|\cdot\|_c: c \in \boldsymbol{R}_+^B\}$.*

Proof. Suppose $B = \boldsymbol{N}$. We have for each $c \in \boldsymbol{R}_+^N$ that $\|\cdot\|_c \leqq p_c$, so $\tau_{\{\|\cdot\|_c\}} \leqq \tau_f$. We fix $c \in \boldsymbol{R}_+^N$.

For $x \in X$ we have:

$$\begin{aligned} p_c(x) &= \textstyle\sum_n c_n \|x_n\| = \sum_n n^{-1}(nc_n \|x_n\|) \\ &\leqq (\textstyle\sum_n n^{-2})^{1/2}(\sum_n (nc_n)^2 \|x_n\|^2)^{1/2} \\ &= (\text{constant}).\ \|x\|_{\{nc_n\}} . \end{aligned}$$

THEOREM 2.6. *If $X = \sum_\beta H_\beta$, then $\mathscr{L}^+(X) \subseteqq \mathscr{L}(X_f)$; hence, $\mathscr{L}^+(X) = \mathscr{L}^*(X_f)$, the algebra of continuous operators on X_f with continuous adjoints.*

Proof. It suffices to show that for each $T \in \mathscr{L}^+(X)$ and $\beta \in B$ the operator $T \circ q_\beta; H_\beta \to X_f$ is continuous (see [Prop. 6.1, p. 54, 7]). Fix $T \in \mathscr{L}^+(X)$, $\beta \in B$ and a seminorm p_c, $c \in \boldsymbol{R}_+^B$, for τ_f. The set $B^\beta = \{\alpha: T_{\alpha\beta} \neq 0\}$ is finite, so for $x_\beta \in H_\beta$ we have:

$$\begin{aligned} p_c(T \circ q_\beta(x_\beta)) &= \textstyle\sum_\alpha c_\alpha \|(T \circ q_\beta(x_\beta))_\alpha\| \\ &= \textstyle\sum_{\alpha \in B^\beta} c_\alpha \|T_{\alpha\beta} x_\beta\| . \end{aligned}$$

Hence, $T \circ q_\beta$ is continuous.

We now turn from direct sums to inductive limits. Let $\{H_\lambda: \lambda \in \Lambda\}$ be a family of Hilbert spaces. Let $H = \sum_\lambda \oplus H_\lambda$. We fix a family $\mathscr{E}$ of subsets of Λ which satisfies (a) $\mathscr{E}$ is closed under finite unions, and (b) all subsets of a member of $\mathscr{E}$ belong to $\mathscr{E}$ (i.e., $\mathscr{E}$ is an ideal in the lattice 2^{Λ}). We let $X = \{x \in H: \mathrm{Supp}\,(x) \in \mathscr{E}\}$, and for $E \in \mathscr{E}$ we let $H_E = \{x \in H: \mathrm{Supp}\,(x) \subseteqq E\}$, *a* Hilbert space, and let i_E be the identity injection of H_E into X. Finally, let τ_f be the final topology on X determined by the family $\{i_E: E \in \mathscr{E}\}$ of injections.

Since X has an inner product the Op^*-algebra $\mathscr{L}^+(X)$ is defined. A subalgebra of $\mathscr{L}^+(X)$ of importance here is $\mathscr{L}_r(X) = \{T \in \mathscr{L}^+(X)$: for each $E \in \mathscr{E}$, $T(H_E) \subseteqq H_E$ and $T^*(H_E) \subseteqq H_E\}$; i.e., $\mathscr{L}_r(X)$ consists of all elements of $\mathscr{L}^+(X)$ which are reduced by each $H_E (E \in \mathscr{E})$.

We denote the topologies on X determined by $\mathscr{L}^+(X)$ and $\mathscr{L}_r(X)$ by τ_+ and τ_r (respectively) and the corresponding families of bounded sets by $\mathscr{S}_+$ and $\mathscr{S}_r$.

ASSUMPTION. Throughout the remainder of this section we assume the existence of an ascending *countable* cofinal (with respect to the partial order $\subseteqq$ on $\mathscr{E}$) subfamily $\mathscr{E}_0 = \{E_n\}_{n=1}^{\infty}$. We let the corresponding Hilbert spaces H_{E_n} and injections i_{E_n} be denoted H_n and $i_n (n \in N)$.

LEMMA 2.7. *The final topology on X generated by the family $\{i_n\}_{n=1}^{\infty}$ is τ_f. Hence, (X, τ_f) is a strict inductive limit of the sequence of Hilbert spaces $\{H_n\}$ and;*

(1) *$\tau_f \mid H_n$ is the norm topology on H_n.*

(2) *$M \subseteqq X$ is τ_f-bounded if, and only if, there exists $n \in N$ such that M is a* (norm) *bounded subset of H_n.*

Proof. That the final topologies are the same can either be easily proved directly or deduced from Proposition 3, p. 159 of [3]. That we have a strict inductive limit and (1) follow from the fact that $\tau_{n+1} \mid H_n = \tau_n$ (trivial if one writes out the norm of an element of H_n considered as an element of H_{n+1}) and a theorem of Dieudonné-Schwartz (see [pp. 159–160, 3]). Claim (2) is another theorem of Dieudonné-Schwartz (see [p. 161, 3]).

THEOREM 2.8. $\mathscr{S}_r = \mathscr{S}_f$

Proof. We show first that $\mathscr{L}_r(X)$ is a subalgebra of $\mathscr{L}(X_f)$. Fix $T \in \mathscr{L}_r(X)$, $n \in N$. We must show that $T \circ i_n = T \mid H_n: H_n \to X_f$ is continuous. But $T(H_n) \subseteqq H_n$ and $\tau_f \mid T_n = \tau_n$. Thus, we must show that $T \mid H_n: H_n \to H_n$ is continuous. Since $T \in \mathscr{L}_r(X)$ we have that $T^*(H_n) \subseteqq H_n$, so $(T \mid H_n)^* = T^* \mid H_n$ and $T \mid H_n$ has an everywhere defined adjoint on H_n, hence is continuous.

Let $M \in \mathscr{S}_f$. Then M is a bounded subset of H_n for some $n \in N$.

For each $T \in \mathscr{L}_r(X)$, $T(M)$ is a bounded subset of H_n since $T \mid H_n$ is continuous on H_n. Hence, $T(M)$ is bounded in H. Since T was arbitrary, $M \in \mathscr{S}_r$.

Suppose $M \notin \mathscr{S}_f$. Case (a). There exist sequences $\{n_j\}$ in N and $\{x_j\}$ in M so that:

(1) $1 = n_1 < n_2 < \cdots$.

(2) $x_j \in H_{n_{j+1}} \backslash H_{n_j} (j = 1, 2, \cdots)$.

Choose $x_1 \in M \backslash H_{n_1} (n_1 = 1)$ let n_2 be sufficiently large that $\operatorname{Supp}(x_1) \subseteqq E_{n_2}$, choose $x_2 \in M \backslash H_{n_2}$, etc. Let $D_j = E_{n_j} \backslash E_{n_{j-1}} (n_0 = 0, E_0 = \varnothing)$, and set $C_j = (\sum_{\lambda \in D_j} \| x_{j,\lambda} \|)^{1/2}$. Define $T: X \to X$ by $(Tx)_\lambda = jC_j^{-1} x_\lambda$ if $\lambda \in D_j$. It is easily verified that $T \in \mathscr{L}_a(X)$, $T^* = T$, and $\operatorname{Supp}(Tx) \subseteqq \operatorname{Supp}(x)$ for $x \in X$. Hence, $T \in \mathscr{L}_r(X)$. Also,

$$\begin{aligned} \| Tx_n \|^2 &= \sum_{j=1}^{\infty} \sum_{\lambda \in D_j} (jC_j^{-1})^2 \| x_{n,\lambda} \|^2 \\ &\geqq (nC_n^{-1})^2 \sum_{\lambda \in D_n} \| x_{n,\lambda} \|^2 = n^2 . \end{aligned}$$

Hence, $\sup_{x \in M} \| Tx \| = \infty$, and $M \notin \mathscr{S}_r$. Case (b). $M \subseteqq H_n$, for some n, but is unbounded. Easy to show that $M \notin \mathscr{S}_r$.

THEOREM 2.9. *$\mathscr{T}_r = \mathscr{T}_b \mid \mathscr{L}_r(X)$, where $\mathscr{T}_b$ is the topology of bounded convergence on the algebra $\mathscr{L}(X_f)$.*

Proof. (1) $\mathscr{T}_r \leqq \mathscr{T}_b$ on $\mathscr{L}_r(X)$: Fix a $\mathscr{T}_r$-neighborhood of 0 in $\mathscr{L}_r(X)$, $\operatorname{Nbd}(M, \varepsilon) = \{T: \| T \|_M < \varepsilon\}$, where $M \in \mathscr{S}_r$, $\varepsilon > 0$. Since $\mathscr{S}_r = \mathscr{S}_f$ there exists $n \in N$ so that $M \subseteqq H_n$ and $\| M \| = \sup_{x \in M} \| x \| < \infty$. Let U be a τ_f-neighborhood of 0 in X so that $U \cap H_n \subseteqq S_n(0, (2 \| M \|)^{-1} \varepsilon)$, the $(2 \| M \|)^{-1}$ ε-ball about 0 in H_n. If $T \in \mathscr{L}_r(X) \cap N\operatorname{bd}(M, U) = \{S \in \mathscr{L}_r(X): S(M) \subseteqq U\}$, then $T(M) \subseteqq U \cap H_n$ and $\| Tx \| < (2 \| M \|)^{-1} \varepsilon$ for $x \in M$. Now

$$\begin{aligned} \| T \|_M &= \sup \{ | (Tx, y) | : x, y \in M \} \\ &\leqq \sup_{x \in M} \{ \sup_{\|y\| \leqq \|M\|} | (Tx, y) | \} \\ &\leqq \sup \| M \| \cdot \| Tx \| \leqq \varepsilon/2 < \varepsilon . \end{aligned}$$

This shows that $\mathscr{T}_r \leqq \mathscr{T}_b$.

(2) $\mathscr{T}_b \leqq \mathscr{T}_r$ on $\mathscr{L}_r(X)$: Fix a $\mathscr{T}_b$-neighborhood of 0 in $\mathscr{L}_r(X)$, $N\operatorname{bd}(M, U)$ where $M \in \mathscr{T}_f$, U is a τ_f-neighborhood of 0 in X. There exists $n \in N$ so that M is a bounded subset of H_n. Let $M_1 = M \cup S_n(0, 1)$, bounded subset of H_n; hence, a τ_f-bounded subset of X. Choose $\varepsilon > 0$ so that $S_n(0, \varepsilon) \subseteqq U \cap H_n$. Suppose $T \in N\operatorname{bd}(M_1, \varepsilon)$. If $x \in M$, then

$$\| Tx \| = \sup_{\|y\| \leqq 1} | (Tx, y) | \leqq \sup \{ | (Tz, y) | : z, y \in M_1 \} = \| T \|_{M_1} < \varepsilon .$$

So $T(M) \subseteqq S_n(0, \varepsilon) \subseteqq U$, and $T \in N\operatorname{bd}(M, U)$. Hence $\mathscr{T}_b \leqq \mathscr{T}_r$.

We now show that $\mathscr{L}^+(X)$ is a subalgebra of $\mathscr{L}(X_f)$. The problem here is that we do not have an obvious charactrization of elements of $\mathscr{L}^+(X)$. We have a fixed cofinal sequence $\{E_n\}$ in $\mathscr{S}$. We let $D_1 = E_1$ and for $n > 1$ we set $D_n = E_n \backslash E_{n-1}$. Let $K_n = \sum_{y \in D_n} \oplus H_\lambda$ and let $Y = \sum_n K_n$.

LEMMA 2.10. *The map* $u: X \to Y$ *defined by* $u(\{x_\lambda\}_{\lambda \in \Lambda}) = \{\{x_\lambda\}_{\lambda \in D_n}\}_{n=1}^\infty$ *has the following properties*:

(1) *u is a linear isomorphism* (onto)

(2) *u is unitary*: $(u(x), u(x')) = (x, x')$ *for* $x, x' \in X$.

(3) $u: (X, \tau_f) \to (Y, \tau_f)$ *is topological.*

Proof. It is easily verified that u is a linear isomorphism (into). If $y \in Y$, then there exists $n \in N$ so that $y_j = 0$ for $j > n$. Then $y_{j\lambda} = 0$ if $\lambda \in D_j, j > n$. Set $x = \{y_{k\lambda}: \lambda \in D_k, k = 1, 2, \cdots\}$. Then $x \in H$ and $\operatorname{Supp}(x) \subseteqq \cup_{j=1}^n D_j = E_n$. Clearly $u(x) = y$, and (1) is proved. That u is unitary depends only on the fact that the series obtained by taking inner products is absolutely convergent so can be rearranged at will.

(3) We show first that u is continuous. We recall Lemma 2.5 and fix a seminorm $\|\cdot\|_c$, $c \in \boldsymbol{R}_+^N$. If $x \in H_n$, then:

$$\begin{aligned}\|u(x)\|_c^2 &= \textstyle\sum_{j=1}^\infty c_j^2 \|u(x)_j\|^2 \\ &= \textstyle\sum_{j=1}^\infty c_j^2 \{\sum_{\lambda \in D_j} \|x_\lambda\|^2\} \\ &= \textstyle\sum_{j=1}^n c_j^2 \{\sum_{\lambda \in D_j} \|x_\lambda\|^2\} \\ &\leqq (\max_{1 \leqq j \leqq n} c_j^2) \|x\|^2 .\end{aligned}$$

Thus, $\|u(x)\|_c \leqq C(n) \|x\|$ for $x \in H_n$ and $u \circ i_n: H_n \to Y_f$ is continuous for arbitrary $n \in N$. Hence, u is continuous $(X_f \to Y_f)$.

Since X_f and Y_f are (LF) spaces (each is a strict inductive limit of Hilbert spaces) and u is a continuous surjection of X_f to Y_f it follows that u is an open map (see [Prop 2.2, p. 78, 7]).

THEOREM 2.11. $\mathscr{L}^+(X) \subseteqq \mathscr{L}(X_f)$; *hence,* $\mathscr{L}^+(X) = \mathscr{L}^*(X_f)$, *the subalgebra of* $L(X_f)$ *which consists of all operators* $T \in \mathscr{L}(X_f)$ *whose adjoint* T^* *exists and belongs to* $\mathscr{L}(X_f)$.

Proof. Let u and Y be as in Lemma 2.10. Since u is a unitary linear isomorphism the induced map $u^*: \mathscr{L}_a(X) \to \mathscr{L}_a(Y)$ defined by $u^*(T) = u \circ T \circ u^{-1}$, which maps $\mathscr{L}_a(X)$ isomorphically onto $\mathscr{L}_a(Y)$, maps $\mathscr{L}^+(X)$ onto $\mathscr{L}^+(Y)$. Also, since $u: X_f \to Y_f$ is topological the same map u^* maps $\mathscr{L}(X_f)$ onto $\mathscr{L}(Y_f)$. Since $\mathscr{L}^+(Y) \subseteqq \mathscr{L}(Y_f)$ (Theorem 2.6) we must have that $\mathscr{L}^+(X) \subseteqq \mathscr{L}(X_f)$.

THEOREM 2.12. $\mathscr{T}_+ \mid \mathscr{L}_r(X) = \mathscr{T}_b \mid \mathscr{L}_r(X)$.

Proof. We show first that $\tau_+ \leqq \tau_f$. It suffices to show that for each $n \in N$ the injection $i_n: H_n \to (X, \tau_+)$ is continuous. Fix a seminorm $\|\cdot\|_T$ for τ_+, where $T \in \mathscr{L}^+(X)$. Since $T \in \mathscr{L}(X_f)$ we have that $T \circ i_n = T \mid H_n: H_n \to X_f$ is continuous; hence, $T \mid H_n: H_n \to H$ is continuous ($\tau_{\text{norm}} \leqq \tau_f$). But then there exists $C_T > 0$ so that $\| T \mid H_n(x) \| \leqq C_T \|x\|$ for $x \in H_n$; i.e., $\|x\|_T \leqq C_T \|x\|$ $(x \in H_n)$.

We have $\mathscr{S}_r = \mathscr{S}_f \subseteqq \mathscr{S}_+ (\tau_+ \leqq \tau_f)$. But since $\mathscr{L}_r(X) \subseteqq \mathscr{L}^+(X)$ we also have $\mathscr{S}_+ \subseteqq \mathscr{S}_r$. Hence, $\mathscr{S}_r = \mathscr{S}_+$ and $\mathscr{T}_r = \mathscr{T}_+$ on $\mathscr{L}_r(X)$. But Theorem 2.9 says that $\mathscr{T}_r = \mathscr{T}_b$ on $\mathscr{L}_r(X)$.

THEOREM 2.13. *$(\mathscr{L}_r(X), \mathscr{T}_r)$ is complete.*

Proof. Let $\{T_\alpha\}$ be a $\mathscr{T}_r$-Cauchy set in $\mathscr{L}_r(X)$. For each $E \in \mathscr{E}$ we let M_E be the unit ball in H_E. Then, $\{M_E\} \subseteqq \mathscr{S}_r$ and if we fix $E \in \mathscr{E}$, then $\{T_\alpha \mid H_E\}$ is a Cauchy net in $\mathfrak{B}(H_E)$:

$$\begin{aligned} \| T_\alpha \mid H_E - T_\beta \mid H_E \| &= \sup \{\| T_\alpha - T_\beta)x\| : x \in H_E, \|x\| \leqq 1\} \\ &= \sup \{| ((T_\alpha - T_\beta)x, y) | : x, y \in M_E\} \\ &= \| T_\alpha - T_\beta \|_{M_E} . \end{aligned}$$

Note also $\| T_\alpha^* \mid H_E - T_\beta^* \mid H_E \| = \|T_\alpha - T_\beta\|_{M_E}$. Thus, $T_\alpha \mid H_E \to T_E \in \mathfrak{B}(H_E)$ and $T_\alpha^* \mid H_E \to S_E \in \mathfrak{B}(H_E)$. We define T and S on X by $Tx = T_E x$ if $x \in H_E$ and $Sx = S_E x$ if $x \in H_E$. If $E \subseteqq F$, then $T_F \mid H_E = T_E$ $(S_F \mid H_E = S_E)$. Hence, T and S are well-defined linear transformations on X. Clearly, both T and S leave each H_E invariant and $S = T^*$. Thus, $T \in \mathscr{L}_r(X)$. That $\mathscr{T}_r\text{-}\lim_\alpha T_\alpha = T$ is easily checked.

THEOREM 2.14. *$(\mathscr{L}_r(X), \mathscr{T}_r)$ is an F^*-algebra with identity. In fact, $(\mathscr{L}_r(X), \mathscr{T}_r) \cong \lim_n \operatorname{inv} \mathfrak{B}(H_n)$.*

Proof. For each $n \in N$ we let M_n be the unit ball in H_n. Then $\{M_n\} \subseteqq \mathscr{S}_r$ and is "essentially" cofinal: if $M \in \mathscr{S}_r$, then M is a bounded subset of some H_n, hence there exists $k \in \boldsymbol{R}_+$ so that $M \subseteqq kM_n$. But then $\| T\|_M \leqq k^2 \| T\|_{M_n} (T \in \mathscr{L}_r^*(X))$. Thus, the topology $\mathscr{T}_r$ is determined by th ascending family $\{\| \cdot \|_{M_n}\}$ of (linear) seminorms, and $(\mathscr{L}_r(X), \mathscr{T}_r)$ is a complete metrizable algebla. As we saw in Theorem 2.13 for $T \in \mathscr{L}_r(X)$ and $n \in N$ we have $\| T\|_{M_n} = \| T^* \|_{M_n} = \| T | H_n \|$. Thus, each $\| \cdot \|_{M_n}$ is a B^*-seminorm, and $(\mathscr{L}_r(X), \mathscr{T}_r)$ is an F^*-algebra with identity. The last part of the conclusion was essentially proved in Theorem 2.13. The map $\rho_n: \mathscr{L}_r(X) \to \mathfrak{B}(H_n)$ is just the restriction map; as is the bonding map $\rho^n: \mathfrak{B}(H_n) \to \mathfrak{B}(H_{n-1}) (n \in N)$.

3. A representation theorem for F^*-algebras. In this section we give three concrete faithful (topologically and algebraically) representations for an abstract F^*-algebra with identity as an algebra of operators on a vector space formed from the irreducible Hilber space representations of the algebra.

We let A be an F^*-algebra with identity and let $\{(\pi_\lambda H_\lambda): \lambda \in \Lambda\}$ be a standard family of irreducible Hilbert space representations of A. Since A is an F^*-algebra the family is complete.

If A is a B^*-algebra and $\{(\pi_\lambda, H_\lambda)$ is a standard family for A, then $\pi: A \to \mathfrak{B}(\sum_\lambda \oplus H_\lambda)$ defined by $\pi(a)\{x_\lambda\} = \{\pi_\lambda(a)x_\lambda)$ is an isometry and *-isomorphism. It is easily seen that for non-B^*-algebras (but still F^*-) this is impossible. In fact, one cannot even define $\pi(a)$ on $\sum_\lambda \oplus H_\lambda$ for all $a \in A$, unless every $a \in A$ has bounded norm: $\sup_n \|a\|_n < \infty$ for some determining family of seminorms. If one moves to the other extreme and defines $\pi(a)$ by the same formula on $X = \sum_\lambda H_\lambda$ (algebraic sum), then $\pi(a)$ makes sense and $\pi: A \to (\mathscr{L}(X_f), \mathscr{T}_b)$ is a continuous *-isomorphism but fails to be topological. This is the case because the final topology on X, hence the topology $\mathscr{T}_b$ on $\mathscr{L}(X_f)$, depends on finite subsets of Λ whereas that of A depends on much larger subsets of Λ. Thus, we must seek a middle ground in order to achieve a faithful representation of A in this manner. Before we introduce the basic concept we first prove a crucial fact about Hilbert space representations of Fréchet *-algebras.

LEMMA 3.1. *Let A be a Fréchet *-algebra with identity, and let $\mu: A \to \mathfrak{B}(H)$ be a representation of A on the Hilbert space H. Then μ is continuous.*

Proof. Fix $\varepsilon > 0$. Let $V = \{a \in A: \|\mu(a)\| \leqq \varepsilon\} = \cap \{V_{x\,y}: \|x\|, \|y\| \leqq 1\}$, where $V_{x\,y} = \{x \in A: |(\mu(a)x, y)| \leqq \varepsilon\}$. For each pair $x, y \in H$ such that $\|x\|, \|y\| \leqq 1$ the set $V_{x,\,y}$ is convex and balanced. Since for each $z \in H$ the map $a \to (\mu(a)z, z)$ is continuous (Do-Shing's Theorem [2]), we have that $a \to (\mu(a)x, y)$ is continuous (polarization formula). Thus, each $V_{x\,y}$ is closed. So V is closed, convex, and balanced. It is easily verified that V is absorbing; hence is a neighborhood of 0 in A.

DEFINITION. Let A be a Fréchet *-algebra with identity and let $\{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ be a stadard family of irreducible Hilbert space representations of A. A subset E of Λ will be called *equicontinuous* if, and only if $\sup_{\lambda \in E} \|\pi_\lambda(a)\| < \infty$ for each $a \in A$. The family of all equicontinuous subsets of Λ will be denoted $\mathscr{E}(\Lambda)$.

LEMMA 3.2. *If A is a Fréchet *-algebra with identity and* $\{(\pi_\lambda,$

H_λ): $\lambda \in \Lambda$} is a standard family for A, then $E \subseteqq \Lambda$ is equicontinuous if, and only if, $\sum_{\lambda \in E} \pi_\lambda$ defines a continuous representation of A on $\sum_{\lambda \in E} \oplus H_\lambda$.

Proof. Suppose $E \subseteqq \Lambda$ is equicontinuous. For $a \in A$ we let $C_a = \sup_{\lambda \in E} ||\pi(a)||$ and define $\pi: A \to \mathfrak{B}(\sum_{\lambda \in E} \oplus H_\lambda)$ by $\pi(a)\{x_\lambda\} = \{\pi_\lambda(a)x_\lambda\}$. Now $||\{\pi_\lambda(a)x_\lambda\}||^2 = \sum_\lambda ||\pi_\lambda(a)x_\lambda||^2 \leqq C_a^2 ||(x)||^2$. So $\pi(a)$ maps $\sum_{\lambda \in E} \oplus H_\lambda$ into itself, and π is a representation of A on $\sum_{\lambda \in E} \oplus H_\lambda$.

Conversely, suppose we can define a representation of A on $\sum_{\lambda \in E} \oplus H_\lambda$ by the direct sum formula. It is clear that $||\pi_\lambda(a)|| \leqq ||\pi(a)||$ for each $\lambda \in E$ and $a \in A$.

LEMMA 3.3. *Let A, $\{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ be as above. Let $\{||\cdot||_n\}$ be a $*$-sequence of seminorms for A. For $n \in N$ we set $E_n = \{\lambda \in \Lambda: ||\pi_\lambda(a)|| \leqq ||a||_n \, (a \in A)\}$. Then $E \subseteqq \Lambda$ is equicontinuous if, and only if, E is contained in some E_n. In particular, the increasing sequence $\{E_n\}$ is cofinal in $\mathscr{E}(\Lambda)$.*

Proof. If $E \subseteqq E_n$ for some n, then clearly $E \in \mathscr{E}(\Lambda)$. Conversely, if $E \in \mathscr{E}(\Lambda)$ then $\pi: A \to \mathfrak{B}(\sum_{\lambda \in E} \oplus H_\lambda)$ defined as in Lemma 3.2 is a continuous representation of A. Hence, there exists $C > 0, n \in N$ so that $||\pi(a)|| \leqq C||a||_n \, (a \in A)$. It is easily verified that we can take $C = 1$, and the condition is satisfied.

We set $H = \sum_{\lambda \in \Lambda} \oplus H_\lambda$ and let $X = \{x \in H: \operatorname{Supp}(x) \in \mathscr{E}(\Lambda)\}$. We are now in the situation of the second part of § 2 with $H_E = \{x: \operatorname{Supp}(x) \subseteqq E\}$, $i_E: H_E \to X$ the natural injection, τ_f the final topology determined by the family $\{i_E: E \in \mathscr{E}(\Lambda)\}$. If $\{||\cdot||_n\}$ is any F^*sequence of seminorms for our F^*-algebra A with identity, then we let $H_n = H_{E_n}$ and $\pi_n: A \to \mathfrak{B}(H_n)$ the induced representation of A on H_n.

LEMMA 3.4. *With the definitions given immediately above for each $n \in N$ and $a \in A$ it is the case that $||a||_n = ||\pi_n(a)||$.*

Proof. In § 1 we indicated that $\{\pi_\lambda: \lambda \in E_n\}$ induces a complete standard family $\{\sigma_\lambda: \lambda \in E_n\}$ of irreducible Hilbert space representations of the B^*-algebra A_n, the completion of $A/\{a: ||a||_n = 0\}$ with respect to the induced norm, and if ρ_n is the natural projection of A into A_n we have $\sigma_\lambda \circ \rho_n = \pi_\lambda (\lambda \in E_n)$. If we let $\sigma_n = \sum_{\lambda \in E_n} \sigma_\lambda: A_n \to \mathfrak{B}(H_n)$, then $||\sigma_n(a_n)|| = ||a_n||$ for each $a_n \in A_n$. But $||\sigma_n(a_n)|| = \sup_{\lambda \in E_n} ||\sigma_\lambda(a_n)||$. Thus, if $a \in A$, then $||a||_n = ||\rho_n a|| = \sup_{\lambda \in E_n} ||\sigma_\lambda(\rho_n a)|| = \sup_{\lambda \in E_n} ||\pi_\lambda(a)|| = ||\pi_n(a)||$.

From the above construction we can infer more. For each $a \in A$ there exists $\lambda_0 \in E_n$ such that $||a||_n = ||\pi_{\lambda_0}(a)||$. This can be proved for A_n by reducing the problem to that for a hermitian elements, then showing that it holds on the algebra generated by the element and extending to the full algebra.

THEOREM 3.5. *Let A be an F^*-algebra with identity, $\{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ a standard family of irreducible Hilbert representations of A, $H = \sum_{\lambda \in \Lambda} \oplus H_\lambda$, and $X = \{x \in H: \text{Supp}(x)$ is equicontinuous$\}$. Let $\pi: A \to \mathscr{L}_a(X)$ be defined by $\pi(a)\{x_\lambda\} = \{\pi_\lambda(a)x_\lambda\}(a \in A, x = \{x_\lambda\} \in X)$. Then*

(1) *For each $a \in A$ the function $\pi(a)$ defined above is indeed in $\mathscr{L}_a(X)$; in fact, $\pi(a) \in \mathscr{L}_r(X)$.*

(2) *$\pi: A \to (\mathscr{L}^*(X_f), \mathscr{T}_b)$ is a topological *-isomorphism* (into).

(3) *$\pi: A \to (\mathscr{L}^+(X), \mathscr{T}_+)$ is a topological *-isomorphism* (into).

(4) *$\pi: A \to (\pi(A), \mathscr{T}_{\pi(A)})$ is a topological *-isomorphism.*

Proof. (1) Fix $a \in A, x \in X$. Then

$$\sum_\lambda ||\pi_\lambda(a)x_\lambda||^2 \leqq \sum \{||\pi_\lambda(a)||^2 \, ||x_\lambda||^2: \lambda \in \text{Supp}(x)\}$$
$$\leqq \sup \{||\pi_\lambda(a)||^2: \lambda \in \text{Supp}(x)\} \cdot ||x||^2 .$$

Thus, $\pi(a)x \in H$ and $\text{Supp}(\pi(a)x) \subseteq \text{Supp}(x) \in \mathscr{E}(\Lambda)$, so $\pi(a)$ maps X into itself. Moreover, if $a \in A$ and $x, y \in X$ we have $(\pi(a)x, y) = (x, \pi(a^*)y)$; so $\pi(a) \in \mathscr{L}^+(X)$. It is clear that $\pi(a) \in \mathscr{L}_r(X)$.

(2) and (3) It is clear that π is a *-isomorphism. Since $\mathscr{T}_+ | \mathscr{L}_r(X) = \mathscr{T}_b | \mathscr{L}_r(X) = \mathscr{T}_r$ (Theorem 2.12) and $\pi(A) \subseteq \mathscr{L}_r(X)$ it is necessary and sufficient that we show $\pi: A \to (\mathscr{L}_r(X), \mathscr{T}_r)$ is topological. We fix an F^*-sequence $\{||\cdot||_n\}$ of seminorms for A, let $\{E_n\}$, $\{H_n\}$, and $\{\pi_n\}$ be the corresponding cofinal sequence in $\mathscr{E}(A)$, Hilbert space sequence in X, and sequence of representations of A (respectively). We note that $\pi_n(a) = \pi(a) | H_n$ for $n \in N, a \in A$. We recall from Theorem 2.14 that $(\mathscr{L}_r(X), \mathscr{T}_r)$ is an F-*algebra with identity and that $\{||\cdot||_{M_n}\}$ is an F^*-sequence of seminorms for $\mathscr{L}_r(X)$, where M_n is the closed unit ball in $H_n(n \in N)$. Moreover, for each $n \in N$ and $T \in \mathscr{L}_r(X)$ we have $||T||_{M_n} = ||T | H_n||$, the norm of the (bounded) restriction of T to H_n. If $a \in A$, then $||a||_n = ||\pi_n(a)||$ (Lemma 3.4) and the latter is $||\pi(a) | H_n|| = ||\pi(a)||_{M_n}$.

We again fix an F^*-sequence $\{||\cdot||_n\}$ of seminorms for A. Let $\{E_n\}$, etc. be as in "(2) and (3)" above. We let $\mathfrak{A} = \pi(A)$, an Op^*-algebra on X with corresponding family $\mathscr{S}_\mathfrak{A}$ of bounded subsets of X. The topology $\mathscr{T}_\mathfrak{A}$ on $\mathfrak{A}$ is defined by the seminorms $\{||\cdot||_M: M \in \mathscr{S}_\mathfrak{A}\}$, where $M \subseteq X$ belongs to $\mathscr{S}_\mathfrak{A}$ if, and only if, $\sup_{x \in M} ||Tx|| < \infty$ for each $T \in \mathfrak{A}$. Lassner's Lemma 5.2 [4] says that $\pi: A \to (\mathfrak{A}, \mathscr{T}_\mathfrak{A})$ is continuous. Fix $n \in N$ and let M_n be the closed unit ball in H_n as above. Since $\mathfrak{A} \subseteq \mathscr{L}^+(X)$, $\mathscr{S}_+ \subseteq \mathscr{S}_\mathfrak{A}$ and $\{M_n\} \subseteq \mathscr{S}_+$; so $\{M_n\} \subseteq \mathscr{S}_\mathfrak{A}$. We know from above that $||a||_n = ||\pi(a)||_{M_n} (a \in A, n \in N)$. This establishes the openness of $\pi: A \to (\mathfrak{A}, \mathscr{T}_{\mathfrak{a}})$.

REMARKS. (1) Do-Shing [2] obtains a representation theorem for LMC^*-algebras (the same as F^*-algebra without the metrizability restriction) which uses essentially the same Hilbert space, but he maps A onto an algebra of unbounded operators with special properties. Also, he does not consider topological properties of the map.

(2) The main problem in studying non-commutative Fréchet *-algebras is the lack of models against which to compare the abstract algebras. A corollary to Do-Shing's theorem on positive functionals on Fréchet *-algebras is that every one induces a cyclic Hilbert space representation, but as we have seen we cannot represent these algebras faithfully on Hilbert spaces. The examples discussed above, the algebras $\mathscr{L}_r(X)$, are quite similar to those considered by E. A. Michael in Appendix A of his memoir [5], where in our case the underlying locally convex space is an inductive limit of Banach (Hilbert) spaces. It seems that the class he defined in [5] might include most examples of noncommutative F-algebras, except those built from a commutative F-algebra and a noncommutative Banach algebra by tensor products, e.g., $C(X, B)$ where X is an appropriate topological space and B is a Banach algebra.

4. Enveloping algebras. In this section we define the enveloping algebra of a Fréchet *-algebra with identity, relate it to inverse limit decompositions of the algebra, and realize it as an algebra of operators naturally constructed from A.

We fix a Fréchet *-algebra with identity, A, and also fix a standard family $\{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ of irreducible Hilbert space representations of A. We recall that $K(A) = \{f: f$ is a positive functional on A, $f(e) = 1\}$.

LEMMA 4.1. *If* $E \subseteqq K(A)$ *and* $\{\|\cdot\|_n\}$ *is a *-sequence of seminorms for* A, *then the following statements are equivalent.*

(1) *E is equicontinuous.*

(2) $\sup_{f \in E} f(a^*a) < \infty \, (a \in A)$.

(3) *There* $n \in N$ *such that* $E \subseteqq K_n(A)$.

Proof. (1) and (2) are clearly equivalent by the uniform boundedness principle for Fréchet spaces: if $E \subseteqq A^*$ and E is pointwise bounded ($\sigma(A^*, A)$-bounded), then E is equicontinuous (see [Theorem 4.2, p. 83, 7]). It is also clear that (1) and (3) are equivalent, since $K_n(A)$ is the intersection with $K(A)$ of the polar of the neighborhood $\{a \in A: \|a\|_n \leqq 1\}$.

DEFINITION. Let $\mathscr{E}(K)$ be all equicontinuous subsets of $K(A)$. For $E \in \mathscr{E}(K)$ we define

$$|a|_E = [\sup\{f(a^*a): f \in E\}]^{1/2} \qquad (a \in A) .$$

THEOREM 4.2. *If A is a Fréchet *-algebra, $\{|\cdot|_E: E \in \mathscr{E}(K)\}$ is the family of maps defined above, then*

(1) *Each $|\cdot|_E$ is a linear seminorm on A.*

(2) *$R^*(A) = \{a \in A: |a|_E = 0$ for each $E \in \mathscr{E}(K)\}$.*

(3) *If $\{||\cdot||_n\}$ is any *-sequence of seminorms for A, then the topology of $(A/R^*(A), \{|\cdot|_E\})$ is determined by the B^*-seminorms $\{|\cdot|_n\}$, where $|\cdot|_n = |\cdot|_{K_n(A)}$. Hence,*

(4) *The completion $E(A)$ of $(A/R^*(A), \{|\cdot|_E)\}$ is an F^*-algebra with identity.*

Proof. (1) and (2) are trivial to verify and (4) follows from (3), which we now prove. Fix a *-sequence $\{||\cdot||_n\}$ of seminorms for A. For each $n \in N$ we set $E_n = \{\lambda \in \Lambda: ||\pi_\lambda(a)|| \leqq ||a||_n\,(a \in A)\}$ and define $\pi_n: A \to \mathfrak{B}(\sum_{\lambda \in E_n} \oplus H_\lambda)$ by $\pi_n(a)(\{\xi\}_\lambda\}_{\lambda \in E_n}) = \{\pi_\lambda(a)\xi_\lambda\}_{\lambda \in E_n}$ for each $a \in A$. We shall show that for each $n \in N$ and $a \in A$ we have $||\pi_n(a)|| = |a|_n$. Fix $n \in N$. For $\lambda \in \Lambda$ we choose a unit vector $\xi_\lambda \in H_\lambda$, define $f_\lambda: A \to C$ by $f_\lambda(a) = (\pi_\lambda(a)\xi_\lambda, \xi_\lambda)$, let K_λ be the completion of $A/\{a: f_\lambda(a^*a) = 0\}$ with respect to the induced inner product $([a]_\lambda, [b]_\lambda) = f_\lambda(b^*a)$, where $[a]_\lambda$ is the coset containing a. Finally, define $\psi_\lambda: A \to \mathfrak{B}(K_\lambda)$ by $\psi_\lambda(a)([b]_\lambda) = [ab]_\lambda$ on $A/\{a: f_\lambda(a^*a) = 0\}$, and extending these norm-continuous operators to K_λ. There exists an isomorphism $U: H_\lambda K_\lambda$ so that $U\pi_\lambda = \psi_\lambda U$. Hence, for $a \in A$ we have $||\pi_\lambda(a)||^2 = ||\psi_\lambda(a)||^2 = \sup\{f_\lambda(b^*a^*ab): f_\lambda(b^*b) = 1\} \geqq f_\lambda(a^*a)$. If $f_\lambda(b^*b) = 1$, then $f_{\lambda,b}: c \to f_\lambda(b^*cb)$ also belongs to $K_n(A)$ (that f_λ does is clear) and $f_{\lambda,b}(a^*a) \leqq |a|_n$. Hence, $||\pi_\lambda(a)|| \leqq |a|_n$ for each $\lambda \in E_n$, and $||\pi_n(a)|| = \sup ||\pi_\lambda(a)||: \lambda \in E_n\} \leqq |a|_n$. Then reverse inequality follows from the fact that $|a|_n = \sup\{f(a^*a): f \in K_n(A), f$ is extreme$\}$.

DEFINITION. We shall call the algebra $E(A)$ in Theorem 4.2 (4) the *enveloping algebra* of A.

THEOREM 4.3. *If $(A, \{||\cdot||_n\})$ is a Fréchet *-algebra with identity and if $\{A_n\}$ is the corresponding inverse limit system of Banach *-algebras with identity, then $E(A) = \lim_n \operatorname{inv}\{E(A_n)\}$.*

Proof. We let ρ_n be the natural map of A into A_n and $\rho^n: A_n \to A_{n-1}(n \geqq 2)$ the induced bonding map. For $n \in N$ we let E_n be the enveloping algebra of A_n and let Ψ_n be the natural map of A_n into E_n. Finally, we let φ be the map of A into $E(A)$.

For $n \in N$ we have the diagram

$$\begin{array}{ccc} A_{n-1} & \xleftarrow{\rho^n} & A_n \\ \Psi_{n-1}\downarrow & & \downarrow\Psi_n \\ E_{n-1} & & E_n \end{array} .$$

Now $\ker(\Psi_n) = R^*(A_n)$ and $\rho^n(R^*(A_n)) \subseteqq R^*(A_{n-1})$. Thus, we have an induced map $\sigma^n\colon E_n \to E_{n-1}$. It is easily verified that $\{E_n, \sigma^n, N\}$ is a dense inverse limit system of B^*-algebras (i.e., the bonding maps have dense range and are norm-decreasing). We let $E = \lim_n \operatorname{inv} \{E_n, \sigma^n, N\}$ and consider E a subset of $\Pi_n E_n$ with the relative product topology.

We define $\tau\colon E(A) \to E$ by first defining τ on $A/R^*(A)$ by the formula $\tau(\varphi a) = \{\Psi_n \rho_n a\}$. If $a \in A$, then $\varphi(a) = 0$ if, and only if, $a \in R^*(A)$ if, and only if, $\rho_n(a) \in R^*(A_n)(n \in N)$ if, and only if, $\Psi_n\rho_n(A) = 0$ $(n \in N)$. Thus, τ is well-defined and one-to-one $A/R^*(A)$. Also, since all the maps involved have dense range it follows that $\tau(A/R^*(A))$ is dense in E. Finally,

$$\begin{aligned} |\tau(\varphi a)|_n^2 &= |\Psi_n \sigma_n(a)|_n^2 = |\rho_n(a)|_n^2 \\ &= \sup\{f_n(\rho_n(a^*a))\colon f \in K(A_n)\} \\ &= \sup\{f(a^*a)\colon f \in K_n(A)\} \\ &= |a|_n^2 \ (n \in N, a \in A) . \end{aligned}$$

Thus, τ is an isometry in each seminorm; hence, extends to a topological map of E onto $E(A)$. It is clear that the map is a $*$-isomorphism.

We now realize $E(A)$ as an algebra of operators on $X = \{x \in \sum_{\lambda \in \Lambda} \oplus H_\lambda\colon \operatorname{Supp}(x)$ is equicontinuous$\}$. We use the same notation as in § 3.

THEOREM 4.4. *Let $(A, \{\|\cdot\|_n\})$ be a Fréchet $*$-algebra with identity and let $(E(A), \{|\cdot|_n\})$ be its enveloping algebra, with natural map φ: $A \to E(A)$. For $a \in A$ we define $\pi(a)$ on X by $\pi(a)\{x_\lambda\} = \{\pi_\lambda(a)x_\lambda\}$. Then $\pi\colon A \to \mathscr{L}_r(X)$ induces a topological $*$-isomorphism $\bar{\sigma}$ of $E(A)$ onto $\overline{\pi(A)}$, where "topological" refers to any of the (equal on $\mathscr{L}_r(X)$) topologies $\mathscr{T}_r$, $\mathscr{T}_+$, or $\mathscr{T}_b$ on $\mathscr{L}_r(X)$ and the closure of $\pi(A)$ is with respect to these topologies.*

Proof. Since for each $a \in A$ and $n \in N$ we have $|a|_n = \|\pi_n(a)\|$ we have that $\ker \pi = R^*(A)$, so there is an induced map $\sigma\colon A/R^*(A) \to \mathscr{L}_r(X)$ so that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\pi} & \mathscr{L}_r(X) \\
{\scriptstyle\varphi}\downarrow & \nearrow{\scriptstyle\sigma} & \\
A/R^*(A) & & \\
\cap\| & & \\
E(A)\,. & &
\end{array}
$$

We have shown in Theorem 2.13 that the topologies $\mathscr{T}_r = \mathscr{T}_b = \mathscr{T}_+$ on $\mathscr{L}_r(X)$ are defined by the sequence $\{\|\cdot\|_{M_n}\}$ of seminorms and that $\|T\|_{M_n} = \|T|H_n\|$. Also, we know that $\pi_n(a)$ is $\pi(a)|H_n$. So from Theorem 4.2 we have $|a|_n = \|\pi_n(a)\|$, and hence, $|a|_n = \|\pi(a)\|_{M_n}$. It follows that σ: $A/R^*(A) \to \mathscr{L}_r(X)$ is topological, and since $(\mathscr{L}_r(X), \mathscr{T}_r)$ is complete σ extends to a topological *-isomorphism of $E(A)$ into $\mathscr{L}_r(X)$.

If A is a Banach *-algebra with identity and $E(A)$ its enveloping algebra, then every Hilbert space representation of A factors through $E(A)$. We conclude our discussion of enveloping algebras by examing this problem for Fréchet *-algebras. We consider only representations in $\mathscr{L}^+(X)$, since this is enough to illustrate the problems involved.

LEMMA 4.5. *If $(A, \{\|\cdot\|_n\})$ is a Fréchet *-algebra with identity, $\{|\cdot|_n\}$ the corresponding sequence of B^*-seminorms on A used to define the topology of $E(A)$, and if μ: $A \to \mathscr{L}^+(X)$ is an essential representation of A on X ($\mu(e) = I$) then, for each $M \in \mathscr{S}_+$ there exists $n \in N$ and $C > 0$ such that $\|\mu(a)\|_M \leqq C|a|_n (a \in A)$.*

Proof. Fix $M \in \mathscr{S}_+$. We let $\|M\| = \sup\{\|x\|: x \in M\}$ ($\|M\| < \infty$, since M is bounded in the Hilbert space completion of X). Since μ is continuous there exist $n \in N$ and $C > 0$ such that $\|\mu(a)\|_M \leqq C\|a\|_n (a \in A)$.

Fix $x \in M$. Then f_x: $a \to (\mu(a)x, x)$ is a positive functional on A. Also, $|f_x(a)| = |(\mu(a)x, x)| \leqq \|\mu(a)\|_M \leqq C\|a\|_n$. Hence, $f_x \in P_n(A)$ for each $x \in M$. Therefore, if $x \in M$ and $x \neq 0$, the positive functional $f_x(e)^{-1}f_x$ belongs to $K_n(A)$ and $f_x(e)^{-1}f_x(a^*a) \leqq |a|_n^2 (a \in A)$. So we have $f_x(a^*a) \leqq f_x(e)|a|_n^2 (x \in M, a \in A)$. But $f_x(e) = (\mu(e)x, x) = \|x\|^2 \leqq \|M\|^2$. Hence, $f_x(a^*a) \leqq \|M\|^2|a|_n^2 (x \in M, a \in A)$.

For $x, y \in M, a \in A$ we have

$$
\begin{aligned}
|(\mu(a)x, y)| &= \|\mu(a)x\|\cdot\|y\| \\
&\leqq \|M\|\cdot(\|\mu(a)x\|^2)^{1/2} \\
&\leqq \|M\|\cdot(\mu(a^*a)x, x)^{1/2} \\
&\leqq \|M\|^2|a|_n\,.
\end{aligned}
$$

Thus, $\| \mu(a) \|_M \leqq \| M \|^2 \, | a |_n (a \in A)$.

THEOREM 4.6. *If* $(A, \{\| \cdot \|_n\})$ *is a Fréchet *-algebra with identity,* $(E(A), (| \cdot |_n\})$ *its enveloping algebra,* φ *the natural map of* A *into* $E(A)$, *and if* $\mu: A \to \mathscr{L}^+(X)$ *is an essential representation of* A *on* X, *then there exists a continuous representation* σ *of* $A/R^*(A)$ *on* X *so that* $\sigma\varphi = \mu$. *If* $\pi(A)$ *is contained in a* $\mathscr{T}_+$*-complete subalgebra of* $\mathscr{L}^+(X)$, *then* σ *extends to a representation of* $E(A)$ *on* X. *In particular, this is the case if* X *is Hilbert space. Hence, all Hilbert space representations of* A *factor through* $E(A)$.

Proof. We need only show that σ can be defined on $A/R^*(A)$ so that $\sigma\varphi = \mu$. The other claims follow from Lemma 4.5. It is sufficient to show that $\ker \varphi \subset \ker \mu$. *If* $a \in \ker \varphi = R^*(A)$ *and if* $x \in X$, *then* $b \to (\mu(b)x, x)$ *is a positive functional on* A; *hence,* $(\mu(a^*a)x, x) = \| \mu(a)x \|^2 = 0$. *Thus,* $\mu(a) = 0$ *and* $a \in \ker \mu$.

EXAMPLE 4.7. We show here that some representations $\mu: A \to \mathscr{L}^+(X)$ fail to factor through $E(A)$.

Let $A = C^\infty(\boldsymbol{R})$ with the topology determined by the seminorms $\| a \|_n = \sum_{k=0}^{n} (k!)^{-1} \| a^{(k)} \|_{n,\infty}$, where $a^{(k)}$ is the kth derivative of a and $\| \cdot \|_{n,\infty}$ is the supremum on $[-n, n]$. Then A is a commutative Fréchet *-algebra with identity (involution is conjugation), and

(1) $| a |_n = \| a \|_{n,\infty} (a \in A, n \in \boldsymbol{N})$

(2) $R^*(A) = 0$, hence

(3) $A/R^*(A) = (A, \{| \cdot |_n\})$ and $E(A)$ is just $C(\boldsymbol{R})$ with the compact-open topology. We use hereafter $| \cdot |_n$ for $\| \cdot \|_{n,\infty}$.

Let $X = C_0^\infty(\boldsymbol{R})$, the compactly-supported C^∞ functions on $\boldsymbol{R}$, considered as a dense subspace of $L^2(\boldsymbol{R})$. We note that if $a \in C(\boldsymbol{R})$ and $f \in C_0^\infty(\boldsymbol{R})$, then there exists $n \in \boldsymbol{N}$ such that $\| af \| \leqq | a |_n \| f \|$ (n depends on f, n is any positive integer so that $\operatorname{Supp}(f) \subseteqq [-n, n]$), and $\| \cdot \|$ is the norm in $L^2(\boldsymbol{R})$.

Define $\mu: A \to \mathscr{L}_a(X)$ by $\mu(a)f = af$. It is clear that this formula actually does define a linear transformation on X and that (1) $\mu(a)^* = \mu(\bar{a})$, (2) μ is a representation of A in $\mathscr{L}^+(X)$, hence, (3) μ; $(A, \{| \cdot |_n\}) \to \mathscr{L}^+(X)$ is continuous (by Theorem 4.6). We now show that μ cannot be extended continuously to $C(\boldsymbol{R})$. We prove (4): if $\tilde{\mu}$ is the extension to $C(\boldsymbol{R})$ of μ and if $f \in X$, then, we must have $\tilde{\mu}(a)f = af$ $(a \in C(\boldsymbol{R}))$. We know that there exists $n \in \boldsymbol{N}$ and $C > 0$ so that $\| \tilde{\mu}(a)f \| \leqq C | a |_n$ for each $a \in C(\boldsymbol{R})$. Fix $a \in C(\boldsymbol{R})$ and choose $\{a_j\} \subseteqq C^\infty(\boldsymbol{R})$ so that $C(\boldsymbol{R}) - \lim_j a_j = a$. Choose $C > 0$ and $n \in \boldsymbol{N}$ as above (for $f \in X$) and such that $\operatorname{Supp}(f) \subseteqq [-n, n]$. Then $\| \tilde{\mu}(a)f - \tilde{\mu}(a_j)f \| \leqq C | a - a_j |_n$. Hence, $\{\tilde{\mu}(a_j)f\}$ converges in $L^2(\boldsymbol{R})$ to $\tilde{\mu}(a)f$. But $\tilde{\mu}(a_j)f = a_j f$ and by our earlier estimate $\| a_j f - af \| \leqq \| f \| \, | a_j - a |_n$. So

$\tilde{\mu}(a_j)f = a_j f$ converges to $\tilde{\mu}(a)f$ and to af. Thus $\tilde{\mu}(a)f = af$ for each $f \in X$. But $C(\boldsymbol{R})\cdot C_0^\infty(\boldsymbol{R}) \nsubseteq C_0^\infty(\boldsymbol{R})$, so μ fails to extend to $C(\boldsymbol{R})$.

REMARKS. In the last example we could have considered μ a representation of $C^\infty(\boldsymbol{R})$ in $\mathscr{L}(C_0^\infty(\boldsymbol{R}), \mathscr{T}_b)$. It is not too difficult to show that μ is continuous when thought of this way. It clearly still fails to extend.

Fainally, we do not know whether representations of A in $(\mathscr{L}^*(X), \mathscr{T}_b)$ where X is a locally convex TVS with a continuous inner product are necessarily continuous, in contrast to representations in $\mathscr{L}^+(X)$. It probably is possible to find an example of a discontinuous representation, since the topology $\mathscr{T}_b$ need not be related to the inner product.

REFERENCES

1. R. M. Brooks, *On locally m-convex *-algebras*, Pacific J. Math., **23** (1967), 5-23.
2. Siu Do-Shing, *С. Дошпн, О по-лунормированных кольцах с инволюцией*, Изв. АН СССР, Сер. Мат., **23** (1959), p. 509-528.
3. J. Horváth, *Topological vector spaces and distributions*, vol. I, Addison-Wesley, Reading, Mass., 1966.
4. G. Lasser, *Topological Algebras of Operators*, (Preprint, Joint Institute for Nuclear Research, Dubna, 1969).
5. E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc., **11**, 1952.
6. C. E. Rickart, *General Theory of Banach algebras*. van Nostrand, Princeton, 1960.
7. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966.

Received September 10, 1970. Research for this paper was supported in part by NSF Grants GP-8346 and GP-18729.

UNIVERSITY OF UTAH