PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 1, 1971

SPECTRAL THEORY FOR A FIRST-ORDER SYMMETRIC SYSTEM OF ORDINARY DIFFERENTIAL OPERATORS

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For a symmetric differential expression associated with a first order system

$$A_0(t)x' + A(t)x, \quad a < t < b$$

where A_0 and A are $n \times n$ matrices and x is an $n \times 1$ vector, a spectral decomposition will be developed. That is, if S is a closed symmetric differential operator determined by the differential system, the explicit nature of the generalized resolutions of the identity for all the self-adjoint extensions of S in any Hilbert space will be determined in terms of a fundamental matrix and spectral matrices associated with these extensions. An important aspect is that these self-adjoint extensions may be defined in Hilbert spaces larger than the natural one $\mathscr{H}$ in which the operator S is defined.

The development proceeds as in Coddington [5]; however, the consideration of systems of differential equations introduces matrix techniques and notation. It is hoped that this formulation will have application to such problems as open end (infinite time) control theory problems, and facilitate the canonical formulation of the associated spectral analysis.

Preliminary definitions. Let $\mathscr{H}$ be a Hilbert space with an inner product $(\ ,\ )$.

(1) *Generalized Resolution of the Identity.* Let $F = \{F(\lambda)\}$ be a family of bounded self-adjoint operators in $\mathscr{H}$, depending on real λ, such that:

(i) $F(\lambda) \geqq F(\mu)$, $\lambda > \mu$,

(ii) $F(\lambda + 0) = F(\lambda)$,

(iii) $F(\lambda) \to I$, as $\lambda \to +\infty$,

$F(\lambda) \to 0$, as $\lambda \to -\infty$,

then F is a generalized resolution of the identity.

The family F is said to be associated with a symmetric operator Z (or F is a "spectral function" for Z, Naimark [7]) if

$$(Zu, v) = \int \lambda d(F(\lambda)u, v),$$

and

$$||Zu||^2 = \int \lambda^2 d(F(\lambda)u, u),$$

for all $u \in \mathscr{D}(Z)$ and $v \in \mathscr{H}$.

(2) *Generalized Resolvent.* Let Z be a symmetric operator and $F = \{F(\lambda)\}$ be an associated generalized resolution of the identity. For $\operatorname{Im} l \neq 0$, let $\mathscr{R} = \{\mathscr{R}(l)\}$ be a family of operators such that

$$(\mathscr{R}(l)u, v) = \int \frac{d(F(\lambda)u, v)}{\lambda - l}.$$

Then $\mathscr{R}$ is a generalized resolvent of Z associated with F. The development for symmetric operators will include the case for self-adjoint operators.

1. Basic vector and matrix definitions. In addition to the usual definitions and notation for the absolute magnitude of a vector, the inner product of two vectors, the norm of a vector and the absolute magnitude of a matrix, the norm of a matrix is defined as

$$||A|| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \int |A_{ij}(t)|^2 dt\right)^{1/2} = \left(\int \text{trace}\,(A^*(t)A(t))dt\right)^{1/2};$$

and a matrix "inner product" is introduced,

$$(A, B) = \int B^*(t)A(t)dt,$$

which is a matrix whose (i, j)th element is

$$\sum_{l=1}^{n} \int \bar{B}_{li}(t)A_{lj}(t)dt.$$

This "inner product" makes sense for any two matrices for which $B^*(t)A(t)$ exists and is integrable.

An inner product of a matrix and a vector can be defined in some situations; it is a special case of the matrix "inner product." For example, if f is an $n \times 1$ vector and G an $n \times n$ matrix,

$$(f, G) = \int G^*(t)f(t)dt.$$

2. The "basic operators" T_0 and T. Let (a, b) be an open interval on the real line (a may be $-\infty$ and/or b may be $+\infty$). A differential operator L is defined by

$$Lx = A_0(t)x' + A(t)x.,$$

where: x is an $n \times 1$ vector, A_0 and A are n by n suitably regular matrix-valued functions (Brauer [2]) and $'$ denotes d/dt. The Lagrange adjoint, L^+, associated with L is defined by

$$\begin{aligned} L^+y &= -(A_0^*(t)y)' + (A^*(t)y) \\ &= -A_0^*(t)y' + (-A_0^{*\prime}(t) + A^*(t)y). \end{aligned}$$

The operator L is formally self-adjoint if $L = L^+$, that is when

$$A_0 = -A_0^* \text{ and } A = -A_0^{*\prime} + A^* = A_0' + A^*.$$

Throughout the remainder of this paper L will be assumed to be formally self-adjoint.

Using the definitions for the inner product of two vectors, and for the norm, a Hilbert space, $\mathscr{H}$, can be defined,

$$\mathscr{H} = \mathscr{L}^2(a, b) = \{u\colon \|u\| < \infty\}.$$

Defining a domain $\mathscr{D}$ in $\mathscr{H}$ by $\mathscr{D} = \{u \in \mathscr{H}$: (i) u is absolutely continuous on every compact subinterval of (a, b), (ii) $Lu \in \mathscr{H}\}$, an operator T, having domain $\mathscr{D}$, can be defined by

$$Tu = Lu, \quad u \in \mathscr{D}.$$

Let, for $u, v \in \mathscr{D}$,

$$\langle uv \rangle = (Lu, v) - (u, L^+v) = (Lu, v) - (u, Lv).$$

Then, similarly for a domain $\mathscr{D}_0$,

$$\mathscr{D}_0 = \{u \in \mathscr{D} \colon \langle uv \rangle = 0 \text{ for all } v \in \mathscr{D}\},$$

an operator T_0 can be defined by

$$T_0 u = Lu, \quad u \in \mathscr{D}_0.$$

The development of the operators T_0 and T is motivated by the fact that T_0 is the smallest closed symmetric operator in $\mathscr{H}$ (associated with the differential operator L) having a domain which contains all vectors which are infinitely differentiable on (a, b) and vanish outside closed bounded subintervals of (a, b). Further, if F_1 is any generalized resolution of the identity for a closed symmetric operator T_1, where $T_0 \subset T_1 \subset T$, then F_1 is a generalized resolution of the identity for T_0, also. Thus, by considering T_0, a maximal set of generalized resolutions of the identity, which are naturally associated with L, can be obtained.

The following theorem provides an important relation between T_0 and T.

THEOREM 2.1. *The operator T_0 is closed, symmetric, and $T_0^* =$*

T, $T^* = T_0$.

Proof. Let

$$K(t, \tau) = \begin{cases} \Phi(t)\Phi^{-1}(\tau)A_0^{-1}(\tau), & t \geqq \tau, \\ 0, & t < \tau, \end{cases}$$

where Φ is a fundamental matrix, that is, a matrix whose columns are independent solutions of $Lx = 0$. Thus, Φ is a nonsingular $n \times n$ matrix such that $L\Phi = 0$. As a function of t, $LK(t, \tau) = 0$, and

$$K(t+, t) - K(t-, t) = \Phi(t+)\Phi^{-1}(t)A_1^{-0}(t) - 0 = A_0^{-1}(t).$$

The representation for K can be simplified

$$(\Phi^* A_0 \Phi)' = 0, \text{ or } \Phi^* A_0 \Phi = D^{-1},$$

where D is a skew-Hermitian constant matrix, and hence

$$\Phi^{-1}A_0^{-1} = D\Phi^*.$$

The matrix K can now be written as

$$K(t, \tau) = \begin{cases} \Phi(t)D\Phi^*(\tau), & t \geqq \tau, \\ 0, & t < \tau. \end{cases}$$

Let Δ be a closed bounded subinterval $[\tilde{a}, \tilde{b}]$ of (a, b). The Hilbert space $\mathscr{L}^2(\Delta)$ is defined by

$$\mathscr{L}^2(\Delta) = \{u: \|u\|_\Delta < \infty\}.$$

For $t \in \Delta$, the vector x defined by

$$x(t) = \int_{\tilde{a}}^{\tilde{b}} K(t, \tau)y(\tau)d\tau = \int_{\tilde{a}}^{t} K(t, \tau)y(\tau)d\tau,$$

where $y \in \mathscr{L}^2(\Delta)$, is such that

(0) $x \in \mathscr{L}^2(\Delta)$,

(i) x is absolutely continuous on Δ,

(ii) $Lx \in \mathscr{L}^2(\Delta)$.

Having verified that for $t \in \Delta$ and $y \in \mathscr{L}^2(\Delta)$ the vector x satisfies conditions (0), (i), and (ii), the proof follows exactly as in Theorem 1 of reference 3.

3. The Green's function G_Δ. In § 5. the generalized resolvents associated with T_0 will be constructed. The generalized resolvent will be developed starting from the Green's function G_Δ associated with certain self-adjoint boundary-value problems on finite subintervals

Δ. The purpose of this section is to derive such Green's functions G_Δ.

Once again, let Δ be a closed bounded subinterval of (a, b), denoted by $[\tilde{a}, \tilde{b}]$. Analogous to previous definitions, a domain $\mathscr{D}_\Delta$ is defined and the associated operator T_Δ, having domain $\mathscr{D}_\Delta$,

$$T_\Delta u = Lu, \quad u \in \mathscr{D}_\Delta.$$

Similarly, for $\mathscr{D}_{0\Delta}$, an operator $T_{0\Delta}$, having domain $\mathscr{D}_{0\Delta}$, is defined by

$$T_{0\Delta} u = Lu, \quad u \in \mathscr{D}_{0\Delta}.$$

[NOTE: The conditions and relations of Theorem 2.1 hold for $T_{0\Delta}$ and T_Δ.]

It will now be shown that abstract self-adjoint boundary conditions can be constructed by considering the self-adjoint extensions of $T_{0\Delta}$. Let

$$\mathscr{E}_\Delta(\pm i) = \{v \in \mathscr{D}_\Delta : T_\Delta v = \pm iv\};$$

It is clear that dim $\mathscr{E}_\Delta(i) = \dim \mathscr{E}_\Delta(-i) = n$. The domain $\mathscr{D}_\Delta$ can be written as a direct sum

$$\mathscr{D}_\Delta = \mathscr{D}_{0\Delta} + \mathscr{E}_\Delta(i) + \mathscr{E}_\Delta(-i).$$

From the theory of the Cayley transform (see Riesz-Nagy [8], for example) every self-adjoint extension, $T_{\Delta U}$, of $T_{0\Delta}$ has a domain

$$\mathscr{D}_{\Delta U} = \mathscr{D}_{0\Delta} + (I - U)\mathscr{E}_\Delta(-i),$$

where U is a unitary mapping from $\mathscr{E}_\Delta(-i)$ onto $\mathscr{E}_\Delta(i)$; and

$$T_{\nabla U} u = Lu, \quad u \in \mathscr{E}_{\Delta U}.$$

Let $\{\varphi_{\Delta i}\}$ $i = 1, \cdots, n$, be an orthonormal basis for $\mathscr{E}_\Delta(i)$; also let $\{\psi_{\Delta i}\}$ $i = 1, \cdots, n$, be an orthonormal basis for $\mathscr{E}_\Delta(-i)$; finally let

$$v_{\Delta j} = \psi_{\Delta j} - U\psi_{\Delta j},$$

and

$$v_{\Delta j*} = \varphi_{\Delta j} - U^*\varphi_{\Delta j}, \quad j = 1, \cdots, n.$$

The following theorem describes the abstract self-adjoint boundary conditions induced by the domain $\mathscr{D}_{\Delta U}$.

THEOREM 3.1. *The domain* $\mathscr{D}_{\Delta U}$ *of* $T_{\Delta U}$ *has the following representation:*

$$\mathscr{D}_{\Delta U} = \{u \in \mathscr{D} : \langle u v_{\Delta j*} \rangle = 0, \mathrm{j} = 1, \cdots, n\}$$

where $\{\langle u v_{\Delta j*} \rangle = 0, j = 1, \cdots, n\}$ *form a self-adjoint set of boundary* *conditions.*

Proof. This follows by direct analogy from the proof of Theorem 3 in Coddington [3].

The set $\{\langle uv_{\Delta j*}\rangle = 0, j = 1, \cdots, n\}$ forms a self-adjoint set of boundary conditions since the $v_{\Delta j*}$ are linearly independent and $\langle v_{\Delta j*}v_{\Delta k*}\rangle = 0$ for all j, k.

The set of self-adjoint boundary conditions $\{\langle uv_{\Delta j*}\rangle = 0, j = 1, \cdots, n\}$ can be represented in matrix form by

$$V_{\Delta*}^*(\tilde{b})A_0(\tilde{b})u(\tilde{b}) - V_{\Delta*}^*(\tilde{a})A_0(\tilde{a})u(\tilde{a}) = 0,$$

where $V_{\Delta*}$ is the matrix whose ith column is the vector $v_{\Delta i*}$. Letting

$$M_\Delta = - V_{\Delta*}^*(\tilde{a})A_0(\tilde{a}),$$

and

$$N_\Delta = V_{\Delta*}^*(\tilde{b})A_0(\tilde{b}),$$

the self-adjoint boundary conditions can be written in standard form

$$U_\Delta u = M_\Delta u(\tilde{a}) + N_\Delta u(\tilde{b}) = 0.$$

The self-adjoint boundary-value problem (on Δ)

$$Lu = lu, \quad U_\Delta u = 0 \quad (bv)$$

will now be considered. The Green's function G_Δ associated with the problem (bv) is a unique function $G_\Delta(t, \tau, l)$ (l not an eigenvalue of (bv)) satisfying the following conditions:

(i) $G_\Delta(t, \tau, l)$ and $\partial/\partial t G_\Delta(t, \tau, l)$ are continuous on $\tilde{a} \leqq t \leqq \tau \leqq \tilde{b}$ and $\tilde{a} \leqq \tau \leqq t \leqq b$, and for each fixed (t, τ) are analytic in l,

(ii) $G_\Delta(t+, \tau, l) - G_\Delta(t-, t, l) = A_0^{-1}(t)$, $\tilde{a} < t < \tilde{b}$,

(iii) G_Δ satisfies $LG_\Delta = lG_\Delta$ (as a function of t),

(iv) G_Δ satisfies $U_\Delta G_\Delta = 0$ (as a function of t),

(v) $G_\Delta(t, \tau, l) = G_\Delta^*(\tau, t, \bar{l})$,

(vi) if $f \in \mathscr{L}^2(\Delta)$ and $Lu = lu + f$,

then,

$$u(t) = \int_\Delta G_\Delta(t, \tau, l)f(\tau)d\tau, \quad U_\Delta u = 0$$

and if

$$\mathscr{G}_\Delta(l)f(t) = \int_\Delta G_\Delta(t, \tau, l)f(\tau)d\tau,$$

then

$$(L - l)\mathscr{G}_\Delta(l)f(t) = f(t),$$

and

$$\|\mathscr{G}_\Delta(l)\|_\Delta \leqq |\text{Im } l|^{-1}.$$

The Green's function will now be constructed starting from the kernel

$$K_\Delta(t, \tau, l) = \begin{cases} \Phi(t, l)D\Phi^*(\tau, l), & t \geqq \tau, \\ 0, & t < \tau, \end{cases}$$

for $t, \tau \in \Delta$; where Φ is a fundamental matrix for $(L - l)u = 0$, having the property that for some $c, \tilde{a} < c < \tilde{b}, \Phi(c, l) = I$. The matrix $D(= A_0^{-1}(c))$ is a constant, skew-Hermitian matrix. From Theorem 8.4 Coddington and Levinson [6], Φ is continuous as a function of (t, l), and for fixed t is an analytic function of $l(\text{Im } l \neq 0)$. Let

$$G_\Delta(t, \tau, l) = K_\Delta(t, \tau, l) + \Phi(t, l)J(\tau, l).$$

Introducing the notation

$$U_\Delta\Phi(l) = M_\Delta\Phi(\tilde{a}, l) + N_\Delta\Phi(\tilde{b}, l),$$

G_Δ can be written as

$$G_\Delta(t, \tau, l) = \begin{cases} \Phi(t, l)(U_\Delta\Phi(l))^{-1}M_\Delta\Phi(\tilde{a}, l)D\Phi^*(\tau, \bar{l}), & t \geqq \tau, \\ - \Phi(t, l)(U_\Delta\Phi(l))^{-1}N_\Delta\Phi(\tilde{b}, l)D\Phi^*(\tau, \bar{l}), & t < \tau. \end{cases}$$

It now follows by direct verification that G_Δ as constructed satisfies the remaining five conditions.

4. The limit function G. In this section it will be shown that a type of limit function G exists for the set $\{G_\Delta\}$, as Δ approaches (a, b).

Let Δ_0, Δ_1, and Δ be closed bounded subintervals of (a, b) such that Δ_0 is properly contained in Δ_1, and Δ_1 is properly contained in Δ; these will be denoted by

$$\Delta_0 = [a_0, b_0], \Delta_1 = [a_1, b_1], \Delta = [\tilde{a}, \tilde{b}].$$

Let μ be a function, having a continuous first derivative, such that for some open interval $\Delta_2, \Delta_0 \subset \Delta_2 \subset \Delta_1$

$$\mu(t) = \begin{cases} 1, & t \in \Delta_2 \\ 0, & t \text{ outside } \Delta_1. \end{cases}$$

Let

$$W_\Delta(t, \tau, l) = G_\Delta(t, \tau, l) - \mu(t)G_{\Delta_1}(t, \tau, l).$$

Then for $t, \tau \in \Delta_0$, $W_\Delta(t, \tau, l)$ is continuous; as a function of t, W_Δ satisfies

$$U_\Delta W_\Delta = M_\Delta W_\Delta(\tilde{a}) + N_\Delta W(\tilde{b}) = 0;$$

and also

$$(L_t - l)W_\Delta(t, \tau, l) = -A_0(t)\mu'(t)G_{\Delta_1}(t, \tau, l), \quad t \neq \tau.$$

Since $\mu'(t) = 0$ for t outside of Δ_1, W_Δ can be written as

$$W_\Delta(t, \tau, l) = -\int_{\Delta 1} G_\Delta(t, s, l)A_0(s)\mu'(s)G_{\Delta_1}(s, \tau, l)ds;$$

(Note: The integral over Δ_1 actually represents the sum of integrals over $[a, t-]$, $[t+, \tau-]$, $[\tau+, b]$ for $\tau > t$), or

$$G_\Delta(t, \tau, l) = \mu(t)G_{\Delta_1}(t, \tau, l) - \int_{\Delta_1} G_\Delta(t, s, l)A_0(s)\mu'(s)G_{\Delta_1}(s, \tau, l)ds.$$

It can be shown that the set $\{W_\Delta\}$ is uniformly bounded and equicontinuous on any compact (t, τ, l) − region, $\operatorname{Im} l \neq 0, t \neq \tau$. Thus, by Ascolis' theorem a uniform limit W exists and from this a limit function G, where

$$G = \mu G_{\Delta_1} + W,$$

and G is a limit function for the set $\{G_\Delta\}$.

THEOREM 4.1. *The function G satisfies the following conditions*:

(i) *$G(t, \tau, l)$ and $\partial/\partial t G(t, \tau, l)$ are continuous on $a < t \leqq \tau < b$ and $a < \tau \leqq t < b$, and for $\operatorname{Im} l \neq 0$ G is analytic in l,*

(ii) *$G(t+, t, l) - G(t-, t, l) = A_0^{-1}(t), a < t < b$,*

(iii) *$L_t G = lG, t \neq \tau$,*

(iv) *$G(t, \tau, l) = G^*(\tau, t, \bar{l})$,*

(v) *$G(t, , l) \in \mathscr{L}^2(a, b), a < t < b$,*

(vi) *If $f \in \mathscr{L}^2(a, b)$, then the vector v defined by*

$$v(t) = \int_a^b G(t, \tau, l)f(\tau)d\tau, \quad \operatorname{Im} l \neq 0,$$

is such that $v \in \mathscr{D}$ and

$$Lv(t) = lv(t) + f(t).$$

Proof. Again, this follows by direct verification.

It is thus seen that G satisfies all the conditions of a Green's function except for satisfying boundary conditions. Further, from property (vi), if

$$\mathscr{G}(l)f(t) = \int_a^b G(t, \tau, l)f(\tau)d\tau,$$

then,

$$(L - l)\mathscr{G}(l)f(t) = f(t)$$

and $\mathscr{G}(l)$ is a right inverse for $L - l$.

5. The generalized resolvent. Having constructed the closed symmetric operator T_0, all its self-adjoint extensions will now be considered. In § 3. the self-adjoint extensions for an operator in $\mathscr{H}$ having equal deficiency indices were considered and these self-adjoint extensions were also in the space $\mathscr{H}$. A spectral analysis of those self-adjoint extensions occurring in $\mathscr{H}$ was carried out, by quite different methods, by Brauer in [2]. The problem to be considered next is for unequal deficiency indices or equivalently, singular problems with equal deficiency indices such that the self-adjoint extensions are outside the original space.

Naimark [7] and others have defined extensions of T_0 for this case in larger Hilbert spaces. Theorem 7. in Straus [12] provides a means for an explicit construction in $\mathscr{H}$ itself. Let $A(l)$ map $\mathscr{E}(-i)$ into $\mathscr{E}(i)$, where $A(l)$ is analytic and $||A(l)|| \leqq 1$ for $\operatorname{Im} l > 0$. Analogously to the case of equal deficiency indices, a domain $\mathscr{D}(l) \subset \mathscr{D}$ is defined by

$$\mathscr{D}(l) = \mathscr{D}_0 + (I - A(l))\mathscr{E}(-i),$$

and an operator $T_{A(l)}$, having domain $\mathscr{D}(l)$ is defined by

$$T_{A(l)}u = Tu, \quad u \in \mathscr{D}(l).$$

Then, $T_0 \subset T_{A(l)} \subset T$, and the generalized resolvent $\mathscr{R}$ can be represented as

$$\mathscr{R}(l) = (T_{A(l)} - lI)^{-1}, \mathscr{R}(\bar{l}) = \mathscr{R}^*(l), \operatorname{Im} l > 0.$$

Further, from Straus [12] every generalized resolvent is generated by such $A(l)$.

Again, analogously to the case for equal deficiency indices, the domain $\mathscr{D}(l)$ can be characterized in an alternate manner which leads to an explicit formulation for the generalized resolvent $\mathscr{R}(l)$. The domain $\mathscr{D}$ can be represented as a direct sum

$$\mathscr{D} = \mathscr{D}_0 + \mathscr{E}(i) + \mathscr{E}(-i);$$

let ω^+ be the dimension of $\mathscr{E}(i)$, and ω^- be the dimension of $\mathscr{E}(-i)$, where $0 \leqq \omega^+, \omega^- \leqq n$; let $\{\varphi_j(i)\}$, $j = 1, \cdots, \omega^+$, be an orthonormal basis for $\mathscr{E}(i)$, and let $\{\psi_k(-i)\}$, $k = 1, \cdots, \omega^-$, be an orthonormal basis for $\mathscr{E}(-i)$; finally, let

$$v_j(l) = \psi_j(-i) - A(l)\psi_j(-i),$$

and

$$v_{j^*}(l) = \varphi_j(i) = A^*(l)\varphi_j(i).$$

THEOREM 5.1. *For* $\operatorname{Im} l > 0$, *the domain* $\mathscr{D}(l)$ *of* $T_{A(l)}$ *can be represented as*

$$\mathscr{D}(l) = \{u \in \mathscr{D}: \langle uv_{j^*}(l)\rangle = 0, \quad j = 1, \cdots, \omega^+\};$$

and the domain of $T^*_{A(l)}$ *is*

$$\mathscr{D}^*(l) = \{w \in \mathscr{D}: \langle wv_k(l)\rangle = 0, \quad k = 1, \cdots, \omega^-\}.$$

Proof. The proof is the same as the proof of Theorem 3.1 with the operator $A(l)$ in place of U.

It will now be shown, again analogous to § 3, that the domain $\mathscr{D}(l)$ induces limiting abstract boundary conditions. For $u \in \mathscr{D}$ and any closed bounded subinterval $[c, d]$ of (a, b)

$$[uv_{j*}](d) - [uv_{j*}](c) = v^*_{j*}(d)A_0(d)u(d) - v^*_{j*}(c)A_0(c)u(c).$$

Since u, v_{j*}, Lu, and Lv_{j*} are each in $\mathscr{L}^2(a, b)$, then $\lim_{d\to b} v^*_{j*}(d)A_0(d)u(d)$ exists, and $\lim_{c\to a} v^*_{j*}(c)A_0(c)u(c)$ exists; these limits will be denoted by $v^*_{j*}(b)A_0(b)u(b)$ and $v^*_{j*}(a)A_0(a)u(a)$. The conditions $\{\langle uv_{j*}\rangle = 0, j = 1, \cdots, \omega^+\}$ can then be represented in matrix form as

$$0 = \langle uV_*\rangle = V^*_*(b, l)A_0(b)u(b) - V^*_*(a, l)A_0(a)u(a)(LB),$$

where V_* is the matrix with v_{j*} in the jth column; these are limiting abstract boundary conditions.

Having obtained the limiting abstract boundary conditions, the following theorem describes a method for the construction of the generalized resolvent $\mathscr{R}(l)$ starting from the integral operator $\mathscr{G}(l)$ developed in § 4.

THEOREM 5.2. *Each generalized resolvent* $\mathscr{R}(l)$ *of* T_0 *is an integral operator of Carleman type, having a kernel* $R(t, \tau, l)$, *which is continuous in* (t, τ, l) *and analytic in* l *in any region for which* $\operatorname{Im} l \neq 0$, *and* $t \neq \tau$.

Proof. The integral operator $\mathscr{G}(l)$ obtained in § 4. is of Carleman type; further $\mathscr{G}(l)$ satisfies the conditions of the theorem except that $\mathscr{G}(l)$ is only a right inverse of $(T - l)$. It will now be shown that a matrix G_1 can be constructed such that the kernel of $\mathscr{R}(l)$ is

$$R(t, \tau, l) = G(t, \tau, l) + G_1(t, \tau, l).$$

For fixed l, $\operatorname{Im} l > 0$, let $\{\theta_i(l)\}$, $i = 1, \cdots, \omega^+$, be an orthonormal basis for $\mathscr{E}(l)$, let $\Theta_+(l)$ be the $n \times \omega^+$ matrix having $\theta_j(l)$ in the jth column, similarly, let $\{\chi_k(\bar{l})\}$, $k = 1, \cdots, \omega^-$, be an orthonormal basis for $\mathscr{E}(\bar{l})$, and $\chi_-(\bar{l})$ be the $n \times \omega^-$ matrix having $\chi_k(\bar{l})$ in the kth column. From the orthonormal property of the $\theta_i(l)$ and the $\chi_k(\bar{l})$,

$$\begin{aligned}(\Theta_+(l), \Theta_+(l)) &= \int_a^b \Theta_+^*(t, l)\Theta_+(t, l)dt \\ &= I_{\omega^+ \times \omega^+},\end{aligned}$$

where $I_{\omega^+ \times \omega^+}$ is the identity matrix of rank ω^+; similarly,

$$(\chi_-(\bar{l}), \chi_-(\bar{l})) = I_{\omega^- \times \omega^-}.$$

For any vector f in $\mathscr{L}^2(a, b)$, $(T - l)(\mathscr{R}(l) - \mathscr{G}(l))f) = f - f = 0$, and thus $(\mathscr{R}(l) - \mathscr{G}(l))f$ is in $\mathscr{E}(l)$. Thus for some $\omega^+ \times 1$ vector $a(f, l)$

$$(\mathscr{R}(l) - \mathscr{G}(l))f = \Theta_+(l)a(f, l).$$

Also

$$((\mathscr{R}(l) - \mathscr{G}(l)f, \Theta_+(l)) = (f, (\mathscr{R}(\bar{l}) - \mathscr{G}(\bar{l}))\Theta_+(l));$$

and, for each column $\theta_k(l)$ of $\Phi_+(l)$, $(T - \bar{l})(\mathscr{R}(\bar{l}) - \mathscr{G}(\bar{l}))\theta_k(l) = \theta_k(l) - \theta_k(l) = 0$. Thus, for some $\omega^+ \times \omega^-$ matrix $B(l)$

$$(\mathscr{R}(\bar{l}) - \mathscr{G}(\bar{l}))\Phi_+(l) = \chi_-(\bar{l})B^*(l).$$

Combining the preceding calculations yields

$$(\mathscr{R}(l) - \mathscr{G}(l))f(t) = \Phi_+(t, l)B(l)(f, \chi_-(\bar{l})) = (f, \chi(\bar{l})B^*(l)\Phi_+^*(t, l));$$

thus,

$$R(t, \tau, l) - G(t, \tau, l) = \Phi_+(t, l)B(l)\chi_-^*(\tau, \bar{l}).$$

Similarly, for some $\omega^+ \times \omega^-$ matrix $H(l)$,

$$R(t, \tau, \bar{l}) - G(t, \tau, \bar{l}) = \chi_-(t, \bar{l})H^*(l)\Phi_+^*(t, \tau)$$

and

$$R^*(\tau, t, \bar{l}) - G^*(\tau, t, \bar{l}) = \Phi_+(t, l)H(l)\chi_-^*(\tau, \bar{l}).$$

Further,

$$\begin{aligned} B(l) &= (\chi_-(\bar{l}), \chi_-(\bar{l})B^*(l)) \\ &= (\chi_-(\bar{l}), (\mathscr{R}(\bar{l}) - \mathscr{G}(\bar{l}))\Phi_+(l)) \\ &= ((\mathscr{R}(l) - \mathscr{G}(l))\chi_-(\bar{l}), \Phi_+(l)) \\ &= (\Theta_+(l)H(l), \Theta_+(l)) \\ &= H(l), \end{aligned}$$

so that

$$R^*(\tau, t, \bar{l}) - G^*(\tau, t, \bar{l}) = R(t, \tau, l) - G(t, \tau, l).$$

Since $\theta_j(l) \in \mathscr{E}(l) \subset \mathscr{D}$, for $j = 1, \cdots, \omega^+$, and $\chi_k(l) \in \mathscr{E}(\bar{l}) \subset \mathscr{D}$, for $k = 1, \cdots, \omega^-$, then $\Theta_+(l)B(l)\chi_-^*(\tau, \bar{l}) \in \mathscr{L}^2(a, b)$ for $a < \tau < b, \operatorname{Im} l \neq 0$. Thus $\mathscr{R}(l)$ is an integral operator of Carleman type.

The operator $\mathscr{R}(l)$ will completely satisfy the condition of the theorem when it is shown that $R(t, \tau, l)$ is analytic in l, $\operatorname{Im} l \neq 0$, and $t \neq \tau$. To facilitate the proof of the analyticity of $R(t, \tau, l)$, analytic bases for $\mathscr{E}(l)$ and $\mathscr{E}(\bar{l})$ will be introduced, as in Coddington [5], related to an arbitrary $l_0, \operatorname{Im} l_0 > 0$.

Matrices $\Psi_-(\bar{l})$and $\Phi_+(l)$ are defined by this process such that the columns of $\Phi_+(l)$ form a basis for $\mathscr{E}(l)$ and the columns of $\Psi_-(\bar{l})$ form a basis for $\mathscr{E}(\bar{l})$; thus for some nonsingular matrix $T(\bar{l})$,

$$\Psi_-(\bar{l}) = \chi_-(\bar{l})T(\bar{l}),$$

and for some nonsingular matrix $S(l)$

$$\Phi_+(l) = \Theta_+(l)S(l).$$

Thus,

$$\begin{aligned} \Theta_+(t, l)B(l)\chi_-^*(\tau, \bar{l}) &= \Phi_+(t, l)S^{-1}(l)B(l)(T^*(\bar{l}))^{-1}\chi_-^*(\tau, l), \\ &= \Phi_+(t, l)C(l)\Psi_-(\tau, \bar{l}), \end{aligned}$$

where $C(l) = S^{-1}(l)B(l)(T^*(\bar{l}))^{-1}$. The matrix $\Phi_+(l)$ is analytic in l and $\Psi_-^*(\bar{l})$ is analytic in l for any compact subset of $\operatorname{Im} l \neq 0$. Thus it remains to show that $C(l)$ satisfies the same conditions of analyticity.

Let Z be an $n \times r$ matrix each of whose columns z_k is in $\mathscr{L}^2(a, b)$, $k = 1, \cdots, r$. Then $\mathscr{R}(l)z_k$ is in $\mathscr{D}(l)$ and thus satisfies the boundary condition (LB),

$$0 = \langle(\mathscr{R}(l)z_k)V_*(l)\rangle;$$

the set $\{0 = \langle(\mathscr{R}(l)z_k)V_*(l)\rangle,\ k = 1, \cdots, r\}$ can be written in matrix form as

$$0 = \langle(\mathscr{R}(l)Z)V_*(l)\rangle.$$

Expanding $\mathscr{R}(l)$, yields

$$0 = \langle(\mathscr{G}(l)Z)V_*(l)\rangle + \langle(\Phi_+(l)C(l)(Z, \Psi_-(\bar{l})))V_*(l)\rangle,$$

also

$$\langle(\Phi_+(l)C(l)(Z, \Psi_-(\bar{l})))V_*(l)\rangle = \langle\Phi_+(l)V_*(l)\rangle C(l)(Z, \Psi_-(\bar{l})).$$

Thus,

$$-\langle(\mathscr{G}(l)Z)V_*(l)\rangle = \langle\Phi_+(l)V_*(l)\rangle C(l)(Z, \Psi_-(\bar{l})).$$

The matrix $C(l)$ will be analytic if $\langle(\mathscr{G}(l)Z)V_*(l)\rangle$ is analytic, and $\langle\Phi_+(l)V_*(l)\rangle$ and $(Z, \Psi_-(\bar{l}))$ are each nonsingular and analytic.

First it can be shown that $\langle\Phi_+(l)V_*(l)\rangle$ is nonsingular and analytic. Next, for $(Z, \Psi_-(\bar{l}))$ to be nonsingular Z must be an $n \times \omega^-$ matrix, it can be verified that for $Z = \Psi_-(-i)$, $(Z, \Psi_-(\bar{l}))$ is nonsingular and analytic. Finally, $\langle(\mathscr{G}(l)\Psi_-(-i)V_*(l)\rangle$ is analytic in l for $|l - l_0| < \operatorname{Im} l_0/2$. Thus

$$C(l) = -\langle\Phi_+(l)V_*(l)\rangle^{-1}\langle(\mathscr{G}(l)\Psi_-(-i)V_*(l)\rangle(\Psi_-(-i), \Psi_-(l))^{-1}$$

is analytic and

$$\Phi_+(t, l)C(l)\Phi_-^*(\tau, \bar{l}) = \Theta_+(t, l)B(l)\chi_-^*(\tau, \bar{l})$$

is analytic in l in a compact subset of $\operatorname{Im} l \neq 0$, $|l - l_0| < \operatorname{Im} l_0/2$. Theorem 5.2 is now proved, the generalized resolvent $\mathscr{R}(l)$ with kernel $\mathscr{R}(t, \tau, l)$ has been constructed.

6. The spectral matrix.

DEFINITION. A matrix ρ, (associated with an eigenvalue problem) is a spectral matrix if it satisfies:

(i) ρ is Hermitian,

(ii) $\rho(\Delta) = \rho(\lambda) - \rho(\mu) \geqq 0$ if $\lambda > \mu$, (where $\Delta = [\mu, \lambda]$),

(iii) ρ is of bounded variation on every finite λ interval.

To develop the spectral matrix associated with the problem $(L - l)u = 0$ with the boundary conditions (LB), and thus associated with the generalized resolvent $\mathscr{R}$ and the generalized resolution of the identity F, the kernel of $\mathscr{R}(l)$ will be split into two parts,

$$R(t, \tau, l) = R_0(t, \tau, l) + R_1(t, \tau, l)$$

where $R_0(t, \tau, l)$ is a certain fundamental matrix for $(L - l)u = 0$. Once again, let Φ be a fundamental matrix for $(L - l)u = 0$, satisfying $\Phi(c, l) = I$, for some c, $a < c < b$. Then, as shown in § 3,

$$[\Phi(t, l)\Phi(t, \bar{l})] = \Phi^*(t, \bar{l})A_0(t)\Phi(t, l) = D^{-1},$$

where D is a nonsingular, constant, skew-Hermitian matrix. Defining

$R_0(t, \tau, l)$ by

$$R_0(t, \tau, l) = \begin{cases} \frac{1}{2}\Phi(t, l)D\Phi^*(\tau, \bar{l}), & t \geqq \tau, \\ \frac{1}{2}\Phi(t, l)D^*\Phi^*(t, \bar{l}), & t < \tau, \end{cases}$$

then,

$$R_0^*(\tau, t, \bar{l}) = R_0(t, \tau, l).$$

Also,

$$R_0(t+, t, l) - R_0(t-, t, l) = A_0^{-1}(t),$$

which is the same jump that $R(t, \tau, l)$ has at $t = \tau$.

Now, let

$$R_1(t, \tau, l) = R(t, \tau, l) - R_0(t, \tau, l).$$

Then as a function of t, R_1 has a continuous first derivative, and $(L_t - l)R_1(t, \tau, l) = 0$.

From the symmetry property $R_1^*(\tau, t, \bar{l}) = R_1(t, \tau, l)$ it follows that for some matrix $\Psi(l)$

$$R_1(t, \tau, l) = \Phi(t, l)\Psi(l)\Phi^*(\tau, \bar{l}).$$

THEOREM 6.1. *The matrix* Ψ *is analytic for* $\operatorname{Im} l > 0$, $\Psi^*(l) = \Psi(\bar{l})$, *and* $\operatorname{Im}\Psi(l)/\operatorname{Im} l > 0$, *where* $\operatorname{Im}\Psi = (\Psi - \Psi^*)/2i$.

Proof. The analyticity of Ψ follows from the choice of $\Phi(c, l) = I$.

Next,

$$R_1(t, \tau, l) = R_1^*(\tau, t, \bar{l}),$$

implying

$$\Phi(t, l)\Psi(l)\Phi^*(\tau, \bar{l}) = \Phi(t, l)\Psi^*(\bar{l})\Phi^*(\tau, \bar{l}),$$

and, since Φ^{-1} exists, $\Psi(l) = \Psi^*(\bar{l})$, or $\Psi^*(l) = \Psi(\bar{l})$.

Let

$$H(t, \tau, l) = \frac{R(t, \tau, l) - R(t, \tau, \bar{l})}{2i},$$

direct computation yields

$$H(c, c, l) = \operatorname{Im}\Psi(l).$$

The proof for $\operatorname{Im}\Psi(l)/\operatorname{Im} l \geqq 0$ now follows as in the proof of Theorem 3 of Coddington [5].

THEOREM 6.2. *The matrix ρ defined by*

$$\rho(\lambda) = \lim_{\varepsilon\to+0}\frac{1}{\pi}\int_0^\lambda \operatorname{Im}\Psi(\nu + i\varepsilon)d\nu$$

exists, is nondecreasing and is of bounded variation on any finite interval.

Proof. This follows directly from Theorem 4 of Coddington [5]. The matrix ρ is the spectral matrix associated with the generalized resolvent $\mathscr{R}$ and the generalized resolution of the identity F.

7. The generalized resolution of the identity. Let ρ be the spectral matrix derived in § 6, let $\Delta = (\mu, \lambda]$ be a finite interval, and let $F(\Delta) = F(\lambda) - F(\mu)$.

THEOREM 7.1. *Let $f \in \mathscr{H}$ and vanish outside a closed bounded subinterval $[c, d]$ of (a, b). If μ and λ are continuity points of F, then*

$$F(\Delta)f(t) = \int_\Delta \Phi(t, \nu)d\rho(\nu)(f, \Phi(\nu)).$$

Proof. It follows from the relationship

$$(\mathscr{R}(l)f, f) = \int_{-\infty}^{\infty}\frac{d(F(\gamma)f, f)}{\gamma - l}$$

that

$$(F(\Delta)f, f) = \lim_{\varepsilon\to+0}\frac{1}{\pi}\int_\Delta (\operatorname{Im}\mathscr{R}(\nu + i\varepsilon)f, f)d\nu$$

at continuity points μ, λ of F. The generalized resolvent $\mathscr{R}(l)$ can be written as $\mathscr{R}(l) = \mathscr{R}_0(l) + \mathscr{R}_1(l)$, where $\mathscr{R}_0(l)$ has kernel $R_0(t, \tau, l)$, and $\mathscr{R}_1(l)$ has kernel $R_1(t, \tau, l)$. Then

$$\mathscr{R}_0(l)f(t) = \int_a^b R_0(t, \tau, l)f(\tau)d\tau = \int_c^d R_0(t, \tau, l)f(\tau)d\tau.$$

However, $(\operatorname{Im}\mathscr{R}_0(\nu + i\varepsilon)f, f)$ tends to zero $\varepsilon \to +0$, uniformly in Δ. Consequently, it follows that

$$(F(\Delta)f, f) = \lim_{\varepsilon\to+0}\frac{1}{\pi}\int_\Delta (\operatorname{Im}\mathscr{R}_1(\nu + i\varepsilon)f, f)d\nu,$$

where

$$\begin{aligned}(\operatorname{Im} \mathscr{R}_1(\nu + i\varepsilon)f, f) &= (\Phi(\nu), f)\left[\frac{\Psi(\nu + i\varepsilon) - \Psi(\nu - i\varepsilon)}{2i}\right](f, \Phi(\nu)) \\ &\quad + \frac{1}{2i}\{\Phi(\nu + i\varepsilon), f)\Psi(\nu + i\varepsilon)(f, (\Psi(\nu - i\varepsilon) - \Phi(\nu))) \\ &\quad + ((\Phi(\nu + i\varepsilon) - \Phi(\nu)), f)\Psi(\nu + i\varepsilon)(f, \Psi(\nu))\} \\ &\quad + \frac{1}{2i}\{(\Phi(\nu), f)\Psi(\nu - i\varepsilon)(f, (\Phi(\nu) - \Phi(\nu + i\varepsilon))) \\ &\quad + ((\Phi(\nu) - \Phi(\nu - i\varepsilon)), f)\Psi(\nu - i\varepsilon)(f, \Phi(\nu + i\varepsilon))\} \\ &= T_1 + T_2 + T_3,\end{aligned}$$

where $T_i = T_i(\nu, \varepsilon, f)$.

In Lemma 3 of Straus [13], it is shown that

$$\lim_{\varepsilon\to+0} \frac{1}{\pi}\int_\Delta T_2(\nu)d\nu = \lim_{\varepsilon\to+0} \frac{1}{\pi}\int_\Delta T_3(\nu)d\nu = 0.$$

Finally, for T_1,

$$\begin{aligned}\lim_{\varepsilon+0}\frac{1}{\pi} \int_\Delta &(\Phi(\nu), f) \operatorname{Im} \Psi(\nu + i\varepsilon)(f, \Phi(\nu))d\nu \\ &= \int_\Delta (\Phi(\nu), f)d\rho(\nu)(f, \Phi(\nu)) \\ &= \int_\Delta (f, \Phi(\nu))^* d\rho(\nu)(f, \Phi(\nu)),\end{aligned}$$

and therefore,

$$(F(\Delta)f, f) = \int_a^b f^*(t)\left[\int_\Delta \Phi(t, \nu)d\rho(\nu)(f, \Phi(\nu))\right]dt.$$

Since this representation must hold for all $f \in \mathscr{H}$ which vanish outside closed finite subintervals of (a, b),

$$F(\Delta)f(t) = \int_\Delta \Phi(t, \nu)d\rho(\nu)f, \Phi(\nu))$$

for all such f.

Thus the generalized resolutions of the identity associated with the first order system of differential operators $Lx(t) = A_0(t)x'(t) + A(t)x(t)$ can be represented explicitly in terms of a certain fundamental matrix Φ and an associated spectral matrix ρ.

8. The expansion and completeness relations. Expansion and completeness relations can be defined in terms of the spectral matrix

ρ and the fundamental matrix Φ. For two vectors $\hat{\alpha}$, $\hat{\beta}$, an inner product is defined in terms of ρ by

$$(\hat{\alpha}, \hat{\beta})_\rho = \int_{-\infty}^{\infty} \hat{\beta}^*(\nu) d\rho(\nu) \hat{\alpha}(\nu).$$

Thus a norm can be defined by

$$\|\hat{\alpha}\|_\rho = (\hat{\alpha}, \hat{\alpha})_\rho^{1\backslash 2}.$$

The Hilbert space $\mathscr{L}^2(\rho)$ is defined by

$$\mathscr{L}^2(\rho) = \{\hat{\alpha} : \|\hat{\alpha}\|_\rho < \infty\}.$$

Defining a mapping from $\mathscr{L}^2(a, b)$ into $\mathscr{L}^2(\rho)$ by

$$\hat{f}(\nu) = (f, \Phi(\nu)) = \int_a^b \Phi^*(t, \nu) f(t) dt,$$

the expansion and completeness relations have the following form:

$$f(t) = (\hat{f}, \Phi^*(t))_\rho = \int_{-\infty}^{\infty} \Phi(t, \nu) d\rho(\nu) \hat{f}(\nu) \quad \text{(expansion)}$$

and

$$\|f\| = \|\hat{f}\|_\rho \quad \text{(completeness).}$$

REFERENCES

1. N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (English Translation) F. Ungar Publishing Co., N. Y., Vol. 1 and Vol. 2, 1961, 1963.
2. F. Brauer, *Spectral theory for linear systems of differential equations*, Pacific J. Math., V. **10** (1960), 17-34.
3. E. A. Coddington, *The spectral representation of ordinary self-adjoint differential operators*, Annals of Math., **60** (July 1954), 192-211.
4. _______, *The spectral matrix and Green's function for singular self-adjoint boundary value problems*, Canad. Math., **6** (1954), 169-185.
5. _______, *Generalized resolutions of the identity for symmetric ordinary differential operators*, Annals of Math., **68** (Sept. 1958) 378-392.
6. _______, and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Co., Inc., 1955.
7. M. A. Naimark, *Spectral functions of a symmetric operator*, Izvest. Akad. Nauk. S. S. S. R. Ser. Mat., **4** (1940), 277-318.
8. F. Riesz and B. Sz-Nagy, *Functional Analysis*, (English Translation) F. Ungar Publishing Co., N. Y., 1955.
9. M. H. Stone, *Linear Transformations in Hilbert Space*, Amer. Math. Soc. Colloqu. Publ., 1932.
10. A. V. Straus, *On generalized resolvents of a symmetric operator*, Dok. Akad. Nauk. S. S. S. R., **71** (1950), 241-244.
11. _______, *On the theory of generalized resolvents of a symmetric operator*, Dok. Akad. Nauk. S. S. S. R., **78** (1951), 217-220.

12. _______, *Generalized resolvents of symmetric operators*, Izvest. Akad. Nauk. S. S. S. R., Ser. Mat., **18** (1954), 51-86.
13. _______, *On spectral functions of differential operators*, Izvest. Akad. Nauk S. S. S. R., Ser. Mat., **19** (1955), 201-220.

Received September 4, 1970 and in revised form June 19, 1971.

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