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MEASURES ON SEMILATTICES

SOLOMON LEADER

New definitions are given for positivity and bounded variation of functions on a semilattice S so that such functions extend to measures (respectively, signed measures) on the σ-algebra generated by some representation of S as a semilattice of sets under intersection. All such representations lie in a Stone space determined by S. Functions on a subsemilattice S of a distributive lattice L which extend to isotone valuations on L are characterized in terms of a partial ordering of finite sequences in S. Functions on a regulated semilattice which correspond to regular Borel measures on the associated locally compact space are characterized in terms of inclusion-exclusion sums.

A semilattice is a commutative, idempotent semigroup $(S, \cdot)$ in which we define the partial ordering $a \leqq b$ to be $ab = a$. So all our semilattices are meet-semilattices [6]. That is, the product of any pair of elements in S is their greatest lower bound.

Every semilattice $(S, \cdot)$ can be represented as a semilattice $(\mathscr{S}, \cap)$ of sets. In particular one can represent each s in S by its set C_s of all lower bounds of s. For s, t in S we have

$$C_s \cap C_t = C_{st}. \tag{1.1}$$

So we get a representation with $\mathscr{S}$ the set of all lower sets C_s. Note that no lower set is a nontrivial union of lower sets. Indeed

$$C_s \subseteqq C_{t_1} \cup \cdots \cup C_{t_n} \text{ implies } C_s \subseteqq C_{t_i} \text{ for some } i. \tag{1.2}$$

The existence of representations of semilattices suggests the study of functions ϕ on S which can be extended under some representation to measures (or signed measures) on the σ-algebra generated by the representation. Motivation for such a study comes from probability theory wherein each bounded measure Φ on the Borel sets in $\boldsymbol{R}^n$ is represented by a distribution function ϕ on $\boldsymbol{R}^n$ according to the relation

$$\phi(s) = \Phi(C_s). \tag{1.3}$$

In this case $\boldsymbol{R}^n$ is a semilattice relative to the coordinate-wise ordering. That is, for $s = (s_1, \cdots, s_n)$ and $t = (t_1, \cdots, t_n)$ in $\boldsymbol{R}^n$, $st = (s_1 \wedge t_1, \cdots, s_n \wedge t_n)$. Explicit characterization of distribution functions ϕ is given essentially by two conditions. The first condition (positive definiteness, usually defined in terms of iterated differences [3]) guarantees that the finitely additive extension whose existence is implied by (1.2) is nonnegative. The second condition (upper continuity) yields the countable additivity of the extension.

S. Newman [4] has defined positive definite functions and functions of bounded variation on a semilattice. We give several equivalent definitions for these concepts below, avoiding combinatorial difficulties by making use of certain structures and objects associated with a semilattice S: ideals, characters, the semigroup ring $Z(S)$ over the integers, the boolean ring $I(S)$ of idempotents in $Z(S)$, the positive cone $Z(S)^+$, the transform representing $Z(S)$ as a function ring on the characters, and the Stone space $X(S)$.

2. Algebraic structures associated with a semilattice.

Given a semilattice S let $Z(S)$ be the semigroup ring of S over the ring Z of integers. That is, $Z(S)$ consists of all integer-valued functions $\boldsymbol{f}$ on S with finite support. $Z(S)$ is a ring under functional addition and the convolution product

$$\boldsymbol{f}*\boldsymbol{g}(t) = \sum_{rs=t} \boldsymbol{f}(r)\boldsymbol{g}(s) \tag{2.1}$$

where the summation runs through all nonzero terms given by ordered pairs (r, s) in S with $rs = t$. Since S is commutative, so is $Z(S)$. S is injected into $Z(S)$ by defining for s, x in S

$$\boldsymbol{s}(x) = \begin{cases} 1 & \text{if } x = s \\ 0 & \text{if } x \neq s. \end{cases} \tag{2.2}$$

Let $\boldsymbol{S}$ be the set of all functions of the form (2.2) with s in S. For all r, s, t in S we have

$$rs = t \quad \text{iff } \boldsymbol{r}*\boldsymbol{s} = \boldsymbol{t}. \tag{2.3}$$

So $(\boldsymbol{S}, *)$ is isomorphic to $(S, \cdot)$. In terms of (2.2) each $\boldsymbol{f}$ in $Z(S)$ is of the form

$$\boldsymbol{f} = \sum_{s \in S} \boldsymbol{f}(s)\boldsymbol{s} \tag{2.4}$$

where we ignore vanishing terms and sum only over the support of $\boldsymbol{f}$. In the form (2.4) the convolution (2.1) is the product induced by the product in S through (2.3) and the distributive law. Since the free commutative group generated by $\boldsymbol{S}$ is the additive group $Z(S)$, every function ϕ on S into an additive group $(G, +)$ has (under identification (2.2) of S with $\boldsymbol{S}$) a unique extension to an additive function on $Z(S)$ into G given by

$$(\phi, \boldsymbol{f}) = \sum_{s \in S} \boldsymbol{f}(s)\, \phi(s). \tag{2.5}$$

If S has a unit 1 (i.e., semigroup identity), then 1 is a unit in $Z(S)$.

A character p on a semilattice S is a function on S into $\{0, 1\}$ such that

$$p(rs) = p(r)p(s) \text{ for all } r, s \text{ in } S. \text{ [5]} \tag{2.6}$$

(Some authors call this a "semicharacter".) The set $T(S)$ of all characters on S is a semilattice (the *dual of* S) under the functional product

$$pq(s) = p(s)\, q(s). \tag{2.7}$$

Each element s of S defines a character p_s on S,

$$p_s(x) = \begin{cases} 1 & \text{if } x \geqq s \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Let $P(S)$ be the set of all such functions (2.8) on S which we call *principal characters.*

For $\boldsymbol{f}$ in $Z(S)$ and p in $T(S)$ define the transform

$$\hat{\boldsymbol{f}}(p) = \sum_{s \in S} \boldsymbol{f}(s)\, p(s) \tag{2.9}$$

which is $(p, \boldsymbol{f})$ in terms of (2.5). Note that $\hat{\boldsymbol{s}}(p) = p(s)$. (2.9) defines a ring homomorphism on $Z(S)$ into a ring of bounded, integervalued functions on $T(S)$. That is,

$$\begin{cases} \widehat{\boldsymbol{f} + \boldsymbol{g}}(p) = \hat{\boldsymbol{f}}(p) + \hat{\boldsymbol{g}}(p) \text{ and} \\ \widehat{\boldsymbol{f} * \boldsymbol{g}}(p) = \hat{\boldsymbol{f}}(p)\, \hat{\boldsymbol{g}}(p). \end{cases} \tag{2.10}$$

As will be shown in Propositions 2 and 3 below, this homomorphism is injective.

Let $I(S)$ be the set of idempotents in $Z(S)$. Since $Z(S)$ is a commutative ring, $I(S)$ is a Boolean ring with $\boldsymbol{f} \wedge \boldsymbol{g} = \boldsymbol{f} * \boldsymbol{g}$ and $\boldsymbol{f} \vee \boldsymbol{g} = \boldsymbol{f} + \boldsymbol{g} - \boldsymbol{f} * \boldsymbol{g}$. The isomorphism of $(S, \cdot)$ with $(\boldsymbol{S}, *)$ imbeds $(S, \cdot)$ as a subsemilattice of $I(S)$.

Let $K(S)$ consist of all members of $Z(S)$ of the form $\boldsymbol{t} - \boldsymbol{s}$ with $s < t$ in S. Let $J(S)$ be the semigroup generated by $K(S)$ under $*$.

LEMMA 1. *Every* $\boldsymbol{f}$ *in* $Z(S)$ *has a representation of the form* $\boldsymbol{f} = \sum_{i=1}^{n} \varepsilon_i \boldsymbol{f}_i$ *where* $\varepsilon_i \in Z$ *and* $\boldsymbol{f}_i \in J(S)$ *with* $\boldsymbol{f}_i \neq 0$ *and* $\boldsymbol{f}_i * \boldsymbol{f}_j = 0$ *for* $i \neq j$. *Moreover, we may assume that the range of* $\hat{\boldsymbol{f}}$ *is* $\{\varepsilon_1, \cdots, \varepsilon_n\}$.

Proof. Choose a nonempty finite subset F of S such that F contains the support of $\boldsymbol{f}$ and at least one zero of $\boldsymbol{f}$ if $\boldsymbol{f}$ has zeros. Let $\boldsymbol{f}_1, \cdots, \boldsymbol{f}_n$ be the atoms of the Boolean ring generated in $I(S)$ by $\boldsymbol{F}$. These atoms belong to $J(S)$ since S is a semilattice. Since each $\boldsymbol{s}$, for s in F, is a sum of some of these atoms we get our representation from (2.4).

Given p in $T(S)$, either $\hat{f}(p) = 0$ or by (2.10) there is some i in our representation such that $\hat{f}_j(p) = 1$ for $j = i$ and 0 for $j \neq i$. In either case $\hat{f}(p) = \varepsilon_i$ for some i. On the other hand we can choose principal $p_1, \cdots, p_n$ such that $\hat{f}_j(p_i) = 1$ for $i = j$ and 0 for $i \neq j$. Then $\hat{f}(p_i) = \varepsilon_i$.

An *ideal* in S is a subset I of S such that

$$r, s \in I \text{ iff } rs \in I. \tag{2.11}$$

(Some authors prefer the terms "filter" or "dual ideal".) There is a one-one correspondence between ideals I_p in S and characters p on S given by the relation

$$I_p = p^{-1}(1). \tag{2.12}$$

Clearly, (2.11) is equivalent to (2.6) under (2.12). I_p is nonempty iff $p \neq 0$. I_p is a principal (i.e., finitely generated) ideal iff p is a principal character (2.8). In terms of (2.12) we can express (2.7) as $I_{pq} = I_p \cap I_q$.

3. Separation.

A set H of ideals in S *separates* S if every principal ideal I is the intersection of all members of H which contain I. *H amply separates* S if for every principal ideal I and finite subset F of S disjoint from I there is some J in H which contains I and is disjoint from F. Using the correspondence (2.12) we can formulate these definitions for H a set of characters on S. *H separates* S if given $r \neq s$ in S there exists p in H with $p(r) \neq p(s)$. *H amply separates* S if given s in S and F a finite subset of S with $st < s$ for all t in F there exists p in H with $p(s) = 1$ and $p(t) = 0$ for all t in F. Clearly, if H amply separates S then H separates S. The converse holds if H is a semilattice under (2.7).

For each s in S let $[s]$ be the set of all p in $T(S)$ such that $p(s) = 1$. For H a subset of $T(S)$ let $[s]_H = H \cap [s]$ and let π_H be the homomorphism on $Z(S)$ defined by (2.9) with p restricted to H.

PROPOSITION 1. *For any subset H of $T(S)$ the following are equivalent:*

(i) *H separates S.*
(ii) *The kernel of π_H on $K(S)$ is trivial.*
(iii) *π_H is an isomorphism on $(K(S), \leqq)$.*
(iv) *π_H is an isomorphism on $(S, *)$*
(v) *$[s]_H = [t]_H$ implies $s = t$.*

Proof. Clear.

PROPOSITION 2. *For any subset H of $T(S)$ the following are equivalent:*

(i) $\hat{f}(H) = \hat{f}(T(S))$ *for every* f *in* $Z(S)$.

(ii) π_H *is a Boolean isomorphism on* $I(S)$.

(iii) π_H *is a semilattice isomorphism on* $J(S)$.

(iv) π_H *is a ring isomorphism on* $Z(S)$.

(v) *The set* $\pi_H(S)$ *of functions on* H *is linearly independent over* Z.

(vi) H *amply separates* S.

(vii) $[s]_H \subseteqq [s_1]_H \cup \cdots \cup [s_n]_H$ implies $[s]_H \subseteqq [s_i]_H$ *for some* i.

Proof. (i) $\Rightarrow$ (ii). Let $\boldsymbol{f} \in I(S)$ and $\pi_H \boldsymbol{f} = \boldsymbol{0}$.

The latter condition means $\hat{\boldsymbol{f}}(H) = \{0\}$. Hence $\boldsymbol{f} = \boldsymbol{0}$ by (i) and Lemma 1.

(ii) $\Rightarrow$ (iii) a fortiori.

(iii) $\Rightarrow$ (iv). Consider $\boldsymbol{f} \neq \boldsymbol{0}$ in $Z(S)$. We contend $\pi_H \boldsymbol{f} \neq \boldsymbol{0}$. We may assume in the representation of $\boldsymbol{f}$ given by Lemma 1 that $\varepsilon_1 \boldsymbol{f}_1 \neq \boldsymbol{0}$. Hence by (iii) there is some $p \in H$ with $\hat{\boldsymbol{f}}_1(p) = 1$. For such p Lemma 1 and (2.10) yield $\hat{\boldsymbol{f}}(p) = \varepsilon_1 \neq 0$. (iv) $\Rightarrow$ (v) since S is a set of independent generators for the additive group $Z(S)$. (v) $\Rightarrow$ (vi). Consider $s, s_1, \cdots, s_n$ in S with $ss_i < s$ for $i = 1, \cdots, n$. In $J(S)$ define

$$\boldsymbol{f} = (\boldsymbol{s} - \boldsymbol{a}_1) * \cdots * (\boldsymbol{s} - \boldsymbol{a}_n) \text{ with } a_i = ss_i. \tag{3.1}$$

Expanding (3.1) we get

$$\boldsymbol{f} = \boldsymbol{s} + \sum_{\alpha \in A} (-1)^{k(\alpha)} \boldsymbol{\pi\alpha} \tag{3.2}$$

where A is the set of all subsequences $\alpha = \{a_{i_1}, \cdots, a_{i_k}\}$ $(1 \leqq i_1 < \cdots < i_k \leqq n)$ of $\alpha_0 = \{a_1, \cdots, a_n\}$, $k(\alpha)$ is the number k of terms in α, and $\boldsymbol{\pi\alpha}$ is the product of all terms in α.

By (v) $\boldsymbol{f}(p)$ cannot vanish identically on H since in (3.2) $\boldsymbol{s}$ has a nonzero coefficient and is distinct from every $\boldsymbol{\pi\alpha}$. So there exists $p \in H$ with $0 \neq \boldsymbol{f}(p) \equiv p(s)[1 - p(s_1)] \cdots [1 - p(s_n)]$. That is, $p(s) = 1$ and $p(s_i) = 0$ for $i = 1, \cdots, n$.

The equivalence of (vi) and (vii) is trivial.

(vi) implies (iii) since $\boldsymbol{f} = \boldsymbol{0}$ in (3.1) iff $s = a_i$ (that is, $s \leqq s_i$) for *some* i.

Finally, (iii) implies (i) by Lemma 1.

PROPOSITION 3. *The set $P(S)$ of principal characters amply separates S.*

Proof. Given $s \in S$ and F a finite subset of S with $st < s$ for all $t \in F$, take p_s in (2.8) to separate s from F.

A character p is irreducible if $qr = p$ in $T(S)$ implies either

$q = p$ or $r = p$.

PROPOSITION 4. *The irreducible characters separate* S.

Proof. Given $st < s$ in S let Q be the set of all q in $T(S)$ such that $q(s) = 1$ and $q(t) = 0$. Q is nonempty since p_s in (2.8) belongs to Q. By Zorn's lemma Q has a maximal member relative to the ordering $p \leqq q$ defined by $p = pq$ in (2.7). For it is clear that the functional maximum of any chain in Q is a character belonging to Q. Now a maximal p in Q must be irreducible. For if $p = qr$ then $1 = p(s) = q(s)r(s)$ and $0 = p(t) = q(t)r(t)$. So either q or r belongs to Q, hence equals p since p is maximal in Q.

4. The positive cone $Z(S)^+$.

PROPOSITION 5. *For any* $\boldsymbol{f}$ *in* $Z(S)$ *the following are equivalent:*
(i) $\sum_{s \geqq a} f(s) \geqq 0$ *for every* $a \in S$.
(ii) $\sum_{s \in I} f(s) \geqq 0$ *for every ideal* I *in* S.
(iii) $\hat{\boldsymbol{f}}(p) \geqq 0$ *for every character* p.
(iv) *There exists an amply separating set* H *of characters such that* $\hat{\boldsymbol{f}}(p) \geqq 0$ *for all* p *in* H.
(v) $\hat{\boldsymbol{f}}$ *has a representation given by Lemma* 1 *with all the coefficients* ε_i *nonnegative.*

Proof. (i) $\Rightarrow$ (ii). Let a be the product of all the elements in the support of $\boldsymbol{f}$ which also belong to I. Then the sums in (i) and (ii) are identical. (ii) implies (i) since (i) is (ii) restricted to principal ideals. (ii) and (iii) are equivalent through (2.9) under the correspondence (2.12).

(iii) implies (iv) a fortiori.

(iv) implies (v) by Lemma 1 and Proposition 2.

(v) implies (iii) by Lemma 1.

Define $Z(S)^+$ to be the set of all $\boldsymbol{f}$ in $Z(S)$ satisfying the conditions in Proposition 5. Then $Z(S)$ is a lattice-ordered ring with $Z(S)^+$ as positive cone. In terms of the representation of Lemma 1 we have

$$\boldsymbol{f}^+ = \sum_{i=1}^{n} \varepsilon_i^+ \boldsymbol{f}_i, \quad \boldsymbol{f}^- = \sum_{i=1}^{n} \varepsilon_i^- \boldsymbol{f}_i, \quad |\boldsymbol{f}| = \sum_{i=1}^{n} |\varepsilon_i| \boldsymbol{f}_i. \tag{4.1}$$

Finally,

$$K(S) \subseteqq J(S) \subseteqq I(S) \subseteqq Z(S)^+. \tag{4.2}$$

5. Positive definite functions.

PROPOSITION 6. *For any real-valued function ϕ on S the following are equivalent:*

(i) *Given $s_1, \cdots, s_n$ in S and $\varepsilon_1 \cdots, \varepsilon_n$ in Z*

$$\sum_{i,j} \varepsilon_i \varepsilon_j \, \phi(s_i s_j) \geqq 0. \tag{5.1}$$

(ii) $(\phi, f^2) \geqq 0$ *for all f in $Z(S)$.*
(iii) $(\phi, f) \geqq 0$ *for all f in $I(S)$.*
(iv) $(\phi, f) \geqq 0$ *for all f in $J(S)$.*
(v) $(\phi, f) \geqq 0$ *for all f in $Z(S)^+$.*
(vi) *Given $a, a_1, \cdots, a_n$ in S with $a_i \leqq a$ for $i = 1, \cdots, n$ then*

$$\sum_{\alpha \in A} (-1)^{k(\alpha)+1} \phi(\pi\alpha) \leqq \phi(a) \tag{5.2}$$

in terms of notation introduced with (3.2).

Proof. (i) is equivalent to (ii) via the relation

$$f = \sum_{i=1}^{n} \varepsilon_i s_i \tag{5.3}$$

for which (ϕ, f^2) is just the sum in (5.1) according to (2.5).

(ii) implies (iii) implies (iv) a fortiori.

To prove (iv) implies (v) apply Lemma 1, (v) of Proposition 5, and additivity of $(\phi, \ )$ on $Z(S)$ to get $(\phi, f) = \sum_{i=1}^{n} \varepsilon_i (\phi, f_i)$ with both factors nonnegative in every term on the right.

(v) implies (ii) since $f^2 \in Z(S)^+$ according to (iii) of Proposition 5 and (2.10).

We call ϕ *positive definite* if it satisfies the conditions of Proposition 6. Note that (iv) becomes Newman's definition [4] if we take f in (3.1) and distribute ϕ into (3.2).

6. Functions of bounded variation.

PROPOSITION 7. *For ϕ a real-valued function on S the following equivalent:*

(i) *There exists $M < \infty$ such that given $s_1, \cdots, s_n$ in S and $\varepsilon_1, \cdots, \varepsilon_n$ in Z satisfying*

$$\sum_{s_i \geqq s} \varepsilon_i = \begin{cases} 0 \\ 1 \end{cases} \text{ for all } s \in S \tag{6.1}$$

then $\left| \sum_{i=1}^{n} \varepsilon_i \, \phi(s_i) \right| \leqq M$.

(ii) $\sup_{f \in I(S)} |(\phi, f)| < \infty$.

(iii) $\sup \sum_{i=1}^{n} |(\phi, \boldsymbol{f}_i)| < \infty$
$\begin{cases} \boldsymbol{f}_i \in J(S) \\ \boldsymbol{f}_i * \boldsymbol{f}_j = 0 \text{ } for \text{ } i \neq j \end{cases}$

((iii) *is essentially Newman's definition of bounded variation* [4].)

Proof. The equivalence of (i) and (ii) follows from the relation (5.3). Under (6.1) $\hat{\boldsymbol{f}}$ for $\boldsymbol{f}$ in (5.3) maps the set P of principal characters into $\{0, 1\}$. By Proposition 3 and the equivalence (i) and (iv) in Proposition 2 (6.1) is equivalent to $\boldsymbol{f} \in I(S)$.
(ii) $\Rightarrow$ (iii). Given $\boldsymbol{f}_i$ in (iii) let $\boldsymbol{f} = \sum_{(\phi, \boldsymbol{f}_i) \vee 0} \boldsymbol{f}_i$ and $\boldsymbol{g} = \sum_{(\phi, \boldsymbol{f}_i) < 0} \boldsymbol{f}_i$. Then $\boldsymbol{f}, \boldsymbol{g} \in I(S)$.
Moreover,

$$(6.2) \qquad (\phi, \boldsymbol{f}) - (\phi, \boldsymbol{g}) = (\phi, \boldsymbol{f} - \boldsymbol{g}) = \sum_{i=1}^{n} |(\phi, \boldsymbol{f}_i)|.$$

By (ii) the left side of (6.2) is bounded.

(iii) $\Rightarrow$ (ii). Each $\boldsymbol{f}$ in $I(S)$ is of the form $\boldsymbol{f} = \sum_{i=1}^{n} \boldsymbol{f}_i$ in the representation given by Lemma 1. Hence $|(\phi, \boldsymbol{f})| = |\sum_{i=1}^{n} (\phi, \boldsymbol{f}_i)| \leqq \sum_{i=1}^{n} |(\phi, \boldsymbol{f}_i)|$.

Let $V(\phi)$ be the supremum in (iii) of Proposition 7. We call $V(\phi)$ the *total variation* of ϕ and call ϕ of *bounded variation* if its total variation is finite. Note that

$$(6.3) \qquad V(\phi) = \sup_{\substack{\boldsymbol{f}, \boldsymbol{g} \in I(S) \\ \boldsymbol{f} * \boldsymbol{g} = \boldsymbol{0}}} (\phi, \boldsymbol{f} - \boldsymbol{g}).$$

For ϕ a real-valued function on S and r an element of S define

$$(6.4) \qquad \phi r(s) = \phi(rs) \text{ for all } s \in S.$$

Then

$$(6.5) \qquad (\phi r, \boldsymbol{f}) = (\phi, \boldsymbol{r} * \boldsymbol{f}) \text{ for all } \boldsymbol{f} \in Z(S).$$

Define

$$(6.6) \qquad |\phi|(r) = V(\phi r) \text{ for all } r \in S.$$

PROPOSITION 8. *For ϕ real-valued on S the following are equivalent:*

(i) $|\phi|(r) < \infty$ *for all* $r \in S$

(ii) *For each* $\boldsymbol{f}$ *in* $I(S)$

$$\sup_{\substack{\boldsymbol{g}, \boldsymbol{h} \in I(S) \\ \boldsymbol{g} + \boldsymbol{h} = \boldsymbol{f}}} (\phi, \boldsymbol{g} - \boldsymbol{h}) < \infty$$

(iii) *For each* $\boldsymbol{f}$ *in* $I(S)$

$$\sup_{\substack{\boldsymbol{f}_1, \cdots, \boldsymbol{f}_n \in J(S) \\ \sum_1^n \boldsymbol{f}_i = \boldsymbol{f}}} \sum_1^n |(\phi, \boldsymbol{f}_i)| < \infty$$

(iv) *For each* $\boldsymbol{f}$ *in* $Z(S)^+$

$$\sup_{\substack{\boldsymbol{g} \in Z(S) \\ |\boldsymbol{g}| \leqq \boldsymbol{f}}} (\phi, \boldsymbol{g}) < \infty$$

where $|\boldsymbol{g}| \leqq \boldsymbol{f}$ *means there exist representations in Lemma* 1 *such that* $\boldsymbol{f} = \sum_1^n \varepsilon_i \boldsymbol{f}_i$ and $\boldsymbol{g} = \sum_1^n \delta_i \boldsymbol{f}_i$ *with* $|\delta_i| \leqq \varepsilon_i$.

(v) ϕ *is the difference of two positive definite functions. There is a Jordan decomposition* $\phi^+ - \phi^- = \phi$ *with* ϕ^+, ϕ^- *positive definite and* $\phi^+ + \phi^- = |\phi|$.

Proof. Setting $\boldsymbol{f} = \boldsymbol{r}$ in (ii), (iii), or (iv) one sees that the supremum there is just (6.6). So (i) is implied by (ii), (iii), or (iv). The converse holds because the supremum in (ii), (iii), (iv) is $(|\phi|, f)$.

Given (i) let $\phi^+ = (|\phi| + \phi)/2$ and $\phi^- = (|\phi| - \phi)/2$. These are positive definite because $|(\phi, \boldsymbol{f})| \leqq (|\phi|, \boldsymbol{f})$ for all $\boldsymbol{f} \in I(S)$. Hence (i) implies (v). The converse is trivial.

COROLLARY 8(a). *ϕ is of bounded variation iff ϕ is the difference of two positive definite functions of bounded variation.*

7. Measures on the Stone space.

Let $(S, \cdot)$ be a semilattice with unit 1. (One can trivially adjoin a unit to a semilattice which has none.) Let $X(S)$ be the set of all characters p on S with $p(1) = 1$. In the notation introduced in § 3, $[s] \subseteqq [1] = X(S)$ for all $s \in S$. Since $X(S)$ contains all the principal characters it amply separates S by Proposition 3 and $\varnothing \neq [st] = [s] \cap [t]$ for all s, t in S.

$X(S)$ is a subset of $\{0, 1\}^S$, the space of all $\{0, 1\}$-functions on S. This space is compact since it is a product of finite spaces. The product topology is determined by pointwise convergence of the functions. So $X(S)$ is a closed subspace of $\{0, 1\}^S$ since a limit of characters with $p(1) = 1$ is again a character with $p(1) = 1$. So $X(S)$ is compact in the topology it inherits from $\{0, 1\}^S$. For p in $X(S)$ and E a subset of $X(S)$ we have $p \in \bar{E}$ iff given any finite subset F of S there exists q in E with $q(t) = p(t)$ for all t in F. Applying this characterization to $[s]$ and its complement we conclude that $[s]$ is both open and closed. The topology is the smallest topology in $X(S)$ for which $[s]$ is open-closed for every s in S.

THEOREM I. *Let ϕ be a real-valued function on a semilattice S with unit* 1. *Define the set function*

$$\Phi([s]) = \phi(s) \text{ for all } s \in S. \tag{7.1}$$

Let $\mathscr{A}$ be the σ-algebra generated in $X(S)$ by the sets $[s]$ with $s \in S$.

(a) *Φ extends to a bounded, countably additive measure on $\mathscr{A}$ iff ϕ is positive definite.*

(b) *Φ extends to a bounded, countably additive, signed measure on $\mathscr{A}$ iff ϕ is of bounded variation.*

Proof. Since $X(S)$ amply separates S, (ii) of Proposition 2 gives through (2.5) a unique extension of Φ in (7.1) to a finitely additive, real-valued function Φ_0 on the Boolean algebra $\mathscr{A}_0$ generated in $2^{X(S)}$ by the sets $[s]$ with $s \in S$. Since every $[s]$ is compact-open and the compact-open subsets of $X(S)$ form a Boolean algebra, every member of $\mathscr{A}_0$ is compact-open. So no member of $\mathscr{A}_0$ can be a union of infinitely many, pairwise disjoint, nonempty members of $\mathscr{A}_0$. Therefore (a) follows from Proposition 6 and the well-known extension theorem for measures on Boolean algebras of sets.

Finally, (b) follows from (a) and Corollary 8(a).

THEOREM II. *Let ϕ be a real-valued function on a semilattice S with unit* 1. *Let H be any set of nonzero characters on S. Let $\mathscr{A}(H)$ be the σ-algebra generated in 2^H by the sets $[s]_H$ with $s \in S$. Define*

$$\Phi([s]_H) = \phi(s). \tag{7.2}$$

Then

(i) *Φ extends to a bounded, countably additive measure on $\mathscr{A}(H)$ iff*

$$\begin{cases} \text{Given any sequence } \{\boldsymbol{f}_n\} \text{ in } Z(S) \text{ such that} \\ \hat{\boldsymbol{f}}_n(p) \downarrow 0 \text{ for all } p \in H, \text{ then } (\phi, \boldsymbol{f}_n) \downarrow 0. \end{cases} \tag{7.3}$$

(ii) *Φ extends to a bounded, countably additive, signed measure in $\mathscr{A}(H)$ iff ϕ is of bounded variation and* (7.3) *holds for $|\phi|$.*

Proof. The Daniell condition (7.3) implies (v) of Proposition 6 through monotoneity of convergence in (7.3) applied to the sequence $\{\boldsymbol{f}, 0, 0, \cdots\}$ for any $\boldsymbol{f} \in Z(S)^+$. So (7.3) implies ϕ is positive definite. So $(\phi, —)$ is a finitely additive measure on the Boolean algebra $I(S)$. Define

$$\Phi(\pi_H \boldsymbol{f}) = (\phi, \boldsymbol{f}) \text{ for } \boldsymbol{f} \in I(S). \tag{7.4}$$

(7.4) is consistent with (7.2) under (2.2) and (2.5). Moreover, (7.4) vanishes for $\pi_H \boldsymbol{f} \equiv \boldsymbol{f} | H = \boldsymbol{0}$ by (7.3). So Φ in (7.4) is a finitely additive measure on the Boolean algebra $\pi_H(I(S))$ which is a subalgebra of 2^H. Since Φ is continuous by (7.3), Φ is countably additive and hence extends to a countably additive measure on the σ-algebra generated by $\pi_H(I(S))$ in 2^H. This σ-algebra contains $\mathscr{A}(H)$ because $[s]_H = (\pi_H \boldsymbol{s})^{-1}(1)$. That is, $\pi_H \boldsymbol{s}$ is the indicator of $[s]_H$.

The last statement implies $\pi_H \boldsymbol{s}$ is measurable, hence integrable, and therefore $\pi_H \boldsymbol{f}$ is integrable for all $\boldsymbol{f} \in Z(S)$. So the converse (7.3) in (i) follows from continuity of the integral.

Finally (ii) follows from (i) applied to ϕ^+, ϕ^- since these satisfy (7.3) whenever $|\phi|$ does.

Note that (7.3) can be put in the explicit form

(7.5) Given sequences $\{s_i\}$ and $\{t_j\}$ in S and increasing sequences $\{M_n\}$ and $\{N_n\}$ of positive integers such that

$$\sum_{i=1}^{M_n} p(s_i) - \sum_{j=1}^{M_n} p(t_j) \downarrow 0 \text{ for all } p \in H,$$

then

$$\sum_{i=1}^{M_n} \phi(s_i) - \sum_{j=1}^{N_n} \phi(t_j) \downarrow 0.$$

8. Measures on distributive lattices. Let $(L, \wedge, \vee)$ be a distributive lattice with distinct lower and upper bounds 0 and 1. We shall usually write ab for $a \wedge b$.

Each finite sequence $\alpha_0 = \{a_1, \cdots, a_n\}$ in L defines a mapping on Z^+ into L, namely

$$\alpha_0(k) = \begin{cases} 1 \text{ for } k = 0 \\ \left\{ \bigvee_{\substack{\alpha \in A \\ k(\alpha)=k}} \pi\alpha \right. \text{ for } 1 \leqq k \leqq n \\ 0 \text{ for } k > n \end{cases} \tag{8.1}$$

(See (3.2) for explanation of notation.)
Note that $\alpha_0(1) = a_1 \vee \cdots \vee a_n$ and $\alpha_0(n) = a_1 \cdots a_n$.

A *measure* ϕ on L is an isotone valuation [1] which vanishes at 0. That is, ϕ is a real-valued function on L such that

$$\begin{cases} \text{(i)} & \phi(0) = 0 \\ \text{(ii)} & \phi(a) \leqq \phi(b) \text{ for } a \leqq b \\ \text{(iii)} & \phi(a \vee b) + \phi(ab) = \phi(a) + \phi(b) \text{ for all } a, b. \end{cases} \tag{8.2}$$

As is well-known (iii) in (8.2) extends by induction to the inclusion-exclusion formula

$$\phi(a_1 \vee \cdots \vee a_n) = \sum_{\infty \in A} (-1)^{k(\alpha)+1} \phi(\pi\alpha) \tag{8.3}$$

in terms of the notation in (3.2). In terms of the injection of a distributive lattice L into the Boolean algebra $\mathscr{A}(L)$ it generates, the measures on L correspond to the finitely additive measures on $\mathscr{A}(L)$.

PROPOSITION 9. *The function defined in* (8.1) *satisfies:*

(i) $\alpha_0(k) \leqq \alpha_0(k-1)$ *for* $k > 0$

(ii) *For* $k > 0$, $n > 1$, *and* α_1 *the sequence* α_0 *with last term deleted,*

$$\alpha_0(k) = \alpha_1(k) \vee a_n\alpha_1(k-1).$$

Proof. (i) holds because each term $\pi\alpha$ of the join in (8.1) defining $\alpha_0(k)$ is a lower bound of a similar term in the join defining $\alpha_0(k-1)$. (ii) follows from (8.1) if we apply the distributive law to split the join in (8.1) into those terms $\pi\alpha$ with α a subsequence of α_1 and those with last term a_n.

PROPOSITION 10. *For* ϕ *a measure on* L *and* $\alpha_0 = \{a_1, \cdots, a_n\}$ *a finite sequence in* L,

$$\sum_{k=1}^{n} \phi(\alpha_0(k)) = \sum_{i=1}^{n} \phi(a_i) \tag{8.4}$$

Proof. Apply (iii) of (8.2) to (ii) of Proposition 9 and sum over k so that, after deleting terms which cancel in pairs, one gets

$$\sum_{k=1}^{n} \phi(\alpha_0(k)) = \sum_{k=1}^{n} \phi(\alpha_1(k)) + \phi(a_n). \tag{8.5}$$

So (8.4), being trivial for $n = 1$, follows by induction from (8.5).

PROPOSITION. 11. *For finite sequences* $\alpha_0 = (a_1, \cdots, a_n)$ *and* $\beta_0 = (b_1, \cdots, b_n)$ *in* L *we have* $\alpha_0(k) \leqq \beta_0(k)$ *for all* k *iff*

$$\sum_{i=1}^{n} h(a_i) \leqq \sum_{j=1}^{m} h(b_j) \text{ for every measure } h \text{ on } L \text{ with range } \{0, 1\}. \tag{8.6}$$

Proof. As is well-known [1] such lattice homomorphisms h separate L, giving a representation of L as a lattice $(L, \cap, \cup)$ of sets. Hence the first condition in Proposition 11 is equivalent to

$$h(\alpha_0(k)) \leqq h(\beta_0(k)) \text{ for all } \{0, 1\}\text{-measures } h \text{ and all } k. \tag{8.7}$$

Applying the lattice homomorphism h to (8.1) we conclude that

$h(\alpha_0(k)) = 1$ iff at least k terms a_i have $h(a_i) = 1$, equivalently

(8.8) $$\sum_{i=1}^{n} h(a_i) \geqq k.$$

So (8.7) is equivalent to $k \leqq \sum_{j=1}^{m} h(b_j)$ for all positive integers k satisfying (8.8). This condition is equivalent to (8.6) since sums of h-values are always nonnegative integers.

PROPOSITION 12. *A real-valued function ϕ on L is a measure iff for all finite sequences $\alpha_0 = (a_1, \cdots, a_n)$ and $\beta_0 = (b_1, \cdots, b_m)$ in L*

(8.9) $$\alpha_0(k) \leqq \beta_0(k) \text{ for all } k \text{ implies } \sum_{i=1}^{n} \phi(a_i) \leqq \sum_{j=1}^{m} \phi(b_j).$$

Proof. (8.9) applied to sequences α_0, β_0 of 0's with $m \neq n$ implies $\phi(0) = 0$ since $\alpha_0(k) \equiv \beta_0(k)$. (8.9) with $m = n = 1$ gives (ii) of (8.2). Finally, (8.9) applied to $\alpha_0 = \{a, b\}$ and $\beta_0 = \{ab, a \vee b\}$ gives (iii) of (8.2) since $\alpha_0(k) \equiv \beta_0(k)$.

Conversely, let ϕ be a measure on L. Then (8.9) follows from (8.4) in Prop. 10 and (ii) of (8.2).

THEOREM III. *Let L be a distributive lattice with 0,1. Let M be any subset of L such that $1 \in M$. A real-valued function ϕ on M extends to a measure on L iff (8.9) holds for all finite sequences α_0 and β_0 in M.*

Proof. By Proposition 12 we need only show that ϕ can be extended to L with (8.9) preserved. Since (8.9) is a property of finite character the axiom of choice yields a maximal extension of ϕ to some subset N of L containing M such that (8.9) holds. We contend $N = L$.

Suppose $N \neq L$. Then we could choose x in $L \sim N$ and define

(8.10) $$\phi(x) = \inf_{(8.11)} \sum_{j=1}^{q} \phi(b_j) - \sum_{i=1}^{p} \phi(a_i)$$

where the infimum is taken over

(8.11) $$\begin{gathered}\text{all } \{a_1, \cdots, a_p\}, \{b_1, \cdots, b_q\} \text{ in } N \text{ with} \\ \{x, a_1, \cdots, a_p\} \leqq \{b_1 \cdots, b_q\}\end{gathered}$$

in terms of the functions defined by (8.1) in L. Note that (8.11) is nonvoid since $\{1\}$, $\{1, 1\} \in (8.11)$. Therefore $\phi(x) < \infty$. On the other hand the inequality in (8.11) implies by Proposition 11 that $\{a_1, \cdots, a_p\} \leqq \{b_1, \cdots, b_q\}$. Hence $\phi(x) \geqq 0$ by (8.9). With the inequality in (8.11) for the hypothesis of (8.9) the conclusion of (8.9) follows from (8.10). On the other hand given

(8.12) $$\{a_{p+1}, \cdots, a_n\} \leqq \{x, b_{q+1}, \cdots, b_m\}$$

with the a_i's and b_j's in N we conclude from (8.11), (8.12) and Proposition 11 that $\{a_1, \cdots, a_p, \cdots, a_n\} \leqq \{b_1, \cdots, b_q \cdots, b_m\}$. So $\sum_{i=1}^{n} \phi(a_i) \leqq \sum_{j=1}^{m} \phi(b_j)$ since (8.9) holds in N. That is,

(8.13) $$\sum_{i=p+1}^{n} \phi(a_i) - \sum_{j=q+1}^{m} \phi(b_j) \leqq \sum_{j=1}^{q} \phi(b_j) - \sum_{i=1}^{p} \phi(a_i).$$

Hence by (8.10) the left side of (8.13) is a lower bound of $\phi(\mathrm{x})$. That is, the conclusion of (8.9) holds with the hypothesis (8.11). Therefore, since (8.9) holds for (8.11) and (8.12) it must hold for all sequences in $\{x, N\}$ by Proposition 11. This contradicts the maximality of N. So $N = L$.

THEOREM IV. *Let $(L, \wedge, \vee)$ be a distributive lattice with distinct* 0.1. *Let $(S, \wedge)$ be a subsemilattice of $(L, \wedge)$ such that $0,1 \in S$. Then a real-valued function ϕ on S extends to a measure on $(L, \wedge, \vee)$ iff both of the following conditions hold:*

(i) *$\phi(0) = 0$ and*

(ii) *For all finite sequences $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_m\}$ in S, $a_1 \vee \cdots \vee a_n \leqq b_1 \vee \cdots \vee b_m$ in L implies*

(8.14) $$\sum_{\alpha \in A} (-1)^{k(\alpha)+1} \phi(\pi\alpha) \leqq \sum_{\beta \in B} (-1)^{k(\alpha)+1} \phi(\pi\beta)$$

in the notation of (3.2)

Proof. Every measure ϕ on L satisfies (i) by definition. ϕ satisfies (ii) since (8.14) is just $\phi(a_1 \vee \cdots \vee a_n) \leqq \phi(b_1 \vee \cdots \vee b_n)$ which holds because ϕ is isotone.

Conversely, let (i) and (ii) hold for ϕ on S. Consider an arbitrary member $\boldsymbol{f} = \boldsymbol{a} * (1 - \boldsymbol{a}_1) * \cdots * (1 - \boldsymbol{a}_n)$ of $J(S)$ with $a, a_1, \cdots, a_n$ in S and $a_i \leqq a$ for all i. In terms of the join in $Z(S)$, $\boldsymbol{f} = \boldsymbol{a} - \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_n$. So for (2.5) we have $(\phi, \boldsymbol{f}) = \phi(a) - (\phi, \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_n)$. With $b_1 = \cdots = b_m = a$ in (ii), (8.14) becomes $(\phi, \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_n) \leqq \phi(a)$. Hence $(\phi, \boldsymbol{f}) \geqq 0$. So (iv) of Proposition 6 holds. That is, ϕ is positive definite. This together with (i) implies that $(\phi, \ )$ of (2.5) is a measure on $I(S)$.

To prove ϕ on S extends to a measure on L we apply Theorem III. We need only show that (8.9) holds for sequences in S. The hypothesis of (8.9) implies $\bigvee_{k(\alpha)=k} \pi\alpha \leqq \bigvee_{k(\alpha)=k} \pi\beta$ in L for all k. Using these inequalities in the hypothesis of (ii) we get

(8.15) $$(\phi, \bigvee_{k(\alpha)=k} \pi\alpha) \leqq (\phi, \bigvee_{k(\alpha)=k} \pi\beta)$$

for all k for the conclusion (8.14). Summing (8.15) over k and subjecting both sides to Proposition 10 in terms of the measure $(\phi,\ )$ on $I(S)$ we get the conclusion of (8.9) from (8.4).

9. Regular measures on a regulated semilattice. A *regulated semilattice* [2] is a triple $(S, \cdot, \ll)$ where

(i) $(S, \cdot)$ is a semilattice with 0 in which $a^\perp = b^\perp$ implies $a = b$, where for E a subset of S the annihilator of E, consisting of all x in S such that $xe = 0$ for all e in E, is denoted by $E^\perp$.

(ii) the regulator $\ll$ is a binary relation on S satisfying:

(A_1) $a \ll b$ implies $a \leqq b$,
(A_2) $0 \ll 0$,
(A_3) If $a \ll b$ and $c \ll d$ then $ac \ll bd$,
(A_4) If $a \ll b \leqq c$ then $a \ll c$,
(A_5) Given $p \ll q \neq 0$ there exist $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ such that $\{a_1, \cdots, a_n\}^\perp \subseteqq p^\perp$, $b_i \neq 0$ for some i, and $a_i \ll b_i \ll q$ for all i,
(A_6) Given a, b, c with $a \ll b$ there exist $d_1, \cdots, d_n$ and $e_1, \cdots, e_n$ such that $\{b, d_1, \cdots, d_n\}^\perp \subseteqq c^\perp$, $ae_i = 0$ and $d_i \ll e_i$ for all i,
(A_7) If $\{a_1, \cdots, a_n\}^\perp \subseteqq p^\perp$ and $a_i \ll q$ for all i, then $p \ll q$.

It was shown in [2] that a regulated semilattice $(S, \cdot, \ll)$ is characterized by the existence of a unique representation $(\mathscr{S}, \cap, \Subset)$ in which $\mathscr{S}$ is a topological base of interiors of compact subsets of a locally compact space X and $A \Subset B$ means the closure of A is interior to B. We consider the following question here: Which functions ϕ on S can be extended, when transferred to $\mathscr{S}$ under the representation, to regular Borel measures on the locally compact space X?

The regulator $\ll$ can be extended to a binary relation on finite sequences α_0, β_0 in S by defining

(9.1) $\beta_0 \ll \alpha_0$ to mean there exists a finite sequence δ in S such that $\delta^\perp \subseteqq \beta_0^\perp$ and for each d_K in δ there is some a_i in α_0 with $d_k \ll a_i$. In terms of the representation it is easily seen that (9.1) means $\bigcup_{A\hat{i} \in \alpha} A_i \Subset \bigcup_{B\hat{j} \in \beta} B_j$.

THEOREM V. *Let ϕ be a real-valued function on a regulated semilattice S. Under the representation of $(S, \cdot, \ll)$ as $(\mathscr{S}, \cap, \Subset)$ in a locally compact space X the function ϕ has a unique extension to a regular Borel measure on X iff the following conditions hold:*

(i) $\phi(0) = 0$

(ii) *For every finite sequence* $\alpha_0 = \{a_1, \cdots, a_n\}$ in S

$$(9.2) \qquad (\phi, \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_n) = \sup_{\beta_0 \ll \alpha_0} (\phi, \boldsymbol{b}_1 \vee \cdots \vee \boldsymbol{b}_m)$$

where the left side of (9.2) *is given explicitly by the left side of* (8.14).

Proof. Let ϕ satisfy (i) and (ii). With $b_1 = \cdots = b_m = 0$ in (ii) we conclude from (i) that (9.2) is nonegative. So ϕ is positive definite since (vi) of Proposition 6 holds.

For each compact subset C of X define

$$\phi_0(C) = \inf_{A_1 \cup \cdots \cup A_n \supseteq C} (\phi, \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_n) \tag{9.3}$$

where A_i is the member of $\mathscr{S}$ representing the member a_i of S. We contend that ϕ_0 is a regular content, hence is the restriction to compact sets of a unique regular Borel measure on X. That is, we must verify the following five conditions for compact sets C and D:

(m_1) $0 \leqq \phi_0(C) < \infty$,
(m_2) $C \subseteqq D$ implies $\phi_0(C) \leqq \phi_0(D)$,

(m_3) $\phi_0(C \cup D) \leqq \phi_0(C) + \phi_0(D)$,
(m_4) $C \cap D = \phi$ implies $\phi_0(C) + \phi_0(D) \leqq \phi_0(C \cup D)$,
(m_5) $\phi_0(C) = \inf_{D \supset C} \phi_0(D)$.
(m_1) and (m_2) follow trivially from (9.3).

Given $a_1, \cdots, a_m, \cdots, a_n$ in S such that

$$A_1 \cup \cdots \supset A_m \supseteqq C \text{ and } A_{m+1} \supset \cdots \supset A_n \supseteqq D \tag{9.4}$$

then

$$\phi_0(C \cup D) \leqq (\phi, \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_m \vee \cdots \vee \boldsymbol{a}_n) \leqq (\phi, \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_m) + (\phi, \boldsymbol{a}_{m+1} \vee \cdots \vee \boldsymbol{a}_n) \tag{9.5}$$

since $A_1 \cup \cdots \cup A_m \cup \cdots \cup A_n \supseteqq C \cup D$ and ϕ is positive definite. Taking infima on the right side of (9.5) we get (m_3). To verify (m_4) consider any $B_1, \cdots, B_k$ in $\mathscr{S}$ which cover $C \cup D$. Using properties of the base $\mathscr{S}$ and the disjointness of C, D we can find $A_1, \cdots, A_m, \cdots, A_n$ in $\mathscr{S}$ such that (9.4) holds, each A_i is contained in some B_j, and $A_i \cap A_j = \varnothing$ for $i \leqq m < j \leqq n$.
So

$$\begin{cases} (\boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_m) * (\boldsymbol{a}_{m+1} \vee \cdots \vee \boldsymbol{a}_n) = 0 \text{ and} \\ \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_n \leqq \boldsymbol{b}_1 \vee \cdots \vee \boldsymbol{b}_k \text{ in } Z(S). \end{cases} \tag{9.6}$$

From (9.3) and (9.4) we get $\phi_0(C) \leqq (\phi, \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_m)$ and $\phi_0(D) \leqq (\phi, \boldsymbol{a}_{m+1} \vee \cdots \vee \boldsymbol{a}_m)$ which yield under addition

$$\phi_0(C) + \phi_0(D) \leqq (\phi, \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_m \vee \cdots \boldsymbol{a}_n) \leqq (\phi, \boldsymbol{b}_1 \vee \cdots \vee \boldsymbol{b}_K) \tag{9.7}$$

by (9.6). Taking an infimum on the right side of (9.7) we get (m_4).

To verify (m_5) consider an arbitrary $\varepsilon > 0$. Choose $a_1, \cdots, a_n$ in (9.3) with $\phi_0(C) > (\phi, \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_n) - \varepsilon$. Choose a compact set D such that $C \Subset D \subseteqq A_1 \cup \cdots \cup A_n$. Then by (9.3) $\phi_0(D) \leqq (\phi, \boldsymbol{a}_1 \vee \cdots \vee \boldsymbol{a}_n) < \phi_0(C) + \varepsilon$. Hence (m_5) holds. Thus ϕ_0 is a regular content.

Let $\bar{\phi}$ be the unique regular Borel measure that extends $\phi_0 \cdot \phi_0(C) = \bar{\phi}(C) = \inf_A \bar{\phi}(A)$ where C is any compact set and the infimum is taken over all finite unions A of members of $\mathscr{S}$ such that $A \supseteqq C$. So ϕ_0 in (9.3) is the only content which extends ϕ.

Conversely, if ϕ has an extension $\bar{\phi}$ then (i) and (ii) must hold since they state that $\bar{\phi}(\varnothing) = 0$ and $\bar{\phi}(A) = \sup_{B \supset A} \bar{\phi}(B)$ for A, B finite unions of members of $\mathscr{S}$.

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