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TWO REMARKS ON ELEMENTARY EMBEDDINGS OF THE UNIVERSE

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The paper contains the following two observations: 1. The existence of the least submodel which admits a given elementary embedding j of the universe. 2. A necessary and sufficient condition on a complete Boolean algebra B that the Cohen extension V^B admits j.

A function j defined on the universe V is an *elementary embedding of the universe* if there is a submodel M such that for any formula φ,

$$(*) \qquad \forall x_1, \cdots, x_n[\varphi(x_1, \cdots, x_n) \longleftrightarrow M \models \varphi(jx_1, \cdots, jx_n)].$$

Let j be an elementary embedding of the universe. If N is a submodel, let $j_N = j | N$ be the restriction of j to N. N *admits* j if

$$(**) \qquad N \models j_N \text{ is an elementary embedding of the universe.}$$

If B is a complete Boolean algebra, let V^B be the Cohen extension of V by B. V^B *admits* j if

$$(***) \qquad V^B \models \text{there exists an elementary embedding } i \text{ of the universe such that } i \supseteq j$$

THEOREM 1. *There is a submodel $L(j)$ which is the least submodel which admits j.*[1]

THEOREM 2. *The Cohen extension V^B admits j if and only if the identity mapping on $j''B$ can be extended to a $j(V)$ − complete homomorphism of $j(B)$ onto $j''B$.*

Before giving the proof, we have a few remarks. The underlying set theory is the axiomatic theory BG of sets and classes of Bernays and Gödel [1]. The formula φ in (*) is supposed to have only set variables. However, if for any class C we let $j(C) = \bigcup_{\alpha \in On} j(C \cap V_\alpha)$, then (*) holds also for formulas having free class variables ("normal formulas" of [1].) Incidentally, "j is an elementary embedding of the universe" is expressible in the language of BG (viz.: j is an ε-isomorphism and $\forall C_1 \forall C_2[\mathscr{F}_i(jC_1, jC_2) = j(\mathscr{F}_i(C_1, C_2))]$ where $\mathscr{F}_i$ are the Gödel operations).

[1] This was observed independently by K. Hrbáček, giving a different proof.

A *submodel* M is a transitive class containing all ordinals which is a model of GB; the classes of M are all those subclasses C of M which satisfy the condition $\forall\alpha(C \cap V_\alpha \in M)$. The submodel M in (*) is unique and $M = j(V)$. It is a known fact that if j is not the identity then there exists a measurable cardinal. And, as proved recently by Kunen [2], $j(V) \neq V$. On the other hand, if there exists a measurable cardinal, then there exists a nontrivial elementary of the universe (cf. Scott [6]).

The notion $L(j)$ differs somewhat from the notion of relative constructibility, introduced by Lévy [4]; in general, $L(j) \supseteqq L[j]$.

A homomorphism is C-complete, if it preserves all Boolean sums $\sum_{i\in I} u_i$ where $\{u_i: i \in I\} \in C$. As usual, $j''B$ is the algebra $\{j(u): u \in B\}$; $j(B)$ is an algebra, $j(B) \supseteqq j''B$, and $j(B)$ is not necessarily complete (although jV-complete).

A similar observation as our Theorem 2 was used recently by J. Silver in his result about extendable cardinals.

As a corollary of Theorem 2, we get the following theorem of Lévy and Solovay [5]: *If κ is measurable and $|B| < \kappa$, then κ is measurable in V^B.*[2]

Let j be a fixed elementary embedding of the universe. First we prove Theorem 1.

Let M be a submodel.

LEMMA 1. *If j_M is a class of M then M admits j.*

Proof. We must show that for any formula φ,

$$(\forall \vec{x} \in M) M \models (\varphi(\vec{x}) \rightarrow jM \models \varphi(j\vec{x})).$$

If $M \models \varphi(\vec{x})$, then since $M \models \varphi(\vec{x})$ is a normal formula, we have $jV \models (jM \models \varphi(j(\vec{x}))$. However, $\models$ is absolute, so that $M \models (jM \models \varphi(j(\vec{x})))$.

LEMMA 2. *If $j \cap M$ is a class of M and if M is closed under j (i.e., $j''M \subseteqq M$), then M admits j.*

Proof. It suffices to show that j_M is a class of M. Obviously, $j_M \cap M = j \cap M$, and because M is closed under j, we have $j_M \subseteqq M$, and $j_M = j_M \cap M = j \cap M$.

Now we define the model $L(j)$:

(i) $L_0(j) = 0$,
(ii) $L_\alpha(j) = \bigcup_{\beta<\alpha} L_\beta(j)$ if α is a limit ordinal,

[2] An example of models which are not mild extensions but still admit j are the models constructed by Kunen and Paris in [3].

(iii) $L_{\alpha+1}(j) = \mathrm{Def}(\langle L_\alpha(j), \varepsilon, j \cap L_\alpha(j)\rangle)$ if α is even,
(iv) $L_{\alpha+1}(j) = L_\alpha(j) \cup [j''L_\alpha(j) \cap \mathscr{P}(L_\alpha(j))]$ if α is odd,
(v) $L(j) = \bigcup_{\alpha \in On} L_\alpha(j)$.

(iii) means that $L_{\alpha+1}(j)$ consists of all subsets of $L_\alpha(j)$ which are definable in $L_\alpha(j)$ from $j \cap L_\alpha(j)$. $\mathscr{P}(L_\alpha(j))$ is the set of all subset of $L_\alpha(j)$.

By standard methods it follows that $L_\alpha(j)$ is a submodel. That $L_\alpha(j)$ satisfies the axiom of choice is proved in Lemma 4.

LEMMA 3. *$i = j \cap L(j)$ is a class of $L(j)$ and*

$$L(j) = L(i) = L^{L(j)}(i).$$

Proof. By induction on α, we prove

$$L_\alpha(j) = L_\alpha(i) = L_\alpha^{L(j)}(i).$$

If α is a limit ordinal or $\alpha = \beta + 1$ with β even, then the proof is standard. Let β be odd:

$$\begin{aligned} x \in L_{\beta+1}(j) &\leftrightarrow x \in L_\beta(j) \vee [x \subseteqq L_\beta(j) \wedge x \in L(j) \wedge (\exists y \in L_\beta(j))[x = j(y)]] \\ &\leftrightarrow x \in L_\beta(i) \vee [x \subseteqq L_\beta(i) \wedge (\exists y \in L_\beta(i))[x = i(y)]] \\ &\leftrightarrow x \in L_{\beta+1}(i) \\ &\leftrightarrow x \in L_{\beta+1}^{L(j)}(i). \end{aligned}$$

COROLLARY. $L(j) \models V = L(i)$.

LEMMA 4. *$L(j) \models$ Axiom of Choice.*

Proof. If $V = L(i)$ then there is a well ordering of the universe, definable from i; hence $L(j) \models$ Axiom of Choice.

LEMMA 5. *$L(j)$ is closed under j.*

Proof. (a) If $X \subseteqq On$ and $X \in L(j)$ then there exists α such that $X \in L_\alpha(j)$ and $j(X) \subseteqq \alpha \subseteqq L_\alpha(j)$; hence $j(X) \in L_{\alpha+1}(j)$ and so $j(X) \in L(j)$. Similarly, if $X \subseteqq On \times On$.

(b) If $X \in L(j)$ is arbitrary, then since $L(j) \models AC$, there exists a well founded relation $R \in L(j)$ on ordinals which is isomorphic to $TC(\{X\})$, the transitive closure of $\{X\}$. Hence $j(TC(\{X\})) = TC(\{jX\})$ is isomorphic to $j(R)$ which is well founded and by (a), $jR \in L(j)$; thus $j(X) \in L(j)$.

LEMMA 6. *If M admits j then*

$$L(j) = L^M(j \cap M) \subseteq M.$$

Proof. Same as of Lemma 3.
Now, Theorem 1 follows.

Let B be a complete Boolean algebra. The *Cohen extension* V^B is the Boolean-valued model of Scott [7] or Vopěnka [8]. There is a natural embedding $x \mapsto \check{x}$ of V into V^B and $C \mapsto \check{C}$ can be defined also for classes, in a natural way (in (***), we should rather write $i \supseteq \check{j}$). More generally, if M is a submodel satisfying the axiom of choice and if $B \in M$ is an M-complete Boolean algebra then M^B is the Cohen extension of M by B.

LEMMA 7. *The condition in Theorem 2 is necessary.*

Proof. Let i be such that

(1) $V^B \models i$ is an elementary embedding of the universe and $i \supseteq \check{j}$.

Let G be the canonical generic ultrafilter on $\check{B}$, i.e.,

(2) $$G \in V^{(B)},\ \mathrm{dom}\,(G) = \{\check{u}\colon u \in B\},\quad G(\check{u}) = u \text{ for all } u \in B.$$

From (1) it follows that

(3) $V^B \models i(G)$ is an $i(\check{V})$-complete ultrafilter on $i(\check{B})$, i.e.,
(4) $V^B \models i(G)$ is a $(jV)^\vee$-complete ultrafilter on $(jB)^\vee$.

Let f be the following function from $j(B)$ into B:

$$f(v) = [\![\check{v} \in i(G)]\!].$$

By (4), f is a $j(V)$-complete homomorphism of $j(B)$ into B and for all $u \in B$, $f(ju) = [\![(ju)^\vee \in i(G)]\!] = [\![i(\check{u}) \in i(G)]\!] = [\![\check{u} \in G]\!] = u$. If we let $h = j \circ f$ then h is a $j(V)$-complete homomorphism of $j(B)$ onto $j''B$ and $h \mid j''B$ is the identity.

LEMMA 8. *The condition is sufficient.*

Proof. Let h be a $j(V)$-complete homomorphism of $j(B)$ onto $j''B$ such that $h(ju) = ju$ for all $u \in B$. We are supposed to find i such that (1) holds. To simplify the considerations, assume that G is some V-complete ultrafilter on B and that $V[G]$ is the universe. (This is possible because

$$V^B \models \check{V}[G] \text{ is the universe},$$

where G is the canonical generic ultrafilter defined in (2).)

Let $i(G) = h_{-1}(j''G)$. We have $i(G) \supseteq j''G$, and

$$i(G) \text{ is a } j(V)\text{-complete ultrafilter on } j(B).$$

Let π_G: $V^B \rightarrow V[G]$ be the G-interpretation of V^B:

$$\pi_G(0) = 0,$$
$$\pi_G(x) = \{\pi_G(y): x(y) \in G\}.$$

Since $j(B) \in j(V)$ is an $j(V)$-complete Boolean algebra, $j(V)^{j(B)} = j(V^B)$ is the Cohen extension of $j(V)$ by $j(B)$; it follows from the definition of $i(G)$ that $i(G)$ is a $j(V)$-complete ultrafilter on $j(B)$. Let π_{iG}: $(jV)^{jB} \rightarrow (jV)[iG]$ be the $i(G)$-interpretation of $(jV)^{jB}$ and let

$$i(\pi_G x) = \pi_{iG}(jx), \text{ for all } x \in V^B.$$

Now we claim that i is a function, i is an elementary embedding of $V[G]$ into $(jV)[iG]$ and that $i \supseteq j$. To prove that, note that for any formula φ and for all $\vec{x} \in V^B$,

$$[\![\varphi(j\vec{x})]\!]^{jV}_{jB} = j[\![\varphi(\vec{x})]\!]^{V}_{B};$$

This can be proved by induction on the rank of $\vec{x}$ and on the complexity of φ. In particular, if $\pi_G x = \pi_G y$, then $[\![x = y]\!]^V_B \in G$, so that $[\![jx = jy]\!]^{jV}_{jB} \in j''G \subseteq i(G)$ and so $i(\pi_G x) = \pi_{iG}(jx) = \pi_{iG}(jy) = i(\pi_G y)$. Similarly, if $V[G] \models \varphi(\pi_G \vec{x})$, then $(jV)[iG] \models \varphi(i(\pi_G \vec{x}))$. *If* $x \in V$, then $i(x) = i(\pi_G \check{x}) = \pi_{iG}(j\check{x}) = j(x)$.

This completes the proof of Theorem 2.

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