

NONLINEAR EQUATIONS OF EVOLUTION

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We begin by considering various kinds of nonlinear operators in a Banach lattice X , i.e., a Banach space which has a compatible lattice structure. With adequate definitions we are able to develop a theory parallel to the theory of nonlinear equations of evolutions in a general Banach space, as carried out by Komura, Kato, Browder and others. Existence and uniqueness theorems about solutions of the equation of evolution $du(t)/dt = -Au(t)$ are developed under conditions on the space X and the operator A . Given a solution $u(t)$ to $du(t)/dt = -Au(t)$ with initial condition $u(0) = v$, where v lies in the domain of A , a semi-group $U(t)$ is defined by $U(t)v = u(t)$, $t \geq 0$.

Conditions are found under which the semi-group is order-preserving, and under which a trajectory $u(t)$ is itself increasing as t increases. Under these conditions, the existence of zeroes of the infinitesimal generator $-A$ is derived, corresponding to fixed points of the semi-group $U(t)$. These results are stronger than those concerning the zeroes of accretive operators in a general Banach space.

Under the special condition that the space X be an algebra whose unit is an order unit, we give conditions on $-A$ in order that it generates a semi-group, without assuming local uniform continuity of A .

Ergodic theory, which concerns the convergence of $1/n \int_0^n U(t)v dt$ as $n \rightarrow \infty$ is developed, having an intimate connection with the nonlinear semi-groups $U(t)$ of the type considered above. Nonlinear ergodic theorems have not so far appeared in the literature.

Furthermore, the existence of a solution to $du(t)/dt = +Au(t)$ under the condition that A preserves order is discussed. These results lead to some new open questions, as for example how to carry over the theory of fixed points and zeroes developed above.

The last two sections deal with equations of evolution outside the Banach lattice. For the case of a lattice, the concept of $x \rightarrow 2J(x^+)$ being a selection of the subgradient of $x \rightarrow \|x^+\|^2$, with J a duality map, was used. With G a closed convex subset of x , we now use the fact that $x \rightarrow 2J(x - Ux)$ is a selection of the subgradient of $x \rightarrow (d(x, G))^2$, where Ux denotes a nearest point to x in G . With G compact, we directly generalise some of the results of the theory of accretive operators, which correspond here to $G = \{0\}$.

Finally, in order to find zeroes of a given operator A , a smoothed

version of $du(t)/dt = + Au(t)$ is considered. Smoothing refers to the fact that Banach spaces $X_2 \subset X_1$ and linear operators $St, t \geq 0$, from X_1 to X_2 are considered, which abstract the idea of smoothing or modification by convolution. This gives results under the conditions of the Nash-Moser inverse function theorem.

1. **Nonlinear Operators in a Banach Lattice.** We recall a *Banach lattice* X is a Banach space X over the real numbers R , which is a lattice under the ordering \leq , such that for x, y and z in X ,

- (a) $x \leq y$ implies $x + z \leq y + z$
- (b) $x \leq y$ implies $ax \leq ay$ for $a \geq 0$ in R
- (c) $|x| \leq |y|$ implies $\|x\| \leq \|y\|$.

A background in Banach lattices is provided by the books of Yosida [24], Schaefer [20] and Birkhoff [2]. Schaefer's notation is mostly followed, in particular $x^+ = \sup(x, 0)$ and $x^- = \sup(-x, 0)$, so that $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

We use where possible nonlinear terminology from Browder [5]. In particular a *duality map* J is a function from X to its dual X^* such that $(Jx, x) = \|x\|^2$ and $\|Jx\| = \|x\|$ for x in X . With an order structure on a Banach space we may consider further properties possessed by J .

DEFINITION. Suppose X is a Banach lattice, and J a duality map. Then J is *positive* if

- (1) $(Jx, y) \geq 0$ if $x \geq 0$ and $y \geq 0$
- (2) $(Jx, y) = 0$ if $x \perp y$.

Since writing this paper as a thesis the author has found Phillips [18] used this duality mapping in the theory of linear nonexpansive semigroups.

We recall: a subset A of X is *order bounded* if it is contained in an *order interval* $[a, b] = \{z \text{ in } X: a \leq z \leq b\}$. X is *countably order complete* (*sigma complete* in Yosida [24]) if for every order bounded countable subset A of X , $\sup A$ and $\inf A$ exist. X is *order complete* if we remove the countability requirement. Following Krasnoselski [13], if X is an ordered Banach space we say the norm is *monotonic* if $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$. Schaefer [20] page 215 shows that X has an equivalent *monotonic* norm iff the positive cone $\{x \text{ in } X: x \geq 0\}$ is normal.

PROPOSITION 1.1. (1) *Suppose X is an ordered Banach space with monotonic norm. Then there is a duality map J satisfying (1) above.*

(2) *Supposing X is a Banach lattice, then there is a duality map satisfying (2) as well, i.e., a positive duality map.*

Proof. (1) Suppose x in K , the positive cone of X , and $\|x\| = 1$. Let B be the open ball center 0 radius 1 and let $B - K$ be the convex open set $\{b - k: b \text{ in } B \text{ and } k \text{ in } K\}$. x is not in $B - K$, for if $x = b - k$, then $b \geq x \geq 0$. Hence, $\|b\| \geq \|x\| = 1$.

By the Hahn-Banach theorem there is a closed hyperplane containing x which does not intersect $B - K$. Hence, there is an element J_1x of X^* with $(J_1x, x) = 1$ and $(J_1x, f) < 1$ if f is in $B - K$. By continuity, $(J_1x, x - k) \leq 1$, hence, $(J_1x, k) \geq 0$ for k in K . For b in B , $(J_1x, b) < 1$, hence, $\|J_1x\| \leq 1$. Therefore, $\|J_1x\| = 1$ since $(J_1x, x) = 1$. This gives a duality map on $K \cup -K$ satisfying (1). If x is not in K or $-K$, we consider the open set B instead of $B - K$ as usual to obtain a duality map $J_1: X \rightarrow X^*$ satisfying (1). The device used above is that of the Brauer-Namioka theorem on extension of positive functionals. (See Schaefer [20], page 227.)

(2) X may not be sigma complete, but X^{**} is; indeed, it is order complete. Hence, for x in X^{**} and for $y \geq 0$ we define

$$P_x(y) = \sup\{\inf(n|x|, y): n \text{ in } Z^+\} \\ = \sup\{[0, y] \cap B_x\}$$

where $B_x = \{x^\perp\}^\perp$ is the band generated by x . For general y we define $P_x(y) = P_x(y^+) - P_x(y^-)$. P_x is a bounded linear operator of norm 1, idempotent, a lattice homomorphism, and $P_x(y) = 0$ if $x \perp y$.

Now the evaluation $e: X \rightarrow X^{**}$ defined by $(e(x), f) = (x, f)$ for f in X^* is a lattice homomorphism and an isometry. If X were sigma complete with J_1 as in (1) a duality map for X , we could define $Jx = P_x^*J_1(x)$.

Given J_1 as in (1) a duality map for X^{**} , we define for x in X , $Jx = e^*P_x^*J_1ex$.

- (a) $(Jx, x) = (J_1ex, P_x(ex)) = \|ex\|^2 = \|x\|^2$;
- (b) $(Jx, y) = (J_1ex, P_x(ey)) \leq \|ex\| \|ey\| = \|x\| \|y\|$;
- (c) if $x, y \geq 0$, then $ex \geq 0$ and $P_x(ey) \geq 0$, hence, $(Jx, y) \geq 0$;
- (d) if $x \perp y$, then $ex \perp ey$, hence, $P_x(ey) = 0$, hence, $(Jx, y) = 0$.

Hence, J is a positive duality map.

COROLLARY. *If X is a Banach lattice and X^* is strictly convex, then the duality map is positive since it is unique.*

If g is a convex real valued function on X , then the subgradient $dg: X \rightarrow$ subsets of X^* is defined by: w is in $dg(x)$ iff for all u in X ,

$$g(u) \geq g(x) + (w, u - x).$$

A selection of a function $F: X \rightarrow$ subsets of Y is a function $f: X \rightarrow Y$ with $f(x)$ in $F(x)$ for x in X . The first part of this next

result is central to further development.

PROPOSITION 1.2. *Suppose X a Banach lattice with positive duality map J . Then $y \rightarrow 2J(y^+)$ is a selection of the subgradient of $y \rightarrow \|y^+\|^2$. Conversely, if $x \rightarrow 2w(x)$ is a selection of the subgradient of $x \rightarrow \|x^+\|^2$ and $w(x) \geq 0$ for all x in X , then $w(x)$ is a permissible value for $J(x^+)$.*

Proof. Let x and y be elements of X .

$$\begin{aligned} (J(x^+), y) &\leq (J(x^+), y^+) \text{ since } J \text{ positive and } y^+ \geq y. \\ (J(x^+), x) &= (J(x^+), x^+ - x^-) \\ &= (J(x^+), x^+) \text{ since } J \text{ positive and } x^+ \perp x^-. \end{aligned}$$

Now $(\|x^+\| - \|y^+\|)^2 \geq 0$, hence,

$$\|y^+\|^2 \geq \|x^+\|^2 + 2(J(x^+), y^+ - x^+).$$

Therefore,

$$\|y^+\|^2 \geq \|x^+\|^2 + 2(J(x^+), y - x),$$

and $x \rightarrow 2J(x^+)$ is a selection of the subgradient of $x \rightarrow \|x^+\|^2$.

Conversely, suppose for y in X we have

$$(1) \quad \|y^+\|^2 \geq \|x^+\|^2 + 2(w(x), y - x).$$

Putting $y = ax$, a real in (1), we have

$$a \|x^+\|^2 - 2(w(x), x)a + 2(w(x), x) - \|x^+\|^2 \geq 0.$$

Hence, the discriminant is ≤ 0 , i.e.,

$$\begin{aligned} 0 &\geq (w(x), x)^2 - 2 \|x^+\| (w(x), x) + \|x^+\|^4 \\ &= ((w(x), x) - \|x^+\|^2)^2. \end{aligned}$$

Hence,

$$\begin{aligned} (w(x), x) &= \|x^+\|^2. \\ ((w(x), x^+)) &= (w(x), x) + (w(x), x^-) \\ &\geq (w(x), x) \text{ since } w(x) \geq 0 \text{ and } x^- \geq 0 \\ &= \|x^+\|^2. \end{aligned}$$

Now given $e > 0$, there is $U(e)$ in X with $\|U(e)\| = \|w(x)\|$ and

$$(w(x), U(e)) \geq \|w(x)\|^2 - e.$$

Putting $y = U(e)$ in (1), we have

$$\begin{aligned} \|U(e)\|^2 + \|x^+\|^2 &\geq 2(w(x), U(e)) \\ &\geq 2\|w(x)\|^2 - 2e. \end{aligned}$$

Therefore,

$$\|w(x)\|^2 \leq \|x^+\|^2 + 2e .$$

Hence,

$$\|w(x)\| \leq \|x^+\| ,$$

giving

$$\|w(x)\| = \|x^+\| \text{ and } (w(x), x^+) = \|x^+\|^2 .$$

A *global ψ system* for a Banach space X is a function $\psi: X \rightarrow X^*$ with $\|\psi(x)\| \leq \|x\|$ and $(\psi(x), x) \geq c\|x\|^2$ for x in X , where $c > 0$ is in R . (See Browder [7], Chapter 3)

PROPOSITION 1.3. *Suppose X a Banach lattice with positive duality map J , then $\psi(z) = \frac{1}{2}(J(z^+) - J(z^-))$ is a global ψ system for X with constant $c = 1/4$.*

Proof. $\|z^+\| \leq \|z\|$ and $\|z^-\| \leq \|z\|$ since X a Banach lattice, hence, $\|\psi(z)\| \leq \|z\|$.

$$\begin{aligned} 2(\psi(z), z) &= (J(z^+) - J(z^-), z^+ - z^-) \\ &= \|z^+\|^2 + \|z^-\|^2 \quad \text{since } J \text{ positive .} \end{aligned}$$

Now

$$2\|z^+\|\|z^-\| \leq \|z^+\|^2 + \|z^-\|^2 .$$

Therefore,

$$\begin{aligned} \|z\|^2 &\leq (\|z^+\| + \|z^-\|)^2 \\ &\leq 2(\|z^+\|^2 + \|z^-\|^2) . \end{aligned}$$

Therefore,

$$(\psi(z), z) \geq \frac{1}{4} \|z\|^2 .$$

DEFINITION. Suppose X a Banach lattice with positive duality map J .

$A: D(A) \rightarrow X$, $D(A) \subset X$, is *T-accretive* if for x and y in $D(A)$, $(Ax - Ay, J(x - y)^+) \geq 0$, and A is *hypermaximal T-accretive* if also

$$R(I + A) = X .$$

$U: D(U) \rightarrow X$, $D(U) \subset X$, is *T-nonexpansive* if for x and y in $D(U)$

$$\|(Ux - Uy)^+\| \leq \|(x - y)^+\| .$$

$A: D(A) \rightarrow X$ is *generalized T-accretive* if there is $k > 0$ in R with

$$(Ax - Ay, J(x - y)^+) \geq -k\|(x - y)^+\|^2 .$$

$U: D(U) \rightarrow X$ is *T-Lipschitz* if there is $k > 0$ in R with

$$\|(Ux - Uy)^+\| \leq k \|(x - y)^+\| .$$

$A: D(A) \rightarrow X$ is *locally generalized T-accretive* if for z in $D(A)$, there is $k_z > 0$, and a neighborhood N_z of z in X , with

$$(Ax - Ay, J(x - y)^+) \geq -k_z \|(x - y)^+\|^2$$

for x and y in $N_z \cap D(A)$.

$U: D(U) \rightarrow X$ is *locally T-Lipschitz* if for z in $D(U)$, there is $k_z > 0$ and a neighborhood N_z of z in X with $\|(Ux - Uy)^+\| \leq k_z \|(x - y)^+\|$ for x and y in $N_z \cap D(U)$.

Given a global ψ system ψ , $A: D(A) \rightarrow X$ is ψ -*accretive* if

$$(Ax - Ay, \psi(x - y)) \geq 0 .$$

$U: D(U) \rightarrow X$ is *monotonic* if $x \geq y$ implies $Ux \geq Uy$.

$U: D(U) \rightarrow X^*$ is *T-monotone* if $(Ux - Uy, (x - y)^+) \geq 0$.

The definition of *T-monotone* functions was given independently by Brezis-Stampacchia [5] in a particular concrete case. *T-monotone* implies *monotone*, but, in general, *T-accretive* does not imply *accretive*. In many function spaces it does, we see later, but we obtain additional information from the *T-accretive* property.

T-accretive functions arise as nonlinear partial differential operators; for example, the conditions (A) of Brower [6] give *T-monotonicity* in the Hilbert space $L^2(G)$, hence, *T-accretivity*, for second order operators.

We now consider certain relations between these classes.

PROPOSITION 1.4. *Suppose X a Banach lattice with positive duality map, and $A: D(A) \rightarrow X$ is T-accretive. Then A is accretive with respect to $\psi(z) = \frac{1}{2}(J(z^+) - J(z^-))$ of Proposition 1.3.*

Proof. Let x and y be in $D(A)$. Now $2\psi(x - y)^+ = J(x - y)^+$ and $2\psi(x - y)^- = J(x - y)^-$ and $2\psi(x - y) = \psi(x - y)^+ - \psi(x - y)^-$. Hence,

$$\begin{aligned} 2(Ax - Ay, \psi(x - y)) &= (Ax - Ay, J(x - y)^+) - (Ax - Ay, J(x - y)^-) \\ &= (Ax - Ay, J(x - y)^+) + (Ay - Ax, J(y - x)^+) \\ &\geq 0 \quad \text{since } A \text{ is } T\text{-accretive.} \end{aligned}$$

We note A is *T-accretive* iff $(Ax - Ay, J_r(x - y)^+) \geq 0$ where J_r is a generalized duality map. Hence, if $J_r(z) = J_r(z^+) - J_r(z^-)$ for z in X , then *T-accretive* maps are *accretive*. This occurs in L^p spaces, $1 < p < \infty$. Yamamuro [23] gives a result on generalized duality maps of this type.

PROPOSITION 1.5. (a) *If U is a (locally) T -Lipschitz function, then $-U$ is (locally) generalized T -accretive and also (locally) Lipschitzian.*

(b) *A T -accretive function is generalized T -accretive and a generalized T -accretive function is locally generalized T -accretive.*

(c) *A T -nonexpansive function is T -Lipschitz and a T -Lipschitz function is locally T -Lipschitz. A T -Lipschitz function is monotonic.*

(d) *Suppose $D(U) \subset X$ is convex. U is locally T -Lipschitz iff there is a continuous function $k: D(U) \times D(U) \rightarrow R$ with*

$$\|(Ux - Uy)^+\| \leq k(x, y) \|(x - y)^+\| .$$

In particular, U is monotonic.

(e) *Suppose $D(A) \subset X$ is convex, and $J(rx) = rJ(x)$ for r in R and x in X . $A: D(A) \rightarrow X$ is locally generalized T -accretive iff there is a continuous function*

$$k: D(A) \times D(A) \rightarrow R \text{ with } (Ax - Ay, J(x - y)^+) \geq -k(x, y) \|(x - y)^+\|^2.$$

Proof. (a), (b), and (c) are straightforward; we prove (d), and the proof of (e) is similar.

We suppose U is locally T -Lipschitz, and x and y are in $D(U)$. We assume $z \rightarrow k_z$ is continuous, using a partition of unity. The line segment joining x and y is compact and contained in $D(U)$. Hence, we divide the line segment:

$$x = x_0 \cdots y = x_n \text{ with } \|(Ux_{i-1} - Ux_i)^+\| \leq k_i \|(x_{i-1} - x_i)^+\|$$

with k_i corresponding to a set containing x_i and x_{i-1} .

Put $k(x, y) = \sup \{k_i : 1 \leq i \leq n\}$.

Now for a and b in a Banach lattice, $a^+ + b^+ \geq (a + b)^+$. Hence, $(\sum a_i)^+ \leq \sum (a_i^+)$ for a finite sum, giving $\|(\sum a_i)^+\| \leq \|\sum (a_i^+)\|$. Hence,

$$\begin{aligned} \|(Ux - Uy)^+\| &= \|(\sum Ux_{i-1} - Ux_i)^+\| \\ &\leq \|\sum (Ux_{i-1} - Ux_i)^+\| \\ &\leq \sum \|(Ux_{i-1} - Ux_i)^+\| \\ &\leq \sum k(x, y) \|(x_{i-1} - x_i)^+\| \\ &= k(x, y) \|(x - y)^+\| . \end{aligned}$$

Then by a partition of unity, we may assume k is continuous. The reverse implication is immediate. U is monotonic for suppose $x \leq y$, then $\|(x - y)^+\| = 0$. Hence, $\|(Ux - Uy)^+\| = 0$, giving $Ux \leq Uy$.

PROPOSITION 1.6. *Suppose X a Banach lattice with positive duality map. If $U: D(U) \rightarrow X$ is T -nonexpansive, then $I - U$ is T -accretive.*

Proof. Let x and y be in $D(U)$. Since $(Ux - Uy)^+ \geq Ux - Uy$,

$$\begin{aligned} & ((x - Ux) - (y - Uy), J(x - y)^+) \\ &= \|(x - y)^+\|^2 - (Ux - Uy, J(x - y)^+) \\ &\geq \|(x - y)^+\|^2 - ((Ux - Uy)^+, J(x - y)^+) \\ &\geq \|(x - y)^+\|^2 - \|(Ux - Uy)^+\| \|(x - y)^+\| \\ &\geq 0. \end{aligned}$$

PROPOSITION 1.7. *Suppose X a Banach lattice with positive duality map J . If $A: D(A) \rightarrow X$ is T -accretive, then for all $d > 0$, $(I + dA)^{-1}: R(I + dA) \rightarrow X$ is T -nonexpansive. If J is demicontinuous from the strong to the weak* topology, then the converse is true.*

Proof. Suppose w and z in $D(A)$, A is T -accretive, and $d > 0$ is given, and $(I + dA)w = x$ and $(I + dA)z = y$. Since A is T -accretive, we have

$$\begin{aligned} \|(w - z)^+\|^2 &\leq (x - y, J(w - z)^+) \\ &\leq ((x - y)^+, J(w - z)^+) \\ &\leq \|(x - y)^+\| \|(w - z)^+\|. \end{aligned}$$

Hence, $\|(w - z)^+\| \leq \|(x - y)^+\|$ and $(I + dA)^{-1}$ is T -nonexpansive.

Conversely, suppose for $d > 0$, $(I + dA)w = x_d$ and $(I + dA)z = y_d$. Then

$$\|(w - z)^+\| \leq \|(x_d - y_d)^+\| \text{ for } d > 0.$$

Hence

$$\begin{aligned} (J(x_d - y_d)^+, w - z) &\leq (J(x_d - y_d)^+, (w - z)^+) \\ &\leq \|J(x_d - y_d)^+\| \|(w - z)^+\| \\ &\leq \|(x_d - y_d)^+\|^2 \\ &= (J(x_d - y_d)^+, x_d - y_d). \end{aligned}$$

Therefore,

$$(J(x_d - y_d)^+, (w - x_d) - (z - y_d)) \leq 0.$$

But $w - x_d = -dAw$ and $z - y_d = -dAz$. Therefore,

$$(J(x_d - y_d)^+, Aw - Az) \geq 0.$$

Now $x_d - y_d \rightarrow w - z$, hence, $(x_d - y_d)^+ \rightarrow (w - z)^+$, hence,

$$(J(w - z)^+, Aw - Az) \geq 0,$$

and A is T -accretive.

We see that A is hypermaximal T -accretive if and only if there exists $d > 0$ with $R(I + dA) = X$ if and only if for all $d > 0$,

$$R(I + dA) = X .$$

PROPOSITION 1.8. *Suppose X a Banach lattice with $A: D(A) \rightarrow X$ hypermaximal T -accretive. Then $R(A + dI) = X$ for $d > 0$.*

Proof. Given y in X , $d > 0$, we want u with $(A + dI)u = y$. Suppose $R(A + d_0I) = X$, $d_0 > 0$. We shall solve

$$(A + dI)(A + d_0I)^{-1}x = y ,$$

and put $u = (A + d_0I)^{-1}x$. We want x with

$$x + d_0^{-1}(d - d_0)(d_0^{-1}A + I)^{-1}x = y .$$

Now $(d_0^{-1}A + I)^{-1}$ is T -nonexpansive, hence, has Lipschitz constant ≤ 2 . Hence, if $|d_0^{-1}(d - d_0)| < 1/2$, we have a solution x by the contraction mapping principle. Repeating this process a finite number of times, we have $A + dI$ is surjective for any $d > 0$.

PROPOSITION 1.9. *Suppose X a Banach lattice with positive duality map, and $A: D(A) \rightarrow X$ is hypermaximal T -accretive. Then for $d > 0$, $A(I + dA)^{-1}$ is T -accretive and Lipschitzian from X to X .*

Proof. $A(I + dA)^{-1} = d^{-1}(I - (I + dA)^{-1})$. By proposition 7, $(I + dA)^{-1}$ is T -nonexpansive. By Proposition 6, $I - (I + dA)^{-1}$ is T -accretive. Hence, $A(I + dA)^{-1}$ is T -accretive. Also, $(I + dA)^{-1}$ is Lipschitzian, hence, $I - (I + dA)^{-1}$ is also, consequently, $A(I + dA)^{-1}$ is Lipschitzian with Lipschitz constant $3/d$.

PROPOSITION 1.10. *Suppose X a Banach lattice with positive duality map, and $A: D(A) \rightarrow X$. Suppose A is*

- (1) T -accretive;

or

- (2) generalized T -accretive;

or

- (3) locally generalized T -accretive.

For v in $D(A)$ and t in R^+ we put $U(t)v = u(t)$, if there is a unique continuous weakly differentiable function $u: [0, t] \rightarrow X$, $u(0) = v$, $du(s)/ds = -Au(s)$ for s in $(0, t)$ and $s \rightarrow \|u(s)\|$ is absolutely continuous. We say $-A$ generates the semigroup $U(t)$. We have $U(t)$ is

- (1) T -nonexpansive;

or

- (2) T -Lipschitz;

or

- (3) locally T -Lipschitz.

Proof. Suppose $0 \leq s \leq t \leq d$, and x and y are in $D(U(d))$. By Proposition 1.2,

$$\begin{aligned} & \| (x(s) - y(s))^+ \|^2 - \| (x(t) - y(t))^+ \|^2 \\ & \geq 2(J(x(t) - y(t))^+, (x(s) - x(t)) - (y(s) - y(t))) . \end{aligned}$$

Dividing by $t - s$ and letting $s \rightarrow t^-$, we have, by the absolute continuity of $\| (x(s) - y(s))^+ \|^2$, for t in $[0, d] - N$, where N is a set of measure zero,

$$\frac{d}{dt} \| (x(t) - y(t))^+ \|^2 \leq -2(J(x(t) - y(t))^+, Ax(t) - Ay(t)) .$$

In case (1), $d \| (x(t) - y(t))^+ \|^2/dt \leq 0$ for t in $[0, d] - N$. Hence,

$$\| (x(d) - y(d))^+ \|^2 \leq \| (x - y)^+ \|^2 ,$$

and $U(d)$ is T -nonexpansive.

In case (2), $d \| (x(t) - y(t))^+ \|^2/dt \leq 2k \| (x(t) - y(t))^+ \|^2$. Hence,

$$\frac{d}{dt} (e^{-2kt} \| (x(t) - y(t))^+ \|^2) \leq 0 .$$

Hence, $\| (U(d)x - U(d)y)^+ \leq \| e^{kd} \| (x - y)^+ \|$, and $U(d)$ is T -Lipschitz. In case (3), we assume d small and x close to y , so that $x(t)$ and $y(t)$ are in a neighborhood N_z where k_z is constant. This puts us back in case (2). Composing with $U(d)$ we obtain the result for general d since $[0, d]$ is compact.

We have a converse of the proposition above.

PROPOSITION 1.11. *Suppose X a Banach lattice with positive duality map, and $A: D(A) \rightarrow X$; is such that $-A$ generates the semigroup $U(t)$ which is*

(1) T -nonexpansive;

or

(2) T -Lipschitz;

or

(3) locally T -Lipschitz.

Then A is

(1) T -accretive;

or

(2) generalized T -accretive;

or

(3) locally generalized T -accretive.

Proof. Given x and y in $D(A)$,

$$\begin{aligned} \|(x(t) - y(t))^+\|^2 &\leq \|(x(0) - y(0))^+\|^2 \\ &\quad + 2(J(x(0) - y(0))^+, (x(t) - y(t)) - (x(0) - y(0))) . \end{aligned}$$

In case (1), $\|(x(t) - y(t))^+\|^2 \leq \|(x(0) - y(0))^+\|^2$. Hence,

$$0 \geq 2(J(x(0) - y(0))^+, (x(t) - x(0)) - (y(t) - y(0))) :$$

dividing by t and letting $t \rightarrow 0^+$ we have A T -accretive.

Cases (2) and (3) are similar.

PROPOSITION 1.12. *Suppose X a Banach lattice with positive duality map. Suppose $A: D(A) \rightarrow X$ generates a semigroup $U(t)$, and A is*

(1) T -accretive;

or

(2) generalized T -accretive;

or

(3) locally generalized T -accretive.

Then

(1) $\|(Au(t))^+\| \leq \|(Au(0))^+\|$;

or

(2) $\|(Au(t))^+\| \leq e^{kt} \|(Au(0))^+\|$ where $(Ax - Ay, J(x - y)^+) \geq -k \|(x - y)^+\|^2$;

or

(3) $\|(Au(t))^+\| \leq e^{K(t)} \|(Au(0))^+\|$, where $(Ax - Ay, J(x - y)^+) \geq -k(y) \|(x - y)^+\|^2$ for x near y and $K(t) = \int_0^t k(u(s)) ds$.

Proof. We prove case (3). Suppose A is locally generalized T -accretive, and u is a continuous function $[0, d] \rightarrow X$ with $du(t)/dt = -Au(t)$. Then

$$\|(u(t + h) - u(t))^- \| \leq \|(u(h) - u(0))^- \|e^{K(t)} .$$

by Proposition 1.10, since $v(t) = u(t + h)$ is a solution to

$$\frac{dv}{dt}(t) = -Av(t), \text{ with } v(0) = u(h) .$$

Dividing by h and letting $h \rightarrow 0^+$, we have

$$\|(Au(t))^+\| \leq \|(Au(0))^+\|e^{K(t)} .$$

Case (1) and (2) are similar, indeed particular cases.

COROLLARY. *Suppose X and A are as in (3) above, with*

$du(t)/dt = -Au(t)$, and $Au(0) \leq 0$. Then $Au(t) \leq 0$ as long as $u(t)$ is defined, and $u(t) \leq u(s)$ if $t \leq s$.

Proof. By (3) of Proposition 1.12, $\|(Au(t)^+\| \leq 0$. Given $t \leq s$,

$$\begin{aligned} u(s) - u(t) &= \int_t^s \frac{du}{dr}(r) dr \\ &= \int_t^s -Au(r) dr \\ &\geq 0. \end{aligned}$$

We recall a function $T: X \rightarrow 2^X$ is g -accretive if there is a map $\phi: X \times X \times X \rightarrow X^*$ with $\phi(u, v, x)$ in $J(u - v)$ for x, u, v in X , such that for y in $T(u)$ and w in $T(v)$, $(\phi(u, v, y - w), y - w) \geq 0$.

PROPOSITION 1.13. *Suppose X a Banach lattice with positive duality map. Then X has an equivalent norm in which T -nonexpansive functions are nonexpansive and T -accretive functions are g -accretive.*

Proof. Set $\|x\|_1 = \|x^+\| + \|x^-\|$. $\|\cdot\|_1$ is an equivalent norm. Suppose $U: D(U) \rightarrow X$ is T -nonexpansive and x and y are in $D(U)$.

$$\begin{aligned} \|Ux - Uy\|_1 &= \|(Ux - Uy)^+\| + \|(Ux - Uy)^-\| \\ &\leq \|(x - y)^+\| + \|(x - y)^-\| \\ &= \|x - y\|_1. \end{aligned}$$

Suppose $A: D(A) \rightarrow X$ is T -accretive. Then for $d > 0$, $(I + dA)^{-1}$ is nonexpansive, hence, nonexpansive in $(X, \|\cdot\|_1)$. Hence, by Theorem 9.1 of Browder [5], A is g -accretive in $(X, \|\cdot\|_1)$.

COROLLARY. *Suppose G is a closed subset of a Banach lattice and $U: G \rightarrow G$ satisfies $\|(Ux - Uy)^+\| \leq a\|(x - y)^+\|$ with $a < 1$, for x and y in G . Then U has a unique fixed point.*

Proof. $\|Ux - Uy\|_1 \leq a\|x - y\|_1$ for x and y in G . The result follows from the contraction mapping principle.

We note that $(X, \|\cdot\|_1, \leq)$ is not a Banach lattice. We have only that $|x| \leq |y|$ implies $\|x\|_1 \leq 2\|y\|_1$. In general, the equivalent norm $\|\cdot\|_1$ does not preserve properties like normal structure, strict convexity, or uniform convexity of X or X^* .

We recall that a Banach space X has property P of Bohnenblust [3] if $a, b, c, d \geq 0$, $a \perp b$, $c \perp d$, $\|a\| = \|c\|$ and $\|b\| = \|d\|$ implies

$$\|a + b\| = \|c + d\|.$$

PROPOSITION 1.14. *Every T -nonexpansive function $U: D(U) \rightarrow X$ is nonexpansive if and only if X has property P .*

Proof. Suppose X has property P and y and x are in $D(U)$. $\|(Ux - Uy)^+\| \leq \|(x - y)^+\|$, hence, $\|(Ux - Uy)^+\| = a \|(x - y)^+\|$ for some a in $[0, 1]$. Similarly, there is b in $[0, 1]$ with $\|(Ux - Uy)^-\| = b \|(x - y)^-\|$. Now $a(x - y)^+ \perp b(x - y)^-$ and $(Ux - Uy)^+ \perp (Ux - Uy)^-$. Hence,

$$\begin{aligned} \|Ux - Uy\| &= \|(Ux - Uy)^+ + (Ux - Uy)^-\| \\ &= \|a(x - y)^+ + b(x - y)^-\| \\ &\leq \|(x - y)^+ + (x - y)^-\| \\ &= \|x - y\|. \end{aligned}$$

Conversely, suppose X does not have property P . Then there are $a, b, c, d \geq 0$ in X with $\|c\| = \|d\|, \|a\| = \|b\|, a \perp c, b \perp d$, and $\|a + c\| < \|b + d\|$. Put $U(0) = 0, U(a - c) = b - d$. Then a calculation shows $U: \{0, a - c\} \rightarrow X$ is T -nonexpansive but not nonexpansive.

The following proposition shows that in the linear case (see Phillips [18] and Sato [19]) semigroups of nonexpansive positive operators are T -nonexpansive. The converse is true too, which could be extended as in Proposition 1.16 on the differentiable case, for example to show a hypermaximal T -accretive operator A is g -accretive, if the resolvents $(I + eA)^{-1}$ are C^1 .

PROPOSITION 1.15. *Suppose E is a lattice subspace of X , and $U: E \rightarrow X$ is a bounded positive linear operator. Then U is T -Lipschitz.*

Proof. Suppose x is in E . $Ux = (Ux^+) + (Ux^-)$ since U is linear. $U(x^+)$ and $U(x^-)$ are ≥ 0 since U is positive. Hence, $U(x^+) \geq U(x)$. Hence, $U(x^+) \geq (Ux)^+$. Hence,

$$\begin{aligned} \|(Ux^+)\| &\leq \|U(x^+)\| \\ &\leq \|U\| \|x^+\|. \end{aligned}$$

We found a locally T -Lipschitz function U (with convex domain) is monotonic. The following proposition shows these properties are equivalent when U is C^1 .

PROPOSITION 1.16. *Suppose $U: G \rightarrow X, G$ an open subset of the Banach lattice X , is C^1 and monotonic. Then U is locally T -Lipschitz.*

Proof. The derivative at x in G , U'_x , is positive. For suppose $h \geq 0$ in X , then for $t \geq 0$, $U(x + th) - U(x) \geq 0$. Therefore,

$$U'_x(h) = \lim t^{-1} (U(x + th) - U(x)) \geq 0 .$$

Given y in G , take B a ball around y in G , and M a positive constant, with $\|U'_x\| \leq M$ for x in B . This may be done since U is C^1 . Then for x in B

$$Ux - Uy = \int_0^1 U'_{tx+(1-t)y}(x - y)dt .$$

Now from the inequality for a finite sum in a Banach lattice

$$(\Sigma a_n)^+ \leq \Sigma (a_n^+) ,$$

we have for a curve $a: [c, d] \rightarrow X$, that

$$\left(\int_c^d a(t)dt \right)^+ \leq \int_c^d a(t)^+ dt .$$

Hence,

$$(Ux - Uy)^+ \leq \int_0^1 (U'_{tx+(1-t)y}(x - y))^+ dt .$$

Hence, by Proposition 1.15,

$$\begin{aligned} \|(Ux - Uy)^+\| &\leq \int_0^1 \|(U'_{tx+(1-t)y}(x - y))^+\| dt \\ &\leq \int_0^1 \|U'_{tx+(1-t)y}\| \|(x - y)^+\| dt \\ &= M \|(x - y)^+\| . \end{aligned}$$

The question arises of the structure of the fixed point set $F(U)$ of a T -nonexpansive function $U: D(U) \rightarrow X$. If X has property P and is strictly convex, then $F(U)$ is convex. The following generalizes a linear theorem of Birkhoff [2], page 391.

PROPOSITION 1.17. *Suppose U is T -nonexpansive $X \rightarrow X$, an AL space. Then $F(U)$ is a sublattice.*

Proof. Suppose $Ux = x$, $Uy = y$, and $z = \sup(x, y)$. U is monotonic, hence, $Uz \geq z$. Suppose $Uz = z + h$, with $h \geq 0$.

$$\begin{aligned} \|Uz - z\| + \|Uz - y\| &\leq \|z - x\| + \|z - y\| \\ &= \|x - y\| . \end{aligned}$$

Hence,

$$\|2h + x - y\| \leq \|x - y\| .$$

Therefore, $h = 0$ and $Uz = z$. Similarly, $\inf(x, y)$ is in $F(U)$.

COROLLARY. *Suppose $A: D(A) \rightarrow X$ is hypermaximal T -accretive, where X is an AL space with positive duality map. Then for x in X , $A^{-1}(x)$ is a sublattice.*

Proof. $A^{-1}(x) = B^{-1}(0)$ where $B: D(A) \rightarrow X$ is defined by $By = Ay - x$. Now $B(y) = 0$ if and only if $(I + B)^{-1}y = y$. But $(I + B)^{-1}$ is T -nonexpansive. Hence, $B^{-1}(0) = F((I + B)^{-1})$ is a sublattice.

2. Existence of solutions to equations of evolution.

THEOREM 2.1. *Suppose X a Banach lattice with X^* uniformly convex. Suppose A is demicontinuous and locally generalized T -accretive from a neighborhood N of v to X .*

Then there is an interval $[0, d]$ and a unique strongly continuous weakly C^1 function

$$u: [0, d] \rightarrow X \text{ with } u(0) = v \text{ and } \frac{du}{dt}(t) = -Au(t).$$

Proof. We have, since A is demicontinuous, a neighborhood N_0 of v and a constant M with $\|Ax\| \leq M$ if x is in N_0 . Since A is locally generalized T -accretive, there is a neighborhood N_1 of v and a constant k with

$$(Ax - Ay, J(x - y)^+) \geq -k \|(x - y)^+\|^2$$

for x and y in N_1 .

We may assume $N = B_R(v) \subset N_0 \cap N_1$. For $e > 0$, we solve:

$$(a) \quad \begin{aligned} \frac{du_e}{dt}(t) &= -Au_e(t - e) & t \geq 0 \\ u_e(t) &= v & t \leq 0. \end{aligned}$$

There is a $d > 0$ such that all solutions of (a) are in N for t in $[0, d]$ independently of e .

Given $e, f > 0$, for t in $[0, d]$, we set

$$q_{e,f}(t) = \|(u_e(t) - u_f(t))^+\|^2.$$

For $0 \leq s \leq t \leq d$, we have

$$q_{e,f}(s) \geq q_{e,f}(t) + 2(J(u_e(t) - u_f(t))^+, (u_e(s) - u_e(t)) - (u_f(s) - u_f(t))).$$

Dividing by $t - s$ and letting $s \rightarrow t^-$,

$$\lim_{s \rightarrow t^-} \frac{q_{e,f}(t) - q_{e,f}(s)}{t - s} \leq 2(J(u_e(t) - u_f(t))^+, \frac{du_e}{dt}(t) - \frac{du_f}{dt}(t)).$$

By the boundedness of du_e/dt , we have $q_{e,f}$ is absolutely continuous. Hence, there is a set N of measure 0 with

$$\frac{d}{dt} q_{e,f}(t) = \lim_{s \rightarrow t^-} \frac{q_{e,f}(t) - q_{e,f}(s)}{t - s}$$

existing for t in $[0, d] - N$. For such t ,

$$\begin{aligned} \frac{d}{dt} q_{e,f}(t) &\leq 2(J(u_e(t - e) - u_f(t - f))^+ - J(u_e(t) - u_f(t))^+), \\ Au_e(t - e) - Au_f(t - f) &- 2(J(u_e(t - e) - u_f(t - f))^+, Au_e(t - e) \\ &- Au_f(t - f)). \end{aligned}$$

For t in $[0, d]$,

$$\|(u_e(t - e) - u_f(t - f))^+\| \leq 2R$$

and

$$\|(u_e(t) - u_f(t))^+\| \leq 2R.$$

Since X^* is uniformly convex, J is uniformly continuous on $B_{2R}(0)$, hence, there is a function $r: R^+ \rightarrow R^+$ with $r(k) \rightarrow 0$ as $k \rightarrow 0$ such that if $\|x\| \leq 2R$, $\|y\| \leq 2R$, then $\|Jx - Jy\| \leq r(\|x - y\|)$. Now $\|u_e(t) - u_e(t - e)\| \leq eM$ and $\|u_d(t) - u_d(t - d)\| \leq dM$ for t in $[0, d]$. Therefore,

$$\|(u_e(t - e) - u_f(t - f)) - (u_e(t) - u_f(t))\| \leq (e + f)M.$$

Now if a, b are in a Banach lattice,

$$|a^+ - b^+| \leq |a - b|, \text{ hence, } \|a^+ - b^+\| \leq \|a - b\|.$$

Therefore,

$$\|J(u_e(t - e) - u_f(t - f))^+ - J(u_e(t) - u_f(t))^+\| \leq r((e + f)M).$$

Therefore for t in $[0, d] - N$,

$$\frac{d}{dt} q_{e,f}(t) \leq 4Mr((e + f)M) + 2k \|(u_e(t - e) - u_f(t - f))^+\|^2.$$

Now

$$\|(u_e(t - e) - u_f(t - f))^+\| \leq (e + f)M + \|(u_e(t) - u_f(t))^+\|.$$

Hence,

$$\|(u_e(t - e) - u_f(t - f))^+\|^2 \leq 2(e + f)^2 M^2 + 2q_{e,f}(t).$$

Hence,

$$\frac{d}{dt} q_{e,f}(t) - 4kq_{e,f}(t) \leq 2(e + f)^2 M^2 + 4Mr((e + f)M).$$

Given $g > 0$ there is $h > 0$ such that for $e, f \leq h$ the $RHS \leq g$. For $e, f \leq h$, t in $[0, d] - N$,

$$\begin{aligned} \frac{d}{dt} (e^{-4kt} q_{e,f}(t)) &= e^{-4kt} \left(\frac{d}{dt} q_{e,f}(t) - 4kq_{e,f}(t) \right) \\ &\leq g . \end{aligned}$$

Now $q_{e,f}(0) = 0$, hence, $e^{-4kt} q_{e,t}(t) \leq gd$ for t in $[0, d]$. Hence $q_{e,f}(t) \leq gde^{4kd}$ for t in $[0, d]$. Hence, $(u_e(t) - u_f(t))^+$ converges to zero uniformly on $[0, d]$ as $e, f \rightarrow 0$. Therefore,

$$(u_e(t) - u_f(t))^- = (u_f(t) - u_e(t))^+$$

converges to zero uniformly on $[0, d]$. Therefore, $u_e(t) - u_f(t)$ converges to zero uniformly on $[0, d]$. Since (u_e) is a Cauchy net of continuous functions $[0, d] \rightarrow B_R(v)$, it converges to a continuous function $u: [0, d] \rightarrow B_R(v)$. Hence, $u_e(t - e)$ also converges to $u(t)$ uniformly on $[0, d]$, and since A is demicontinuous, $Au_e(t - e)$ converges weakly to $Au(t)$.

Suppose f is in X^* , then for t in $[0, d]$

$$(u_e(t), f) = (v, f) - \int_0^t (Au_e(s - e), f) ds .$$

Now $|Au_e(s - e), f| \leq M\|f\|$, hence, the integral converges to $\int_0^t (Au(s), f) ds$. $(u_e(t), f)$ converges to $(u(t), f)$. Hence,

$$(u(t), f) = (v, f) - \int_0^t (Au(s), f) ds .$$

Hence, u is weakly differentiable with derivative $du(t)/dt = -Au(t)$, and since A is demicontinuous, u is weakly C^1 . Uniqueness follows from Proposition 1.10.

THEOREM 2.2. *Suppose X a Banach lattice with X^* uniformly convex. Suppose $A_0: X \rightarrow X$ is locally generalized T -accretive and demicontinuous. Suppose $A: D(A) \rightarrow X$ is hypermaximal T -accretive. Suppose v is in $D(A)$. Then there is an interval $[0, h]$ and a unique strongly continuous weakly C^1 function $u: [0, h] \rightarrow X$ with $u(0) = v$ and $du(t)/dt = -A_1u(t)$, where $A_1 = A_0 + A$.*

Proof. We have, for $d > 0$,

$$A(I + dA)^{-1} = d^{-1}(I - (I + dA)^{-1})$$

is T -accretive and Lipschitzian, with domain $R(I + dA) = X$. Hence, $A_{1,d} = A_0 + A(I + dA)^{-1}$ is demicontinuous and locally generalized T -accretive: $X \rightarrow X$.

By Theorem 2.1, there is a unique solution u_d of

$$(a) \quad \frac{d}{dt} u_d(t) = -A_{1,d}u_d$$

$$u_d(0) = v$$

on some interval $[0, h]$.

By Proposition 1.12, we have a bound on the derivative of $u_d(t)$ depending only on the locally generalized T -accretive part of $A_{1,d}$ and on t . Hence, we have solutions of (a) for an interval $[0, h]$ independent of d , with $\|du_d(t)/dt\| \leq M$ for t in $[0, h]$ for some constant M . Taking h small, we may assume there are positive constants k, M_0, R , with all solutions $u_d(t)$ in $B_R(v)$ for t in $[0, h]$, with $\|A_0x\| \leq M_0$ and

$$(A_0x - A_0y, J(x - y)^+) \geq -k \|(x - y)^+\|^2$$

for x and y in $B_R(v)$.

Let

$$v_d(t) = (I + dA)^{-1}u_d(t) .$$

Then

$$-Av_d(t) = \frac{d}{dt} u_d(t) + A_0u_d(t) .$$

Hence,

$$\|Av_d(t)\| \leq M + M_0$$

for t in $[0, h]$. Now

$$u_d(t) - v_d(t) = dAv_d(t) ,$$

giving

$$\|u_d(t) - v_d(t)\| \leq d(M + M_0) .$$

Now

$$\|(u_d(t) - u_e(t))\| \leq 2R ,$$

hence,

$$\begin{aligned} \|(v_d(t) - v_e(t))\| &\leq \|v_d(t) - u_d(t)\| + \|v_d(t) - u_e(t)\| + 2R \\ &\leq 2R + (e + d)(M + M_0) \\ &\leq 4R, \text{ if } e, d \leq (M + M_0)^{-1}R . \end{aligned}$$

Since X^* is uniformly convex, there is a function $r: R^+ \rightarrow R^+$ with $r(k) \rightarrow 0$ as $k \rightarrow 0$ such that for $\|y\|, \|x\| \leq 4R$, $\|Jx - Jy\| \leq r(k)$ if $\|x - y\| \leq k$. Given $d, e \leq (M + M_0)^{-1}R$,

$$\|v_d(t) - v_e(t) - u_d(t) + u_e(t)\| \leq (e + d)(M + M_0) .$$

Hence,

$$\|(v_d(t) - v_e(t))^+ - (u_d(t) + u_e(t))^+\| \leq (e + d)(M + M_0) .$$

Hence,

$$\|J(v_d(t) - v_e(t))^+ - J(u_d(t) - u_e(t))^+\| \leq r((e + d)(M + M_0)) .$$

We set $q_{d,e}(t) = \|(u_d(t) - u_e(t))^+\|^2$. Now $q_{d,e}$ is absolutely continuous

on $[0, h]$, hence, there is a set N with measure 0 such that for t in $[0, h] - N$, $dq_{d,e}(t)/dt$ exists, and as in Theorem 2.1, for such t we have

$$\begin{aligned} \frac{d}{dt}q_{d,e}(t) &\leq 2(J(u_d(t) - u_e(t))^+, \frac{du_d}{dt}(t) - \frac{du_e}{dt}(t)) \\ &= -2(J(u_d(t) - u_e(t))^+, A_0u_d(t) - A_0u_e(t)) \\ &\quad - 2(J(v_d(t) - v_e(t))^+, Av_d(t) - Av_e(t)) \\ &\quad + 2(J(v_d(t) - v_e(t))^+ - J(u_d(t) - u_e(t))^+, Av_d(t) - Av_e(t)) \\ &\leq 2k \|(u_d(t) - u_e(t))^+\|^2 \\ &\quad + 2 \|J(v_d(t) - v_e(t))^+ - J(u_d(t) - u_e(t))^+\| \cdot \|Av_d(t) - Av_e(t)\| \\ &\leq 4(M + M_0)r((e + d)(M + M_0)) + 2kq_{d,e}(t) . \end{aligned}$$

Hence, for t in $[0, h] - N$,

$$\begin{aligned} \frac{d}{dt}(e^{-2kt}q_{d,e}(t)) &= e^{-2kt}\left(\frac{d}{dt}q_{d,e}(t) - 2kq_{d,e}(t)\right) \\ &\leq e^{-2kt}4(M + M_0)r((e + d)(M + M_0)) . \end{aligned}$$

Given $g > 0$ there is an $f > 0$ with $f \leq (M + M_0)^{-1}R$ such that for $d, e \leq f$, the *RHS* $\leq g$.

Since $q_{d,e}(0) = 0$,

$$e^{-2kt}q_{d,e}(t) \leq gd$$

for t in $[0, h]$, hence,

$$q_{d,e}(t) \leq gde^{2kd} .$$

Hence, $(u_d(t) - u_e(t))^+$ converges to zero uniformly on $[0, h]$ as $d, e \rightarrow 0$. As in Theorem 2.1, u_d converges to a continuous u . Since

$$\|u_d(t) - v_d(t)\| \leq d(M + M_0) ,$$

$v_d(t)$ also converges to $u(t)$ uniformly on $[0, h]$.

We claim $u(t)$ lies in $D(A_i)$ for t in $[0, h]$ and $t \rightarrow A_1u(t)$ is weakly continuous.

Since $u_d(t) \rightarrow u(t)$ and A_0 is demicontinuous, $A_0u_d(t) \rightarrow A_0u(t)$. A is hypermaximal ψ -accretive, where ψ is the ψ system of Proposition 1.3. $v_d(t) \rightarrow u(t)$, and $Av_d(t)$ is bounded, hence, has a weak cluster point w .

Given v in $D(A)$,

$$(Av_d(t) - Av, \psi(v_d(t) - v)) \geq 0 .$$

Hence,

$$(w - Av, \psi(u(t) - v)) \geq 0 .$$

Since A is maximal ψ -accretive, $u(t)$ is in $D(A)$ and $Au(t) = w$. If $s \rightarrow t$, then $Au(s)$ has a weak cluster point w_0 , and $u(s) \rightarrow u(t)$.

Given v in $D(A)$,

$$(Au(s) - Av, \psi(u(s) - v)) \geq 0 .$$

Hence,

$$(w_0 - Av, \psi(u(t) - v)) \geq 0 ,$$

hence, $w_0 = Au(t)$, hence, $Au(s) \rightarrow Au(t)$. Hence, $u(t)$ is in $D(A_1)$ and $A_1u(s) \rightarrow A_1u(t)$ as $s \rightarrow t$.

As in Theorem 2.1, we have for t in $[0, d]$, f in X^* ,

$$(f, u(t)) = (f, v) - \int_0^t (f, Au(s))ds .$$

Hence, u is continuous, weakly C^1 , and unique by Proposition 1.10.

THEOREM 2.3. *Suppose X a Banach lattice with positive duality map. Let N be a neighborhood of v in X . Let $A: N \rightarrow X$ be locally uniformly continuous and locally generalized T -accretive. Then there is a unique strongly C^1 function $u: [0, h] \rightarrow N$ for some $h > 0$ with $u(0) = v$ and $du(t)/dt = -Au(t)$.*

Proof. For some neighborhood M of v and positive constant k , $A + kI$ is T -accretive $M \rightarrow X$. By Proposition 1.13, $A + kI$ is g -accretive in $(x, \|\cdot\|_1)$. Hence, for x and y in M ,

$$(Ax - Ay, \phi(x, y, Ax - Ay)) \geq -k \|x - y\|_1^2 .$$

The result follows from Theorem 9.7 of Browder [5].

We note that we could extend Theorem 2.2 to include a second hypermaximal T -accretive operator A_2 such that $\|A_2x\| \leq k \|Ax\|$ with $k < 1$. We could also have considered the case of the generator A being multivalued, and considered the temporally inhomogeneous problem, $du(t)/dt = -T_t u(t)$.

THEOREM 2.4. *Suppose X a Banach lattice with X^* uniformly convex. Suppose $A_0: X \rightarrow X$ is demicontinuous and T -accretive, and $A_1: D(A_1) \rightarrow X$ is hypermaximal T -accretive. Then $A_0 + A_1$ is hypermaximal T -accretive.*

Proof. Given w in X , we have $x \rightarrow A_0x + x - w$ is demicontinuous and T -accretive: $X \rightarrow X$, hence, by Theorem 2.2, $A_0 + I - w + A_1$ generates a semigroup $U(t)$.

Given x and y in $D(A_1)$, for t in $[0, h] - N$ as in Theorem 2.1, we have

$$\begin{aligned} \frac{d}{dt} \|(x(t) - y(t))^+\|^2 &\leq 2(J(x(t) - y(t))^+, \frac{dx}{dt}(t) - \frac{dy}{dt}(t)) \\ &\leq -2 \|(x(t) - y(t))^+\|^2 . \end{aligned}$$

Putting $x(t) = y(t + h)$ and letting $h \rightarrow 0^+$, we obtain

$$\left\| \left(\frac{dy}{dt}(t) \right)^+ \right\| \leq e^{-t} \left\| \left(\frac{dy}{dt}(0) \right)^+ \right\|,$$

and similarly,

$$\left\| \left(\frac{dy}{dt}(t) \right)^- \right\| \leq e^{-t} \left\| \left(\frac{dy}{dt}(0) \right)^- \right\|.$$

Hence, if there is a solution $y(t)$ to $dy(t)/dt = -(A_0 + I + A_1 - w)y(t)$ on an interval $[0, k)$, then by the bound on the derivative $y(t)$ converges as $t \rightarrow k^-$, hence, there is a solution on an interval $[0, k+h]$ for some $h > 0$, hence, on all of R^+ .

As $t \rightarrow \infty$, $y(t)$ is Cauchy, hence, $y(t)$ converges to z for some z in X . Since $A_1 y(t)$ is bounded and A_1 is hypermaximal T -accretive, z is in $D(A_1)$ and $A_1 y(t)$ converges weakly to Az . Hence, z is in $D(U(t))$ for $t \geq 0$.

Now for $t \geq 0$, $U(t)$ is continuous, hence,

$$\begin{aligned} U(t)z &= U(t) \lim_{s \rightarrow \infty} U(s)y \\ &= \lim_{s \rightarrow \infty} U(t + s)y \\ &= z. \end{aligned}$$

Hence,

$$\begin{aligned} A_0 z + z + A_1 z - w &= \lim_{t \rightarrow \infty} \frac{U(t)z - z}{t} \\ &= 0. \end{aligned}$$

Hence, $R((A_0 + A_1) + I)$ contains w , hence,

$$R((A_0 + A_1) + I) = X.$$

COROLLARY. *Suppose X a Banach lattice with positive duality map. Suppose $A: X \rightarrow X$ is locally uniformly continuous and T -accretive. Then A is hypermaximal T -accretive.*

Proof. By Theorem 2.3, $-(A + 1)$ generates a semigroup, and the proof proceeds as in Theorem 2.4.

3. Surjectivity of T -accretive operators.

THEOREM 3.1. *Let X be a Banach lattice with X^* uniformly convex. Let $A_1: D(A_1) \rightarrow X$ be hypermaximal T -accretive. Let $A_2: X \rightarrow X$ be demicontinuous and locally generalized T -accretive. Let $A = A_1 + A_2$.*

Suppose either

$$\begin{aligned} \frac{d}{dt} \|(z(t) - w(t))^+\|^2 &\leq 2(J(z(t) - w(t))^+, \frac{dz}{dt}(t) - \frac{dw}{dt}(t)) \\ &\leq 2k \|(z(t) - w(t))^+\|^2 + 2(J(z(t) - w(t))^+, b - d) \end{aligned}$$

(where k is the local constant associated with A and h)

$$\leq 2k \|(z(t) - w(t))^+\|^2 \quad \text{since } b \leq d .$$

Hence, $z(t) \leq w(t)$ in the interval $[0, h]$, hence, for all t in R^+ . Therefore, $z \leq w$. Now we have solutions to

$$\frac{dr}{dt}(t) = -Ar(t) + c \quad r(0) = z$$

and

$$\frac{ds}{dt}(t) = -As(t) + e \quad s(0) = w .$$

$r(t)$ decreases to r with $Ar = c$ and $s(t)$ decreases to s with $As = e$. We have except on a set of measure 0,

$$\begin{aligned} \frac{d}{dt} \|(r(t) - s(t))^+\|^2 &\leq 2k \|(r(t) - s(t))^+\|^2 + 2(J(r(t) - s(t))^+, c - e) \\ &\leq 2k \|(r(t) - s(t))^+\|^2 \quad \text{since } c \leq e . \end{aligned}$$

Now $\|(r(0) - s(0))^+\| = 0$, giving $r(t) \leq s(t)$ for all t in R^+ ; hence, $r \leq s$.

The function taking c to r by this construction takes $e \geq c$ to $s \geq r$ and hence, is monotonic.

COROLLARY 2. *Under the conditions of Theorem 3.1, if we have x and y such that $Ax \leq x \leq y \leq Ay$, then there is a fixed point of A in $[x, y]$.*

Proof. Starting from x as in Corollary I, we have a monotonic selection B of A^{-1} . $BA(x) = x$, and we claim BAy is in $[x, y]$. $BAy = \lim_{t \rightarrow \infty} x(t)$, where $x(0) = x$ and $dx(t)/dt = -Ax(t) + Ay$. Hence, $x \leq BAy$.

Except on a set of measure 0,

$$\begin{aligned} \frac{d}{dt} \|(x(t) - y)^+\|^2 &\leq 2(J(x(t) - y)^+, \frac{dx}{dt}(t)) \\ &= -2(J(x(t) - y)^+, Ax(t) - Ay) \\ &\leq 2k(x(t)) \|(x(t) - y)^+\|^2 . \end{aligned}$$

Hence, $x(t) \leq y$ for all t in R^+ . Hence, $BAy \leq y$.

Then $[Ax, Ay]$ is invariant under B , and is a complete lattice since X is reflexive.

The Tarski fixed point theorem says: Let L be a complete lattice. Then a monotonic function $L \rightarrow L$ has a fixed point.

Hence, there is z in $[Ax, Ay]$ with $Bz = z$, hence, $Az = z$. Since $B[Ax, Ay] \subset [x, y]$ by construction, z is in $[x, y]$.

THEOREM 3.2. *Suppose X a reflexive Banach lattice with positive duality map. Suppose $A: D(A) \rightarrow X$ is hypermaximal T -accretive with A^{-1} locally bounded. Then*

- (a) $R(A) = X$;
- (b) A^{-1} is monotonic if it is single-valued and demicontinuous.

Proof. (a) We consider $B = A(I + A)^{-1}: X \rightarrow X$. B is T -accretive and Lipschitzian, and B^{-1} is locally bounded. We show $R(B) = X$.

Now Theorem 5.3 of Browder [5] gives: Suppose X a Banach space with a ψ system, A a hypermaximal ψ -accretive function $D(A) \rightarrow X$, and A^{-1} locally bounded, then $clR(A) = X$.

Since B is ψ -accretive with respect to the ψ system of Proposition 1.3, we have only to show $R(B)$ is closed. By translation we have only to show that if there is a sequence (x_n) in X with $Bx_n \rightarrow 0$, then 0 is in $R(B)$.

Now Bx_n converges to 0 relatively uniformly (see Yosida [24], page 370). In particular, there is a subsequence (x_m) and $z \geq 0$ with $|Bx_m| \leq m^{-1}z$.

Suppose $B^{-1}(B_{3r}(0))$ is bounded. If m large, then $\|m^{-1}z\| \leq r$. Put $D(x) = B(x) - m^{-1}z$ for x in X . Then $D(x_m) \leq 0$, hence, if $x(0) = x_m$ and $dx(t)/dt = -Dx(t)$, then $Dx(t) \leq 0$, and $x(t)$ is increasing. Also,

$$\|(Dx(t))\| \leq \|Dx_m\| \leq \|B(x_m)\| + \|m^{-1}z\| \leq 2r .$$

Hence,

$$\|Bx(t)\| \leq \|Dx(t)\| + \|m^{-1}z\| \leq 3r .$$

Hence, $x(t)$ is bounded, and by Dini's Theorem, as in Theorem 3.1, we have $x(t)$ defined on R^+ and $x(t)$ converging strongly to w . Again, as in Theorem 3.1, $U(t)w = w$ and, hence, $D(w) = 0$, giving $Bw = m^{-1}z$ and $w \geq x_m$.

Similarly, putting $C(x) = B(x) + m^{-1}z$, we obtain v with $Bv = -m^{-1}z$ and $v \leq x_m$. Then if

$$v(0) = v, \frac{dv}{dt}(t) = -Bv(t)$$

$$w(0) = w, \frac{dw}{dt}(t) = -Bw(t) ,$$

we have $v(t)$ increasing, $w(t)$ decreasing, and $v(t) \leq w(t)$. Hence, $v(t)$ is bounded, hence, converges strongly to x , with $B(x) = 0$.

(b) For $d > 0$, $(I + d^{-1}A)^{-1}$ is T -nonexpansive. Hence,

$$(A + dI)^{-1} = d^{-1}(I + d^{-1}A)^{-1}$$

is T -Lipschitz, hence, monotonic. Given y in X , let $(A + d_n I)x_n = y$ for a sequence $d_n \rightarrow 0^+$. By Lemma 3.3, x_n may be assumed bounded. Hence, $d_n x_n \rightarrow 0$, hence, $Ax_n \rightarrow y$, hence, $x_n \rightarrow A^{-1}y$. Suppose $w \leq y$, and $(A + d_n I)z_n = w$. Since $(A + d_n I)^{-1}$ is monotonic, $z_n \leq x_n$. Since the positive cone is weakly closed, and $z_n \rightarrow A^{-1}w$, $A^{-1}w \leq A^{-1}y$.

We recall a Banach lattice X with positive cone k is uniformly monotone if for $\epsilon > 0$, there is $d > 0$ such that if f and g are in K , $\|f\| = 1$ and $\|f + g\| \leq 1 + d$, then $\|g\| \leq \epsilon$. A theorem of Birkhoff [2], page 371, says that a bounded subset of a uniformly monotone Banach lattice directed under \leq is convergent. If X has this property it is fully regular (Krasnoselski [13]).

COROLLARY. *Theorem 3.2 holds if X is fully regular.*

LEMMA 3.3. *If $A: D(A) \rightarrow X$, $D(A) \subset X$, a Banach space with ψ system, is hypermaximal ψ -accretive and A^{-1} is locally bounded, then given w in $cl(R(A))$, there is a neighborhood N of w , a bounded set B , and $d_0 > 0$ with $(A + dI)^{-1}N \subset B$ for $d \leq d_0$.*

Proof. Suppose not, then there are sequences (z_n) , (d_n) with $d_n \rightarrow 0^+$, $\|z_n\| \rightarrow \infty$ and $(A + d_n I)z_n \rightarrow w$. Suppose

$$A^{-1}(B_{3r}(w)) \subset B_h(0) \subset B_k(0)$$

with $h < k$. Take s in $B_h(0)$ with $A(s)$ in $B_r(w)$. Take d_0 with $d_0 k \leq r$. If n large, z_n is outside $B_k(0)$, and $(A + d_n I)z_n$ is in $B_r(w)$ and $d_n \leq d_0$. Then $(A + d_n I)s$ is in $B_{2r}(w)$. But $(A + d_n I)^{-1}B_{2r}(w)$ is connected, hence, it contains s_1 with $\|s_1\| = k$. Then

$$A(s_1) = (A + d_n I)(s_1) - d_n s_1$$

is in $B_{3r}(w)$, contradicting $A^{-1}(B_{3r}(w)) \subset B_h(0)$.

THEOREM 3.4. *Suppose X an order complete lattice, and A an order bounded subset of X such that $B \subset A$ implies $\sup B$ and $\inf B$ are in A . Suppose $\{U_t: t \text{ in } T\}$ is a commuting directed set of monotonic functions, with $U_t: D(U_t) \rightarrow X$. Suppose that for a in A , $D(U_t)$ eventually contains a , in which case we may say a as in $D(U_t)$ for small t . Then there is c in A with $U_t c = c$ for small t .*

Proof. Let $M = \{x \text{ in } A: U_t x \geq x \text{ for small } t\}$. M is nonempty

since $\inf A$ is in M . Let $c = \sup M$. Then c is in A . Given x in M , for small t both x and c are in $D(U_t)$, hence, $U_t c \geq U_t x \geq x$. Hence, c is in M .

We claim $U_t c$ is in M for small t . Given s small in T , $U_s c \geq c$, hence, for small t , $U_t U_s c \geq U_t c$. Therefore, $U_s(U_t c) \geq U_t c$, giving $U_t c$ in M . Therefore, $U_t c \leq c$ for small t since $c = \sup M$. Therefore, $U_t c = c$ for small t .

This generalization of the Tarski fixed point theorem also holds for $U_t: A \rightarrow 2^A$, with U_t monotonic and $U_t(x)$ closed under sup for x in A . We say $U: X \rightarrow 2^X$ is monotonic if for $x \leq y$, we have:

(a) for w in $U(x)$, there is z in $U(y)$ with $w \leq z$;

and

(b) for z in $U(y)$, there is w in $U(x)$ with $w \leq z$.

PROPOSITION 3.5. *Suppose X is an order complete Banach lattice with positive duality map. Suppose $A: X \rightarrow X$ is locally uniformly continuous and locally generalized T -accretive. Then for $a \leq b$ in X : $[A(a), A(b)] \subset A[a, b]$.*

Proof. We assume $Aa \leq Ab$, since otherwise $[A(a), A(b)]$ is empty, and suppose y satisfies $A(a) \leq y \leq A(b)$. Consider $A_y(x) = Ax - y$ for x in X . A_y is locally generalized T -accretive and locally uniformly continuous, hence, $-A_y$ generates a semigroup $U(t)$. $A_y(a) \leq 0$, hence, $A_y U(t)a \leq 0$ and $U(t)a$ is increasing. $A_y(b) \geq 0$, hence, $A_y U(t)b \geq 0$ and $U(t)b$ is decreasing. By monotonicity of $U(t)$, if x is in $[a, b]$, then for small $t \geq 0$, we have

$$b \geq U(t)b \geq U(t)x \geq U(t)a \geq a.$$

Since $\{U(t)\}$ is a commuting family of monotonic functions, and $[a, b]$ is an order bounded set closed under inf and sup on subsets, and for small t any given x is in $D(U(t))$, and if x is in $[a, b]$, $U(t)x$ is in $[a, b]$ for t small, we have by Theorem 3.4 an element c of $[a, b]$ with $U(t)c = c$ for small t . Therefore, $A_y(c) = 0$, giving $A(c) = y$.

In §4 we will consider operators on the space $C(M)$ of continuous real valued functions on a compact T_2 space M . Such a space is order complete if and only if M is extremally disconnected, i.e., the closure of an open subset is open. A particular example of this case is the dual of an (AL) space, also studied in §4. If X is an order complete Banach lattice whose positive cone has nonempty interior, then it is isomorphic as a Banach lattice to $C(M)$ where M is extremally disconnected.

THEOREM 3.6. *Suppose X is an order complete Banach lattice with positive duality map whose cone K has nonempty interior. Suppose $B: D(B) \rightarrow X$ is hypermaximal T -accretive and B^{-1} is locally bounded. Then B is surjective.*

Proof. We have $A = B(I + B)^{-1}$ is T -accretive and Lipschitzian, $X \rightarrow X$. A^{-1} is locally bounded. By the Corollary to Theorem 2.4, A is hypermaximal T -accretive. To show $R(B) = X$, it is enough to show $R(A) = X$, and by Proposition 3.5, it is enough to show that for y in X there are a, b in X , with $a \leq b$ and $A(a) \leq y \leq A(b)$.

We take $r > 0$ in R, w and v in X , with $B_r(w)$ in $y + K$ and $B_r(v)$ in $y - K$. By Lemma 3.3, we have, decreasing r is necessary, positive constants d_0 and k such that $(A + dI)^{-1}B_r(w) \subset B_k(0)$ and

$$(A + dI)^{-1}B_r(v) \subset B_k(0) .$$

Take $d \leq \min \{d_0, rk^{-1}\}$ and take $b = (A + dI)^{-1}w$, and $a = (A + dI)^{-1}v$. Then $a \leq b$ since $(A + dI)^{-1}$ is monotonic.

$\|db\| \leq r$, and $\|da\| \leq r$, hence, Aa is in $B_r(v)$ and Ab is in $B_r(w)$, hence, $Aa \leq y \leq Ab$.

We complete this section with some results on fixed points of T -nonexpansive functions.

PROPOSITION 3.7. *Let G be a closed bounded convex subset of a reflexive Banach lattice X . Let $N_e(G) = \{x \text{ in } X: d(x, G) \leq e\}$. Suppose $U: N_e(G) \rightarrow G$ is locally T -Lipschitz. Then $(I - U)N_e(G)$ is closed.*

Proof. By translation, as in Theorem 3.2, it is enough to suppose x_n is a sequence in $N_e(G)$ with $|(I - U)x_n| \leq n^{-1}z$ for some $z \geq 0$ in X . Since X is reflexive, we may impose an equivalent norm in which X^* is strictly convex. Take n large so that $\|n^{-1}z\| < e/8$. Now $A = I - U - n^{-1}z$ is locally generalized T -accretive, and locally Lipschitzian, and $-A$ generates a semigroup $\{U(t): t \text{ in } R^+\}$. Suppose $x(t)$ is in $N_e(G)$ but $d(x(t), G) \geq e/2$. Let $Vx(t)$ be a nearest point to $x(t)$ on G . Then we have, except on a set of measure 0, by Lemma 3.8,

$$\begin{aligned} \frac{d}{dt} \|x(t) - Vx(t)\|^2 &\leq 2(J(x(t) - Vx(t)), \frac{dx}{dt}(t)) \\ &= 2(J(x(t) - Vx(t), Ux(t) - x(t) + n^{-1}z)) \\ &\leq 2(J(x(t) - Vx(t), n^{-1}z) - 2\|x(t) - Vx(t)\|^2) \\ &\leq 2e^2/8 - 2e^2/4 \\ &= -e^2/4 . \end{aligned}$$

Hence, $N_e(G)$ is invariant under $\{U(t): t \text{ in } R^+\}$.

Since $Ax_n \leq 0$, $U(t)x_n$ is increasing as t increases, and, hence, converges to x in $N_e(G)$ with $x - Ux = z/n$. Similarly, we obtain y with $y - Uy = -z/n$ and $y \leq x_n \leq x$. By Proposition 3.5, 0 is in $(I - U)[y, x]$. By construction, $(I - U)z = 0$ where $z = \lim z(t)$, $z(0) = y$, and $dz(t)/dt = (U - I)z(t)$. Hence, z is in $N_e(G)$.

LEMMA 3.8. *Suppose G is a closed convex subset of a Banach space X with X^* strictly convex. Let Ux be a nearest point to x on G . Then $2J(I - U)$ is a selection of the subgradient of $x \rightarrow (d(x, G))^2$.*

Proof. (a) We claim $(J(x - Ux), z - Ux) \leq 0$ for x in X and z in G . Let $z(t) = Ux + t(z - Ux)$ for t in $[0, 1]$. Then $z(t)$ is in G , hence,

$$\|x - z(t)\|^2 \geq \|x - Ux\|^2.$$

Now

$$\|x - Ux\|^2 \geq \|z - z(t)\|^2 + 2(J(x - z(t)), z(t) - Ux)$$

since J is the duality map. Hence,

$$\begin{aligned} 0 &\geq 2(J(x - z(t)), z(t) - Ux) \\ &= 2t(J(x - z(t)), z - Ux). \end{aligned}$$

Hence, $(J(x - z(t)), z - Ux) \leq 0$, and letting $t \rightarrow 0^+$, we have $z(t) \rightarrow Ux$, and since J is demicontinuous, we have the result.

(b) Given x and y in X , $(\|x - Ux\| - \|x - Uy\|)^2 \geq 0$. Hence,

$$\|y - Uy\|^2 + \|x - Ux\|^2 - 2(J(x - Ux), y - Uy) \geq 0.$$

Hence,

$$\begin{aligned} \|y - Uy\|^2 &\geq \|x - Ux\|^2 + 2(J(x - Ux), y - Uy - x + Ux) \\ &\geq \|x - Ux\|^2 + 2(J(x - Ux), y - x) \end{aligned}$$

by (a), setting $z = Uy$.

PROPOSITION 3.9. *Let G be a closed bounded convex subset of a reflexive Banach lattice X . Suppose there is $e > 0$ with $U: N_e(G) \rightarrow G$ T -nonexpansive. Then U has a fixed point.*

Proof. Fix y in G ; for p in $(0, 1)$, we set $U_p x = pUx + (I - p)y$. U_p takes $N_e(G)$ to itself, and $\|(U_p x - U_p z)^+\| = p\|(Ux - Uz)^+\|$ for x and z in $N_e(G)$. By the Corollary to Proposition 1.13, U_p has a fixed point, x_p . Letting $p \rightarrow 1$, we have $x_p - Ux_p \rightarrow 0$, since G is bounded. By Proposition 3.7, $(I - U)N_e(G)$ is closed, hence, contains 0 .

COROLLARY. *Suppose there is a T -nonexpansive retraction $R: N_e(G) \rightarrow G$ where G is a closed bounded convex subset of a reflexive Banach lattice. Suppose $U: G \rightarrow G$ is T -nonexpansive. Then U has a fixed point.*

Proof. $UR: N_e(G) \rightarrow G$ is T -nonexpansive, giving x in G with $URx = x$. But $Rx = x$, hence, $Ux = x$.

4. **Equations of evolution in $C(M)$.** In the following we suppose $X = C(M)$, the space of real valued continuous functions on a compact T^2 space M .

DEFINITION. T , a relation on X , is *S-accretive* if xTu and yTv implies $(u - v)(x - y) \geq 0$. Operators of this type were considered by Kacurovskii [9]. We see T is *S-accretive* if and only if T^{-1} is *S-accretive*. We will, for simplicity, restrict the discussion to single valued operators.

DEFINITION. $T: D(T) \rightarrow X, D(T) \subset X$, is *hypermaximal S-accretive*, if *S-accretive*, i.e., $(Tx - Ty)(x - y) \geq 0$ for x, y in $D(T)$, and $R(I + A) = X$. $U: D(U) \rightarrow X, D(U) \subset X$, is *S-nonexpansive* if $|Ux - Uy| \leq |x - y|$ for x, y in $D(U)$.

PROPOSITION 4.1. *If U is S-nonexpansive, then $I - U$ is S-accretive.*

Proof. Given x, y in $D(U)$,

$$\begin{aligned} ((x - Ux) - (y - Uy))(x - y) &\geq |x - y|^2 - |Ux - Uy||x - y| \\ &\geq 0. \end{aligned}$$

PROPOSITION 4.2. *T is S-accretive if and only if $(I + eT)^{-1}$ is S-nonexpansive for $e > 0$.*

Proof. Suppose T is *S-accretive*, and $e > 0$. Suppose $(I + eT)w = x$, $(I + eT)v = y$. Then $(x - y)(w - v) \geq (w - v)^2$. Hence, $|x - y| \geq |w - v|$, and $(I + eT)^{-1}$ is *S-nonexpansive*.

Suppose T is not *S-accretive*, then there is w and v in X and s in M with $w(s) > v(s)$ and $Tw(s) < Tv(s)$. Then there is an $e > 0$ with

$$w(s) + eTw(s) = v(s) + eTv(s).$$

Hence, $v(s) = w(s)$, a contradiction.

PROPOSITION 4.3. *If $T: D(T) \rightarrow X$ generates a semigroup $U(t)$,*

then T is S -accretive if and only if $U(t)$ is S -nonexpansive for t in \mathbb{R}^+ .

Proof. If

$$\begin{aligned}\frac{d}{dt}U(t)x &= -TU(t)x, \\ \frac{d}{dt}U(t)y &= -TU(t)y,\end{aligned}$$

then

$$\frac{d}{dt}(U(t)x - U(t)y)^2 = -2(U(t)x - U(t)y)(TU(t)x - TU(t)y).$$

The l.h.s. ≤ 0 if and only if $U(t)$ is S -nonexpansive.

The r.h.s. ≤ 0 if and only if T is S -accretive.

PROPOSITION 4.4. *If $dx(t)/dt = -Tx(t)$ and T is S -accretive, then $|Tx(t)|$ is decreasing.*

Proof.

$$|x(t+h) - x(t)| \leq |x(s+h) - x(s)|$$

for $s \leq t$, since $U(t)$ is S -nonexpansive. Dividing by h and letting $h \rightarrow 0^+$,

$$|Tx(t)| \leq |Tx(s)|.$$

PROPOSITION 4.5. *If $T: X \rightarrow X$ is S -accretive and continuous from line segments to the topology of pointwise convergence, then T is continuous and monotonic.*

Proof. Given u_0 in X , $(u_0 + n^{-1}u)$ converges to u_0 where u is the unit of X . Hence, $T(u_0 + n^{-1}u)$ converges pointwise to $T(u_0)$. If $m > n$, then

$$(T(u_0 + m^{-1}u) - T(u_0 + n^{-1}u))(m^{-1} - n^{-1})u \geq 0,$$

hence, $T(u_0 + n^{-1}u)$ is decreasing. Therefore, by Dini's theorem, $T(u_0 + n^{-1}u)$ converges strongly to $T(u_0)$. Similarly, $T(u_0 - n^{-1}u)$ converges to $T(u_0)$. Given n , if (u_k) converges to u_0 , then eventually

$$u_0 - (2n)^{-1}u \leq u_k \leq u_0 + (2n)^{-1}u,$$

giving

$$T(u_0 - n^{-1}u) \leq T(u_k) \leq T(u_0 + n^{-1}u).$$

Hence, $T(u_k)$ converges to $T(u_0)$ strongly, and T is continuous. If

$x \leq y$, then $y + n^{-1}u - x \geq n^{-1}u$, hence, $T(y + n^{-1}u) \geq T(x)$. Therefore, $T(y) \geq T(x)$ and T is monotonic.

PROPOSITION 4.6. *If $T: D(T) \rightarrow X$ is hypermaximal S -accretive, then T is maximal S -accretive.*

Proof. Suppose $(w - Tv)(u - v) \geq 0$ for v in $D(T)$. We want to show that u is in $D(T)$ and $w = T(u)$.

Given a in X , and $t > 0$, choose v_t with

$$(I + T)v_t = u + w + ta .$$

Then

$$(u + w - v_t - Tv_t)(u - v_t) \geq 0 .$$

Hence,

$$(-ta)(u - v_t) \geq 0 .$$

Hence,

$$a(u - v_t) \leq 0 .$$

Since $(I + T)^{-1}$ is S -nonexpansive, and

$$v_t = (I + T)^{-1}(u + w + ta) ,$$

we have

$$v_t \rightarrow (I + T)^{-1}(u + w) \text{ as } t \rightarrow 0 .$$

Therefore,

$$a(u - (I + T)^{-1}(u + w)) \leq 0 .$$

Hence,

$$u - (I + T)^{-1}(u + w) = 0 ,$$

giving u in $D(I + T) = D(T)$, and $T(u) = w$.

PROPOSITION 4.7. *If T is S -accretive, then T is accretive and T -accretive.*

Proof. If x in X , the Jx is the set of bounded real Baire measures m on M with support in $|x|^{-1}(\|x\|)$, with total variation $\|x\|$, positive where x is positive and negative where x is negative.

If x, y are in $D(T)$, and m is in $J(x - y)$, then $(Tx - Ty)(x - y) \geq 0$ implies $Tx - Ty$ is positive on the support of the positive part of m and negative on the support of the negative part. Hence,

$$(Tx - Ty, m) \geq 0 .$$

If n is in $J(x - y)^+$, then $Tx - Ty$ is positive on support of n . Hence, $(Tx - Ty, n) \geq 0$.

THEOREM 4.8. *If $T: X \rightarrow X$ is continuous from line segments to pointwise convergence, and S -accretive, then $-T$ generates a strongly C^1 semigroup on $X \times R^+$.*

Proof. Given u_0 in X , there is for $e > 0$ a solution of

$$(a) \quad \begin{aligned} \frac{du_e}{dt}(t) &= -Tu_e(t-e) & t \geq 0 \\ u_e(t) &= u_0 & t \leq 0 \end{aligned}$$

since by Proposition 4.5, T is continuous.

There is an $r > 0$ and $M > 0$ in R with $\|T(x)\| \leq M$ if x is in $B_r(u_0)$. Then there is an interval $[0, h]$ of R such that all solutions of (a) are in $B_r(u_0)$ for $t \leq h$.

Suppose $e, d > 0$.

$$\begin{aligned} \frac{d}{dt}(u_e(t) - u_d(t))^2 &= -2(u_e(t) - u_d(t))(Tu_e(t-e) - Tu_d(t-d)) \\ &\leq 2((u_e(t-e) - (u_e(t) - u_d(t) - u_d(t-d) - u_d(t)))(Tu_e(t-e) - Tu_d(t-d))) \end{aligned}$$

since T is S -accretive.

Since

$$\begin{aligned} \|u_e(t-e) - u_e(t)\| &\leq eM, \\ \|u_d(t-d) - u_d(t)\| &\leq dM, \end{aligned}$$

and

$$\|Tu_e(t-e) - Tu_d(t-d)\| \leq 2M,$$

we have the r.h.s. $\leq 4M^2(d+e)u$ where u is the unit of X .

Now $(u_e(0) - u_d(0))^2 = 0$.

Hence,

$$0 \leq (u_e(t) - u_d(t))^2 \leq h4M^2(d+e)u$$

for t in $[0, h]$.

Hence, u_e is a Cauchy net of continuous functions $[0, h] \rightarrow X$, hence, convergent to a continuous function $u: [0, h] \rightarrow X$. Also, $u_e(s-e)$ converges to $u(s)$ uniformly on $[0, h]$. Hence, $Tu_e(s-e)$ converges to $Tu(s)$ on $[0, h]$.

Now for $e > 0$, and for $t \leq h$, by (a),

$$u_e(t) = u_0 - \int_0^t Tu_e(s-e)ds.$$

Hence,

$$u(t) = u_0 - \int_0^t Tu(s)ds,$$

giving $u(t)$ a C^1 function: $[0, h] \rightarrow X$.

Suppose $u(t)$ is defined on $[0, k)$. By Proposition 4.4, $|Tu(t)| \leq |Tu_0|$, hence, if $t_n \rightarrow k^-$,

$$\begin{aligned} \|u(t_n) - u(t_m)\| &\leq \int_{t_m}^{t_n} \|Tu(t)\| dt \\ &\leq (t_n - t_m) \|Tu_0\|, \end{aligned}$$

giving $u(t_n)$ a Cauchy sequence, hence, convergent to $u(k)$.

By the first part of the proof there is a solution on an interval $[k, k + h]$, with initial point $u(k)$, hence, there is a solution on R^+ .

COROLLARY. *If T is continuous and S -accretive: $X \rightarrow X$, then T is hypermaximal S -accretive.*

Proof. Since $T + I$ is continuous and S -accretive, $-(T + I)$ generates a C^1 semigroup $U(t)$ for t in R^+ . Take y and x in X , then

$$\frac{d}{dt}(U(t)x - U(t)y)^2 \leq -2(U(t)x - U(t)y)^2.$$

Therefore,

$$|U(t)x - U(t)y| \leq e^{-t} |x - y|.$$

Letting $y = U(h)x$ and letting $h \rightarrow 0$,

$$|TU(t)x| \leq e^{-t} |Tx|.$$

Therefore,

$$|U(t)x - U(s)x| \leq |e^{-s} - e^{-t}| |Tx|,$$

hence, $U(t)x$ is a Cauchy net, hence, convergent to an element z of X , as $t \rightarrow \infty$. Then $U(t)z = z$ for t in R^+ by continuity. Hence, $(T + I)z = 0$. By translation we have $R(T + I) = X$.

THEOREM 4.9. *If T is continuous, S -accretive, and proper $X \rightarrow X$, then T is surjective.*

Proof. By Proposition 4.7, T is accretive, and by the Corollary above, T is hypermaximal accretive. Since T is proper, T^{-1} is locally bounded and T takes closed balls to closed sets.

Hence, by Theorem 5.3 of Browder [7], $R(T) = X$.

THEOREM 4.10. *If X is the dual of an (AL) space, $T_0: X \rightarrow X$ is continuous and S -accretive, $T_1: D(T_1) \rightarrow X$ is hypermaximal S -accretive, $T = T_0 + T_1$, then $-T$ generates a weak* C^1 semigroup on $D(T_1) \times R^+$.*

Proof. Given u_0 in $D(T_1)$, we have, by Theorem 4.8, for $e > 0$

a solution for t in R^+ of

$$(a) \quad \frac{du_e}{dt}(t) = -T_e u_e(t)$$

$$u_e(t) = u_0,$$

where

$$T_e = T_0 + T_1(I + eT_1)^{-1} = T_0 + e^{-1}(I - eT_1)^{-1}.$$

We let $v_e(t) = (I + eT_1)^{-1}u_e(t)$.

Now since T_e is S -accretive,

$$\left| \frac{du_e}{dt}(t) \right| \leq \left| \frac{du_e}{dt}(0) \right| \leq | -T_0 u_0 | + |T_1 u_0|.$$

There is $M > 0$ and $r > 0$ in R such that if x is in $B_r(u_0)$, then $\|T_0(x)\| \leq M$. There is an $h > 0$ such that if $t \leq h$, then the solution of (a) is in $B_r(u_0)$ for t in $[0, h]$, independently of e .

Hence, if t is in $[0, h]$, $e > 0$, then

$$|T_1 v_e(t)| \leq \left| \frac{d}{dt} u_e(t) \right| + M \leq |T_0 u_0| + |T_1 u_0| + M = M_1.$$

Now $v_e(t) - u_e(t) = eT_1 v_e(t)$. Hence, $|v_e(t) - u_e(t)| \leq eM_1$ for t in $[0, h]$. Given $e, d > 0$,

$$\begin{aligned} \frac{d}{dt}(u_e(t) - u_d(t))^2 &\leq -2(u_e(t) - u_d(t))(T_1 v_e(t) - T_1 v_d(t)) \\ &\leq 2((v_e(t) - v_d(t) - u_e(t) + u_d(t))(T_1 v_e(t) - T_1 v_d(t)) \\ &\leq 4M_1^2(e + d). \end{aligned}$$

Hence,

$$(u_e(t) - u_d(t))^2 \leq h4M_1^2(e + d)u$$

for t in $[0, h]$.

Hence, $u_e(t)$ converges to $u(t)$ uniformly on $[0, h]$, where u is continuous. Hence, $v_e(t)$ converges uniformly to $u(t)$ on $[0, h]$. From (a) we have for t in $[0, h]$,

$$u_e(t) = u_0 - \int_0^t T_e u_e(s) ds.$$

$$T_e u_e(s) = T_0 u_e(s) + T_1 v_e(s).$$

We have $T_0 u_e(s)$ converging to $T_0 u(s)$. $T_1 v_e(s)$ is bounded, hence, there is a weak* cluster point w . For v in $D(T_1)$, $(T_1 v_e(s) - T_1 v)(v_e(s) - v) \geq 0$. Hence, $(w - T_1 v)(u(s) - v) \geq 0$ since the positive cone is weak* closed. Since T_1 is maximal S -accretive, $u(s)$ is in $D(T_1)$ and $w = T_1 u(s)$. Hence,

$$\begin{aligned}
 u(t) &= u_0 - \int_0^t T_0 u(s) ds - \int_0^t T_1 u(s) ds \\
 &= u_0 - \int_0^t T u(s) ds .
 \end{aligned}$$

Hence, $u(t)$ is weak* C^1 on the interval $[0, h]$.

As in Theorem 4.8, there is a solution $u(t)$ on all of R^+ .

COROLLARY. *If $X, T_0,$ and T_1 are as in Theorem 4.10, then $T_0 + T_1$ is hypermaximal S -accretive.*

The proof is the same as the Colollary to Theorem 4.8.

We note that in this section we used the fact that the Banach lattice X is an algebra whose unit is an order unit u . Stone's algebra theorem says that X is isomorphic with the Banach lattice $C(M)$ of continuous real valued functions on M , where M is the set of multiplicative positive linear forms f satisfying $f(u) = 1$.

5. Ergodic theory.

THEOREM 5.1. *Suppose X is a uniformly convex Banach lattice with positive cone K . Suppose $U: K \rightarrow K$ is nonlinear, with $W: K \rightarrow K$ linear, and $Ux \leq Wx$ for x in K . Suppose $\|W\| \leq 1$. Then for x in $K, S_n x = n^{-1} \sum_1^n U^i x$ is convergent to x_0 in K with $Ux_0 \leq x_0$.*

Proof. Suppose G is the convex hull of $\{U^i x: i \text{ in } Z^+\}$. Let $m = \inf \{\|h\|: h \text{ in } G\}$. Take $e > 0$. Then there exists g in G with $\|g\| \leq m + e$, with $g = \sum_1^k a_i U^i x, \sum a_i = 1, a_i \geq 0$. For n in Z^+ let $T_n = n^{-1} \sum_1^n W^i$. Then $\|T_n g\| \leq m + e$ since $\|W\| \leq 1$.

$$\begin{aligned}
 W^h g &= \sum_{i=1}^k a_i W^h U^i x \\
 &\geq \sum_{i=1}^k U^{i+h} x:
 \end{aligned}$$

Hence,

$$\begin{aligned}
 T_n g &\geq n^{-1} \sum_{h=1}^n \sum_{i=1}^k a_i U^{i+h} x \\
 &= n^{-1} \sum_{h=i}^n U^h x - n^{-1} \sum_{i=1}^k b_i U^i x + n^{-1} \sum_{i=n+1}^{n+k} c_i U^i x
 \end{aligned}$$

where b_i and c_i are in $[0, 1]$.

Hence,

$$S_n x \leq T_n g + n^{-1} \left(\sum_{i=1}^k U^i x + \sum_{i=n+1}^{n+k} U^i x \right) .$$

Hence,

$$\|S_n x\| \leq m + 2e \quad \text{if} \quad kn^{-1} \|x\| \leq e/2 .$$

But $S_n x$ is in G , giving $\|S_n x\| \geq m$.

By the uniform convexity of X , $S_n x$ converges, and to the point x_0 of $\text{cl}(G)$ with minimum norm. Now $G + K$ is invariant under W , and, hence, $\text{cl}(G + K)$ is too. Therefore, $Wx_0 = x_0$. Therefore $Ux_0 \leq x_0$.

COROLLARY. *Suppose X is a uniformly convex Banach lattice, and $U: X \rightarrow X$ is linear and positive, with $\|U\| \leq 1$. Then for x in X , $S_n x$ is convergent.*

Proof. If x is in K , the result follows from the theorem on putting $U = W$. For general x , $S_n x = S_n(x^+) - S_n(x^-)$.

THEOREM 5.2. *Suppose X is a uniformly convex Banach lattice with positive cone K . Suppose $\{U(t): t \text{ in } \mathbb{R}^+\}$ is a one parameter semigroup of nonlinear operators $U(t): K \rightarrow K$. Suppose $\{W(t): t \text{ in } \mathbb{R}^+\}$ is a one parameter semigroup of positive nonexpansive linear operators. Suppose $U(t)x \leq W(t)x$ for x in K and t in \mathbb{R}^+ . Then for x in K ,*

$$S_t x = t^{-1} \int_0^t U(s)x$$

is convergent to x_0 in K with $U(t)x_0 \leq x_0$ for t in \mathbb{R}^+ .

The proof is the same as in the case of the discrete semigroup of Theorem 5.1.

THEOREM 5.3. *Suppose X is a Banach lattice with X and X^* uniformly convex. Suppose $A: D(A) \rightarrow X$ is the sum of a hypermaximal T -accretive and a demicontinuous generalized T -accretive function, and $A(0) = 0$. Suppose $B: D(B) \rightarrow X$ is linear and hypermaximal T -accretive. Suppose $D(A) \subset D(B)$ and $K \subset \text{cl}(D(A))$, and for x in $K \cap D(A)$ we have $Ax \geq Bx$. Let $U(t)$ be the semigroup generated by $-A$. Then for x in K ,*

$$S_t x = t^{-1} \int_0^t U(t)x$$

converges to x_0 in K , with $Ax_0 \geq 0$ if x_0 is in $D(A)$.

Proof. Let $W(t)$ be the semigroup generated by $-B$. Then $W(t)$ is linear and T -nonexpansive, hence, $\|W(t)x\| \leq \|x\|$ for x in K . Suppose x is in $D(A) \cap K$. Let $x(t) = U(t)x$ and $y(t) = W(t)x$. Then, except on a set of measure 0,

$$\begin{aligned} \frac{d}{dt} \|(x(t) - y(t))^+\|^2 &\leq 2\left(J((x(t) - y(t))^+), \frac{d}{dt}x(t) - \frac{d}{dt}y(t)\right) \\ &= -2(J((x(t) - y(t))^+), Ax(t) - Bx(t)) \\ &\quad - 2(J((x(t) - y(t))^+), Bx(t) - By(t)) \end{aligned}$$

since $x(t)$ is in $D(A)$ and, hence, in $D(B)$.

Since $Ax(t) \geq Bx(t)$ and B is T -accretive the r.h.s. is ≤ 0 . Therefore, $x(t) \leq y(t)$ for t in R^+ . Extending $U(t)$ and $W(t)$ to K by uniform continuity, we have $U(t)x \leq W(t)x$ for every x in K and t in R^+ . By Theorem 5.2, $S_t x$ converges to x_0 in K with $U(t)x_0 \leq x_0$ for t in R^+ . If x_0 is in $D(A)$, we have in the weak topology

$$-Ax_0 = \lim t^{-1}(U(t)x_0 - x_0),$$

giving $Ax_0 \geq 0$.

COROLLARY. *The result holds if $K \subset \text{cl}(D(B))$ and $D(B) \subset D(A)$. In this case, x_0 is in $D(A)$.*

Proof. With notation as above,

$$\begin{aligned} \frac{d}{dt} \|(x(t) - y(t))^+\|^2 &= -2(J(x(t) - y(t))^+, Ax(t) - By(t)) \\ &= -2(J(x(t) - y(t))^+, Ax(t) - Ay(t)) \\ &\quad - 2(J(x(t) - y(t))^+, Ay(t) - By(t)) \end{aligned}$$

since $y(t)$ is in $D(B)$ and, hence, in $D(A)$,

$$\leq 2k \|(x(t) - y(t))^+\|^2$$

since A is generalized T -accretive.

Since $x(0) = y(0) = x$, we have $x(t) \leq y(t)$ for t in R^+ . As above, $S_t x$ converges to x_0 , where x_0 is in $F(W(t))$ for $t \geq 0$. Now x_0 is in $D(B)$ since $\lim t^{-1}(W(t)x_0 - x_0)$ exists. Therefore, x_0 is in $D(A)$, giving $A(x_0) \geq 0$.

THEOREM 5.4. *Let W be a positive linear operator in $L^2(S, B, m)$ with $\|W\| \leq 1$. Suppose W extends to a positive linear operator W_1 in $L^1(S, B, m)$ with $\|W_1\| < 1$. Suppose $m(S) < \infty$, and $W(1) = 1$. Let K be the positive cone in $L^2(S, B, m)$. Let $U: K \rightarrow K$ satisfy $Ux \leq Wx$ for all x . Then for x in K , $S_n x = n^{-1} \sum_{i=1}^n U^i x$ converges to x_0 in K and $\sup(S_n x, x_0)$ converges to x_0 m a.e.*

Proof. By Theorem 5.1, $S_n x$ and $T_n x$ converge to x_0 in K . $T_n x$ converges to x_0 m a.e., by the individual ergodic theorem (Yoshida [22], page 388). For all n we have

$$x_0 \leq \sup(S_n x, x_0) \leq \sup(T_n x, x_0) .$$

Hence, $\sup(S_n x, x_0)$ converges to x_0 *m a.e.*

PROPOSITION 5.5. *Given the conditions of Theorem 5.1, let $S_0: K \rightarrow K$ be defined by $S_0 x = x_0$. Suppose U is continuous and monotonic. Then*

$$S_0 U x = S_0 x \geq U S_0 x \geq S_0^2 x = S_0^n x \quad (n \geq 2),$$

and $R(S_0^2) = F(U)$.

Proof.

$$\begin{aligned} S_n U x &= n^{-1} \sum_{i=2}^{n+1} U^i x \\ &= S_n x + n^{-1} (U^{n+1} x - U x) . \end{aligned}$$

The second term converges to 0; therefore, $S_n U x$ converges to $S_0 x$. We showed in Theorem 5.1 that $S_0 x \geq U S_0 x$. Hence, $U^n S_0 x$ is a decreasing sequence, since U is monotonic. $U^n S_0 x$ is convergent since X is uniformly convex, and to a fixed point z of U since U is continuous. Therefore, $S_n S_0 x$ converges to z , giving $S_0^2 x = z$, and $S_0^2 x \leq U S_0 x$. Since z is in $F(U)$, $S_n z = z$. Hence, $S_0 z = z$, giving $S_0^n x = S_0^2 x$ for $n \geq 2$. This also gives $R(S_0^2) = F(U)$.

6. Monotonic generators. In the theory of equations of evolution the property of being locally T -Lipschitz was the weakest condition stronger than monotonicity. If $A: X \rightarrow X$ is locally T -Lipschitz, there is a local solution u for u_0 in X to

$$\begin{aligned} \text{(a)} \quad & u(0) = u_0 \\ & \frac{d}{dt} u(t) = Au(t) . \end{aligned}$$

Supposing A is merely monotonic, we would like a solution of (a).

DEFINITION. Let X be a topological lattice. $A: X \rightarrow X$ is locally order bounded if for v in X there is a neighborhood N of v with $A(N)$ order bounded.

THEOREM 6.1. *Suppose X is an order complete Banach lattice. Let $A: X \rightarrow X$ be monotonic, continuous, and locally order bounded. Then for u_0 in X there is a strongly C^1 solution u to (a) on some interval $[0, h]$.*

Proof. For $d > 0$ the space of equivalence classes of integrable

functions $f: [0, d] \rightarrow X$ is an order complete Banach lattice, $L^1([0, d], X)$. For f integrable $[0, d] \rightarrow X$, Af is integrable $[0, d] \rightarrow X$, giving an operator $B: L^1([0, d], X) \rightarrow L^1([0, d], X)$. If $f(s) \geq g(s)$ a.e., then $Af(s) \geq Ag(s)$ a.e. Therefore, B is monotonic. Suppose f is integrable $[0, d] \rightarrow X$, then $t \rightarrow u_0 + \int_0^t f(s)ds$ is integrable $[0, d] \rightarrow X$, thus giving an operator $C: L^1([0, d], X) \rightarrow L^1([0, d], X)$. If $f(s) \geq g(s)$ a.e., then

$$u_0 + \int_0^t f(s)ds \geq u_0 + \int_0^t g(s)ds$$

for all t in $[0, d]$. Therefore, C is monotonic.

There is a fixed point of CB by the Tarski fixed point theorem if there is an order interval $[a, b]$ invariant under CB . We want a_1 in X and $d > 0$ with

$$\begin{aligned} a_1 &\leq u_0 + \int_0^t A(a_1)ds \\ &= u_0 + tA(a_1) \end{aligned}$$

for t in $[0, d]$.

If $A(X)$ is order bounded, there is $g \geq 0$ with $A(x)$ in $[-g, g]$ for x in X . Take $a_1 = u_0 - g$, and $d = 1$. Then if a is the equivalence class of $t \rightarrow a_1$, we have $CBa \geq a$. Taking b , the class of $t \rightarrow u_0 + g$, we have $CBb \leq b$. Therefore, there exists $u: [0, 1] \rightarrow X$ with

$$u(t) = \int_0^t Au(s)ds .$$

In the general case, we have, for $h \geq 0$, $u_h: [0, 1] \rightarrow X$ with

$$u_h(t) = \int_0^t \sup(-h, \inf(h, Au_h(s)))ds .$$

Since A is locally order bounded, there exist $r > 0$ and $g \geq 0$ with $A(B_r(u_0)) \subset [-g, g]$. Now $\|u_g(t) - u_0\| \leq t\|g\|$. Take $d \leq r/\|g\|$. Then $u_g(t)$ is in $B_r(u_0)$ for t in $[0, d]$. Hence, $Au_g(t)$ is in $[-g, g]$. Therefore, for t in $[0, d]$ we have

$$u_g(t) = \int_0^t Au_g(s)ds .$$

Then $u = u_g$ is differentiable with $du(t) \setminus dt = Au(t)$, and $u(0) = u_0$.

COROLLARY 1. *Suppose X is an order complete Banach lattice with order unit. Let $A: X \rightarrow X$ be monotonic and continuous. Then for u_0 in X there is a strongly C^1 solution to (a) on some interval $[0, d]$.*

Proof. A is locally order bounded since continuous.

THEOREM 6.2. *Suppose X is an order complete Banach lattice with positive cone K , such that bounded subsets of X directed under \leq are convergent. Suppose $A: X \rightarrow X$ is monotonic, continuous, and there is a neighborhood N of u_0 with $A(N) \subset K$. Then there is a strongly C^1 solution (a) on some interval $[0, d]$.*

Proof. Take M and r such that $\|Ax\| \leq M$ for x in $B_r(u_0)$. Then take $d \leq r/M$. For t in $[0, d]$, take $u_1(t) = u_0$, and define inductively

$$u_{n+1}(t) = u_0 + \int_0^t Au_n(s) ds.$$

Then (u_n) is an increasing bounded sequence in $L^1([0, d], X)$, hence, convergent to an element u of $L^1([0, d], X)$. For s in $[0, d]$, $u_n(s)$ is an increasing sequence convergent to $u(s)$, and $Au_n(s)$ is an increasing sequence convergent to $Au(s)$. Hence,

$$u(t) = u_0 + \int_0^t Au(s) ds$$

for t in $[0, d]$, giving the result.

We could use the same technique to study the temporally inhomogeneous problem, and, indeed, the functional equation $du(t)/dt = f(P_t u, t)$, where $P_t u$ is the function $[-t, 0] \rightarrow X$ defined by $P_t u(s) = u(t + s)$. Also, the theory could be stated in terms of a locally convex vector lattice.

7. G-accretive functions. If G is a closed convex subset of a reflexive Banach space X , there is a multivalued map $x \rightarrow Ux$ taking x to points of G which are nearest to x . If X is strictly convex, then U is singlevalued. Suppose X is a reflexive Banach lattice and $G = -K$, the negative cone. Then Ux contains $-x^-$, $(I - U)x$ contains x^+ , and $x \rightarrow 2J(x^+)$ being a selection of the subgradient of $x \rightarrow \|x^+\|^2$ is a particular case of the following result.

PROPOSITION 7.1. *Let G be a closed convex subset of a Banach space with strictly convex dual. Then $2J(I - U)$ is a subset of the subgradient of $x \rightarrow (d(x, G))^2$.*

This was proved as Lemma 3.8.

DEFINITION. Let G be a subset of a Banach space X with duality map J , and U be a function taking points in X to closest points in G . We say $V: D(V) \rightarrow X$, $D(V) \subset X$, is G -nonexpansive if

$$d(Vx - Vy, G) \leq d(x - y, G)$$

for x and y in $D(V)$. We say $A: D(A) \rightarrow X, D(A) \subset X$, is G -accretive if for x and y in $D(A)$, $(J(I - U)(x - y), Ax - Ay) \geq 0$.

If $G = \{0\}$, then $I - U = I$, and we are back in the nonexpansive and accretive case. If $G = -K$, then we are back in the T -nonexpansive and T -accretive case. When G is a compact convex circled subset of X , we can directly generalize some results on accretive operators.

PROPOSITION 7.2. *Let G be a closed convex subset of a Banach space X with strictly convex dual. Suppose u and v are strongly continuous and weakly differentiable $R \rightarrow X$, with $du(s)/ds = -Tu(s)$ and $dv(s)/ds = -Tv(s)$. Then*

$$(J(I - U)(u(t) - v(t)), Tu(t) - Tv(t)) \geq 0$$

if and only if $d(u(t) - v(t), G)$ is decreasing.

Proof. Let $q(t) = \|(I - U)(u(t) - v(t))\|^2$. By Proposition 7.1,

$$q(t) - q(s) \leq 2(J(I - U)(u(t) - v(t), u(t) - v(t) - u(s) + v(s)) .$$

Dividing by $t - s$ and letting $s \rightarrow t^-$, we obtain

$$\overline{\lim}_{s \rightarrow t^-} \left\{ \frac{q(t) - q(s)}{t - s} \right\} \leq -2(J(I - U)(u(t) - v(t), Tu(t) - Tv(t)) .$$

Since the l.h.s. is ≤ 0 if and only if $d(u(t) - v(t), G)$ is decreasing, one implication follows.

The technique of Proposition 1.11 gives the converse.

THEOREM 7.3. *Suppose X is a Banach space with X^* strictly convex, and G compact convex circled subset of X . Suppose $T: X \rightarrow X$ is G -accretive and locally uniformly continuous. Then $-T$ generates a strongly C^1 solution to $dx(t)/dt = -Tx(t), x(0) = x_0$.*

Proof. (a) Given x_0 in X and $e > 0$, we have a solution to

$$\frac{d}{dt}x_e(t) = -Tx_e(t - e) \quad t \geq 0$$

$$x_e(0) = x_0 .$$

Now there exist positive constants r and M such that $\|T(B_r(x_0))\| \leq M$. Then there is $d > 0$ such that all solutions $x_e(t)$ are in $B_r(x_0)$ for t in $[0, d]$. For e and $f > 0$ and t in $[0, d]$, we set

$$q_{e,f}(t) = \|(I - U)(x_e(t) - x_f(t))\|^2$$

$$\begin{aligned} \overline{\lim}_{s \rightarrow t^-} \left\{ \frac{q_{e,f}(t) - q_{e,f}(s)}{t - s} \right\} &\leq 2 \left(J(I - U)(x_e(t) - x_f(t)), \frac{d}{dt}x_e(t) - \frac{d}{dt}x_f(t) \right) \\ &= -2(J(I - U)(x_e(t) - x_f(t), Tx_e(t - e) - Tx_f(t - f)) . \end{aligned}$$

(b) By the local uniform continuity of T we may assume r is small enough that T is uniformly continuous on $B_r(x_0)$. Then there is a function $s: R^+ \rightarrow R^+$ with $s(k) \rightarrow 0$ as $k \rightarrow 0$, such that $\|Tx - Ty\| \leq s(k)$ if x and y are in $B_r(x_0)$ and $\|x - y\| \leq k$. Now

$$\|x_e(t - e) - x_e(t)\| \leq eM, \quad \|x_f(t - f) - x_f(t)\| \leq fM,$$

and $\|x_e(t) - x_f(t)\| \leq 2r$. It follows that

$$\|Tx_e(t - e) - Tx_e(t) - Tx_f(t - f) + Tx_f(t)\| < [s(eM) + s(fM)] .$$

Hence,

$$\begin{aligned} &-2(J(I - U)(x_e(t) - x_f(t)), Tx_e(t - e) - Tx_f(t - f)) \\ &= -2(J(I - U)(x_e(t) - x_f(t), Tx_e(t) - Tx_f(t)) \\ &\quad + 2(J(I - U)(x_e(t) - x_f(t), Tx_e(t) - Tx_f(t) - Tx_e(t - e) \\ &\quad + Tx_f(t - f)) \leq \|2J(I - U)(x_e(t) - x_f(t))\| \|Tx_e(t) - Tx_f(t) \\ &\quad - Tx_e(t - e) + Tx_f(t - f)\| \leq 4r(s(eM) + s(fM)) . \end{aligned}$$

Now given $g > 0$, there is $h > 0$ such that if $e, f \leq h$, then

$$4r(s(eM) + s(fM)) \leq g .$$

Hence, for $e, f \leq h$, we have

$$\overline{\lim}_{s \rightarrow t^-} \left\{ \frac{q_{e,f}(t) - q_{e,f}(s)}{t - s} \right\} \leq g .$$

Since $q_{e,f}(0) = 0$, we have $q_{e,f}(t) \leq gt \leq gd$ for t in $[0, d]$. Hence, $(I - U)(x_e(t) - x_f(t)) \rightarrow 0$ uniformly on $[0, d]$ as $e, f \rightarrow 0$.

(c) We claim $F = \{U(x_e - x_f), e, f \leq 1\}$ is relatively compact in $C([0, d], X)$. $\{x_e\}$ is an equicontinuous family since

$$\|x_e(t) - x_e(s)\| \leq M|t - s| .$$

Hence, $\{x_e - x_f: e, f \leq 1\}$ is equicontinuous. U is Lipschitzian, hence, F is equicontinuous. Since $F[0, d]$ is relatively compact in X , F is relatively compact by the Arzela-Ascoli theorem.

Take a sequence $e(n) \rightarrow 0$ in R . Writing $x_{e(n)} = x_n$, there is for n fixed a subsequence $U(x_n - x_{m'})$ of $U(x_n - x_m)$ convergent to z_n in $\text{cl}(F)$ as $m' \rightarrow \infty$. There is a subsequence $z_{n'} \rightarrow z$ in $\text{cl}(F)$ as $n' \rightarrow \infty$. Since $(I - U)(x_{n'} - x_{m'}) \rightarrow 0$, we can find, given $\epsilon > 0$, a sequence $x_{n'}(i) = x(i)$ such that $\|(x(i + 1) - x(i)) - z\| \leq \epsilon 2^{-i}$. Therefore,

$$\|x(i) - x(1) - iz\| \leq \epsilon .$$

But $\|x(i) - x(1)\| \leq 2r$ for all i , so that $z = 0$. Hence, $x(i)$ is a Cauchy sequence, hence, converges to a continuous function $x: [0, d] \rightarrow B_r(x_0)$.

(d) For each i we have

$$x(i)(t) = x_0 - \int_0^t Tx(i)(s - e(n'(i)))ds .$$

Now $x(i)(s - e(n'(i))) \rightarrow x(s)$ uniformly on $[0, d]$ as $i \rightarrow \infty$. Hence, $Tx(i)(s - e(n'(i))) \rightarrow Tx(s)$, and we have

$$x(t) = x_0 - \int_0^t Tx(s)ds$$

for t in $[0, d]$. Hence, x is C^1 , and $dx(t)/dt = -Tx(t)$.

We have uniqueness to within the compact set G . That is, if $x(t)$ and $y(t)$ are solutions, then $x(t) - y(t)$ is an element of G , since $(I - U)(x(t) - y(t)) = 0$.

COROLLARY (Peano). *Suppose $f: R^n \rightarrow R^n$ is continuous and x_0 is an element of R^n . Then there is a solution to $dx(t)/dt = f(x(t))$, $x(0) = x_0$.*

Proof. Take G a large set so that the notion of G -accretivity is void for points near x_0 . The local uniform continuity of f follows from the local compactness of R^n .

THEOREM 7.4. *Suppose X is a Banach space with X^* uniformly convex, and G a compact convex circled subset of X . Suppose $T: X \rightarrow X$ is G -accretive and demicontinuous. Then $-T$ generates a weakly C^1 solution to $dx(t)/dt = -Tx(t)$, $x(0) = x_0$.*

Proof. The proof is the same as in Theorem 7.3 except in (d) where the integral is taken in the weak topology, and in (b) where the uniform continuity of J is used, as is shown below.

(b)

$$\overline{\lim}_{s \rightarrow t^-} \left\{ \frac{q_{e,f}(t) - q_{e,f}(s)}{t - s} \right\} \leq -2(J(I - U)(x_e(t) - x_f(t), Tx_e(t - e) - Tx_f(t - f)))$$

as in Theorem 7.3. Since T is G -accretive, we have

$$\begin{aligned} \text{r.h.s.} &\leq 2(J(I - U)(x_e(t - e) - x_f(t - f)) \\ &\quad - J(I - U)(x_e(t) - x_f(t), Tx_e(t - e) - Tx_f(t - f))) \\ &\leq 4M \|J(I - U)(x_e(t - e) - x_f(t - f) - J(I - U)(x_e(t) - x_f(t)))\| \\ &\leq 4M s(2(e + f)M) \end{aligned}$$

where $s: R^+ \rightarrow R^+$ satisfies $s(k) \rightarrow 0$ as $k \rightarrow 0$, and $\|Jx - Jy\| \leq s(k)$

if $\|x - y\| \leq k$, for x and y in $B_{kr}(0)$.

Given $g > 0$, we take $h > 0$ such that if $e, f \leq h$, then $4Ms(2(e + f)M) < g$. Since $q_{e,f}(0) = 0$, $q_{e,f}(t) \leq tg \leq dg$ for t in $[0, d]$. Therefore, $(I - U)(x_e(t) - x_f(t)) \rightarrow 0$ uniformly on $[0, d]$ as $e, f \rightarrow 0$.

PROPOSITION 7.5. *Under the conditions of Theorem 7.3 or 7.4, the solution $x(t)$ is defined for all t in R^+ .*

Proof. It is enough to suppose there is a solution $x(t)$ on $[0, k)$, and show $x(t)$ is convergent as $t \rightarrow k^-$. Take h in $(0, k)$. Then $\{x(t): 0 \leq t < h\}$ is relatively compact. We recall that given a bounded subset A of a complete metric space X , the ball measure of noncompactness of A , written $C(A)$, is $\inf\{a > 0: A \text{ can be covered by a finite number of balls of radius } \leq a\}$. $C(A) = 0$ if and only if $\text{cl}(A)$ is compact. The concept is developed in Nussbaum [17] and Ambrosetti [1]. We recall $f: X \rightarrow X$ is a C - k -set contraction if $C(f(A)) \leq kC(A)$ for all bounded subsets A of X .

Given $a > 0$, we may, since $C\{x(t): 0 \leq t < h\} = 0$, take $t_1 \cdots t_n$ in $[0, h)$ such that $\{x(t): 0 \leq t < h\} \subset \cup \{B_a(x_i): 1 \leq i \leq n\}$ where $x_i = x(t_i)$. Suppose y in $\{x(t): 0 \leq t < h\}$ is in $B_a(x_i)$. Since

$$\begin{aligned} \frac{d}{dt} \|(I - U)(x_i(t) - y(t))\|^2 \\ \leq -2(J(I - U)(x(t) - y(t)), Tx_i(t) - Ty(t)) \leq 0, \end{aligned}$$

$y(t)$ is in $B_a(x_i(t) + G)$ for $t \leq k - h$, where $x_i(t)$ is any solution to $du(t)/dt = -Tu(t)$ with $u(0) = x_i$. Hence,

$$\{x(t): k - h \leq t < k\} \subset \cup \{B_a(x_i(t) + G): 0 \leq t \leq k - h, 1 \leq i \leq n\}.$$

Hence, $C\{x(t): k - h \leq t < k\} \leq a$. Since a was arbitrary,

$$C\{x(t): k - h \leq t < k\} = 0.$$

Hence, $x(t)$ converges as $t \rightarrow k^-$.

The semigroup $U(t)$ generated by $-T$ is taken as $U(t)x =$ all $x(t)$ such that $x(0) = x$ and $dx(s)/ds = -Tx(s)$. The property

$$U(s)U(t) = U(t + s)$$

holds, but homological properties of $U(t)$ are unclear; for example, the existence of a continuous selection.

PROPOSITION 7.6. *Under the conditions of Theorem 7.3 or 7.4, $(dI + T)$ generates a semigroup $\{U(t): t \text{ in } R^+\}$ with $U(t)$ a $C - e^{-dt}$ -set*

contraction, for $d, t \geq 0$.

Proof. We note I is G -accretive, since

$$\begin{aligned} (J(I - U)x, x) &= (J(I - U)x, (I - U)x) + (J(I - U)x, Ux - 0) \\ &\geq \|(I - U)x\|^2 \quad \text{since } 0 \text{ is in } G. \end{aligned}$$

Hence, $(dI + T)$ generates a semigroup $U(t)$. Given $t \geq 0$, suppose $A \subset \cup \{B_i(x_i): 1 \leq i \leq n\}$. Given a in A , suppose a is in $B_i(x_i)$.

$$\frac{d}{dt} \|(I - U)(a(t) - x_i(t))\|^2 \leq -2d \|(I - U)(a(t) - x_i(t))\|^2.$$

Hence,

$$\begin{aligned} \|(I - U)(a(t) - x_i(t))\| &\leq e^{-dt} \|(I - U)(a - x_i)\| \\ &\leq e^{-dt} \|a - x_i\| \leq be^{-dt}. \end{aligned}$$

Therefore, $a(t)$ is in $B_k(x_i(t) + G)$ where $k = e^{-dt}b$. Hence,

$$U(t)A \subset \cup \{B_k(x_i(t) + G): 1 \leq i \leq n\}.$$

Hence, $U(t)$ is a $C - e^{-dt}$ -set contraction, since $C(U(t)A) \leq e^{-dt}C(A)$.

PROPOSITION 7.7. *Under the conditions of Proposition 7.6, with $d > 0$, there is a compact convex set closed under $U(1)$.*

Proof. We have, if $dx(t)/dt = -(dT + I)x(t)$, that

$$\begin{aligned} \frac{d}{dt} \|(I - U)x(t)\|^2 &\leq 2(J(I - U)x(t), \frac{dx}{dt}(t)) \\ &= -2(J(I - U)x(t), dx(t) + Tx(t)) \\ &\leq -2d \|(I - U)x(t)\|^2 - 2(J(I - U)x(t), Tx(t)) \\ &\leq -2d \|(I - U)x(t)\|^2 + 2 \|(I - U)x(t)\| \|T(0)\| \\ &\leq 0 \quad \text{if } \|(I - U)x(t)\| \geq d^{-1} \|T(0)\|. \end{aligned}$$

Hence, $\{x \text{ in } X: \|x\| \leq 2d^{-1} \|T(0)\| + \text{diameter}(G)\}$ K_1 is closed under $\{U(t): t \text{ in } R^+\}$. Take x in K_1 and let $H = \{U(t)x: t \text{ in } R^+\}$. If $C(H) > 0$, $C(H) = C(U(1)H) \leq e^{-d}C(H)$ since

$$H = U(1)H \cup \{U(s)x: 0 \leq s \leq 1\}.$$

Hence, $C(H) = 0$, and H is relatively compact. Let $L = \{K: K \text{ a closed convex subset of } K_1, U(1)K \subset K \text{ and } \text{cl}(H) \cap K \text{ is nonempty}\}$. Since $\text{cl}(H)$ is compact, if $\{K_a\}$ is a chain in L , then $\cap (K_a \cap \text{cl}(H))$ is nonempty, hence, $\cap K_a$ is in L . Therefore, by Zorn's lemma, there is a minimal element K_0 of L . Let K_2 be the convex closure of $U(1)K_0$. K_2 is in L . If $C(K_0) > 0$, then

$$C(K_2) = C(U(l)K_0) < C(K_0) ,$$

contradicting the minimality of K_0 . Hence, $C(K_0) = 0$, and K_0 is compact.

PROPOSITION 7.8. *Under the conditions of Theorem 7.3 or 7.4, $R(T + dI) = X$ for $d > 0$.*

Proof. By Proposition 7.7 the fixed point set of $U(1): K_1 \rightarrow K_1, F(U(1))$, is nonempty. $F(U(1))$ is compact since $U(1)$ is a $C - e^{-d}$ -set contraction by Proposition 7.6

The same proof shows that for n in Z^+ the set $F(U(2^{-n}))$ is nonempty and compact. Moreover, $F(U(2^{-n})) \subset F(U(2^{-m}))$ for $n > m$. It follows that there is an element u_0 of K_1 having the property that for all n in $Z^+, U(2^{-n})u_0 = u_0$. Therefore, $(dI + T)u_0 = 0$. By translation of T we have $R(dI + T) = X$.

THEOREM 7.9. *Under the conditions of Theorem 7.3 or 7.4, T^{-1} locally bounded implies $R(T)$ is dense in X , and T proper implies $R(T) = X$.*

Proof. We consider the operators $S: x \rightarrow T(x + G)$ and $S_e: x \rightarrow (eI + T)(x + G)$. To show $R(T)$ is dense in X , we use $R(T) = R(S)$ and show $R(S)$ is dense. By Proposition 7.8, $R(S_e) = R(eI + T) = X$. For y in X ,

$$S_e^{-1}(y) = (eI + T)^{-1}y + G .$$

We show S_e is semi-invertible, that is,

- (a) for y in $X, S_e^{-1}(y)$ is connected, and
- (b) S_e^{-1} is u.s.c.

Suppose a and b are in $S_e^{-1}(y)$. Then there exist u and v with $a - u$ in G and $b - v$ in G , such that $(eI + T)u = y$ and $(eI + T)v = y$. It follows that

$$\begin{aligned} 0 &= (eu + Tu - ev + Tv, J(I - U)(u - v)) \\ &\geq e \|(I - U)(u - v)\|^2 . \end{aligned}$$

Hence, $u - v$ is in G . Now $(u + G) \cup (v + G)$ is connected, and is contained in $S_e^{-1}(y)$, and contains a and b . Hence, $S_e^{-1}(y)$ is connected. To show S_e^{-1} is u.s.c., it is enough to show $y \rightarrow (eI + T)^{-1}y$ is u.s.c. We want to show that if N is a neighborhood of $(eI + T)^{-1}y$, and

$(eI + T)z_n$ converges to y , then x_n is eventually in N . Take z in $(eI + T)^{-1}y$. Then $(I - U)(z_n - z)$ converges to 0. Hence, a subsequence, $z_{n'}$, is convergent to some element w of X . Then $(eI + T)z_{n'}$ converges weakly to $(eI + T)w$. Hence, w is in $(eI + T)^{-1}y$. Hence, $z_{n'}$ is eventually in N . Hence, z_n is eventually in N .

Now S_e converges to S uniformly on bounded sets, i.e., if $B \subset X$ is bounded, then for $a > 0$ there is $e_0 > 0$ such that for x in B and $e \leq e_0$,

$$S_e(x) \subset B_a S(x)$$

and

$$S(x) \subset B_a S_e(x) .$$

Now Theorem 5.1 of Browder [7] says: Let X be a topological space with a bounding system of subsets $\{B_n\}$, Y a connected, locally connected uniform space. Let S be a mapping of $X \rightarrow 2^Y$, and suppose there exists a directed set $\{S_e: e \in E\}$ converging uniformly to S on each B_n . Suppose for each y_0 in Y , there is a neighborhood N of y_0 with $S^{-1}(N)$ bounded with respect to the given bounding system. Then

- (a) $R(S)$ is dense in Y , and
- (b) if $S(B_n)$ is closed for each n , then $R(S) = Y$.

We consider the bounding system $\{B_n(0): n \text{ in } Z^+\}$ on X . To apply the theorem above, we want S^{-1} locally bounded, but

$$S^{-1}(N) = T^{-1}(N) + G ,$$

which is bounded if $T^{-1}(N)$ is. By (a), $R(S) = R(T)$ is dense in X . If T is proper, $T(X)$ is closed, and, therefore, $R(T) = X$.

8. Solutions by smoothing. We recall that to show the existence of a zero of $A: D(A) \rightarrow X$ such that $R(A + I) = X$ and

$$(Ax - Ay, J(x - y)) \geq c \|x - y\|^2, c > 0 ,$$

we take an arbitrary v in X and solve $u(0) = v, du(t)/dt = -Au(t)$, and find $u(t)$ is convergent to x in X with $Ax = 0$.

The same approach is used in this section to give zeros of A under the conditions of the Nash-Moser inverse function theorem. For this, see Nash [16] and Moser [14, 15]. The results are simpler and stronger than those given by the smoothed Newton's method, and are obtained by adapting the following basic result.

THEOREM 8.1. *Let X be a Banach space and $f: B \rightarrow X$ a C^1 function, where $B = \{x \text{ in } X: \|x\| < 1\}$. Suppose $L: B \rightarrow \{T \text{ in } L(X, X): \|T\| \leq M\}$ is locally Lipschitzian, and Lx is a right inverse of f'_x in $L(X, X)$ for x in B . Suppose $\|f(0)\| < M^{-1}$. Then 0 is in $f(B)$.*

Proof. Consider

$$\begin{aligned} \frac{du}{dt}(t) &= -Lu(t)fu(t) \\ u(0) &= 0. \end{aligned}$$

There is a strong solution $u(t)$ as long as $\|u(t)\| < 1$, since for x and y in B

$$\|L(x)f(x) - L(y)f(y)\| \leq \|L(x)(f(x) - f(y))\| + \|L(x) - L(y)\| \|f(y)\|.$$

Now

$$\begin{aligned} \frac{d}{dt} fu(t) &= f'_{u(t)} \frac{du}{dt}(t) \\ &= -f'_{u(t)} Lu(t)fu(t) \\ &= -fu(t). \end{aligned}$$

The function $t \rightarrow \|fu(t)\|^2$ is absolutely continuous; hence, $d\|fu(t)\|^2/dt$ exists except on a set of measure 0, and for any duality map J ,

$$\begin{aligned} \frac{d}{dt} \|fu(t)\|^2 &\leq 2(Jfu(t), \frac{d}{dt} fu(t)) \\ &= -2(Jfu(t), fu(t)) \\ &= -2\|fu(t)\|^2. \end{aligned}$$

Therefore,

$$\|fu(t)\|^2 \leq \|fu(0)\|^2 e^{-2t} \quad \text{for } t \geq 0.$$

Hence,

$$\left\| \frac{du}{dt}(t) \right\| \leq M \|f(0)\| e^{-t}.$$

Hence,

$$\begin{aligned} \|u(t)\| &\leq \int_0^t \left\| \frac{du}{ds}(s) \right\| ds \\ &\leq M \|f(0)\| (1 - e^{-t}) \\ &\leq M \|f(0)\|. \end{aligned}$$

Hence, $u(t)$ is in B and defined for all t in R^+ . Since

$$\|u(t) - u(s)\| \leq M \|f(0)\| (e^{-t} - e^{-s}), u(t)$$

is Cauchy, hence, there is x in B with $u(t) \rightarrow x$ as $t \rightarrow \infty$. Since $fu(t) \rightarrow 0$, we have $fx = 0$ by continuity.

Before giving the main theorems we study smoothing operators. For an exposition of classical smoothing operators, see Schwartz [21]. For comparison, we give the Nash-Moser theorem as stated by Schwartz.

Let K be a compact n -dimensional manifold and C^r the space of r times continuously differentiable functions $u: K \rightarrow R$ with norm

$$|u|_r = \max_{|a| \leq r} \max_{x \text{ in } K} |D^a u(x)|.$$

Let f be a function $D(f) \rightarrow C^{m-a}$ where $D(f) = \{x \text{ in } C^m: |x|_m \leq 1\}$. Suppose there is a constant M such that:

- (1) f has two continuous Frechet derivatives, both bounded by M ;
- (2) there exists a map L with domain $D(f)$ and range $L(C^m, C^{m-a})$, with:

- (a) $|L(u)h|_{m-a} \leq M|h|_m \quad u \text{ in } D(f), h \text{ in } C^m$
- (b) $f'_u L(u)h = h \quad u \text{ in } D(f), h \text{ in } C^{m+a}$
- (c) $|L(u)f(u)|_{m+9a} \leq M(1 + |u|_{m+10a})$;
- (3) $|f(0)|_{m+9a} \leq 2^{-40} M^{-202}$.

Then $f(D(f))$ contains the origin.

DEFINITION. Suppose we have two Banach spaces X_1 and X_2 such that the inclusion $i: X_2 \rightarrow X_1$ is bounded and $i(X_2)$ is dense in X_1 . A smoothing for X_1 and X_2 is a family $\{S(t): t \geq t_0\} \subset L(X_1, X_2)$ with constants M and t_0 and a function $h: R^+ \rightarrow R^+$ such that

- (a) $\|S(t)\| \leq M$ in $L(X_1, X_1)$ and $L(X_2, X_2)$ for all t ;
- (b) $\|S(t)x\|_2 \leq Mh(t)\|x\|_1$ for x in X_1 ;
- (c) $\|S(t)x - x\|_1 \leq Mt^{-1}\|x\|_2$ for x in X_2 .

PROPOSITION 8.2. Suppose $X_2 = D(A)$ where A is an unbounded closed linear operator in X_1 , and for x in X_2 , $\|x\|_2 = \|x\|_1 + \|Ax\|_1$ or an equivalent norm. Suppose there are constants b , t_0 , and M such that $(I - t^{-1}(A - bI))^{-1}$ is in $L(X_1, X_1)$ with norm $\leq M$ for all $t \geq t_0$. Then we have a smoothing for X_1 and X_2 .

Proof. The conditions on A imply that A generates a semigroup $\{U(t): t \text{ in } R^+\}$ with $\|U(t)\| \leq Me^{bt}$, $U(t)U(s) = U(t+s)$, $U(0) = I$, and $U(t)x$ converges strongly to x as $t \rightarrow 0^+$.

There are a couple of ways of looking at smoothing.

(1) We parody the simplest smoothing by convolution where x is a locally integrable function $R \rightarrow R$, and $U(t)x(s) = x(s+t)$. Let f be a C^∞ function $R \rightarrow R^+$ with compact support in $\{s \text{ in } R: s > 0\}$, such that $\int f(s)ds = 1$. For u in R^+ , set $f_u(t) = u^{-1}f(ut)$ and put $S(u)x = \int_{R^+} f_u(t)U(t)xdt$ for x in X . It is straightforward to show $\{S(u): u > 0\}$ is a smoothing for X_1 and X_2 .

(2) We use the same construction, but with $f(s) = e^{-s}$ for $s \geq 0$. In this case, $S(u) = (I - u^{-1}(A - bI))^{-1}$, a simpler expression. We check that this gives a smoothing. Let x be an element of X_1 .

$$S(u)x + u^{-1}(A - bI)S(u)x = x.$$

Hence, $S(u)$ is a bounded linear operator $X_1 \rightarrow X_2$.

(a) By definition, $\|S(u)\| \leq M$ in $L(X_1, X_1)$. Let x be in X_2 .

$$\begin{aligned} \|S(u)x\|_2 &= \|S(u)x\|_1 + \|AS(u)x\|_1 \\ &\leq M\|x\|_1 + M\|Ax\|_1 \\ &= M\|x\|_2. \end{aligned}$$

Hence, $\|S(u)\| \leq M$ in $L(X_2, X_2)$.

(b) Let x be in X_1 .

$$\begin{aligned} \|S(u)x\|_2 &= \|S(u)x\|_1 + \|(A - bI)S(u)x\|_1 + \|bS(u)x\|_1 \\ &\leq (1 + b)M\|x\|_1 + \|u(I - S(u))x\|_1 \\ &\leq ((1 + b)M + u(1 + M))\|x\|_1. \end{aligned}$$

(c) Let x be in X_2 .

$$\begin{aligned} \|S(u)x - x\|_1 &= u^{-1}\|(A - bI)S(u)x\|_1 \\ &\leq Mu^{-1}\|(A - bI)x\|_1 \\ &\leq Mu^{-1}(1 + b)\|x\|_2. \end{aligned}$$

We note that as in case (1), $S(u)x$ is differentiable in u for fixed x in X_1 ; $dS(u)x/du = -A(S(u))^{-2}t^{-2}x$.

PROPOSITION 8.3. *Let X_1 have a Schauder basis. Then we have a smoothing for X_1 and X_2 , if the inclusion i is compact.*

Proof. Suppose $\{x_n: n \geq 1\}$ is a normal Schauder basis for X_1 . As in Schaefer [20], page 115, the inverse mapping theorem tells us that there exists C such that for all M and all $x = \sum_{n=1}^{\infty} a_n x_n$ in X_1 ,

$$\left\| \sum_{n=1}^M a_n x_n \right\|_1 \leq C \|x\|_1.$$

Since X_2 is dense in X_1 , there exists for each $n \geq 1$ an element y_n of X_2 with $\|y_n - x_n\|_1 \leq (C2^{n+1})^{-1}$, and $\|y_n\|_1 = 1$. Define $B: X_1 \rightarrow X_1$ by $B(\sum a_n x_n) = \sum a_n y_n$. B is an isomorphism of X_1 since for $x = \sum a_n x_n$

$$\begin{aligned} \|Bx - x\|_1 &= \|\sum a_n (y_n - x_n)\|_1 \\ &\leq C \|x\|_1 \sum \|y_n - x_n\|_1 \\ &\leq 2^{-1} \|x\|_1. \end{aligned}$$

Therefore, $\{y_n: n \geq 1\}$ is a normal Schauder basis for X_1 . Rearranging, we may assume $k_n = \|y_n\|_2$ is an increasing sequence. For $x = \sum a_n y_n$ in X_1 and u in R^+ , we put $T(u)x = \sum_{k_n \leq u} a_n y_n$. $T(u)$ is a bounded linear operator $X_1 \rightarrow X_2$.

(a) There exists M such that $\|T(u)x\|_1 \leq M \|x\|_1$ for x in X_1 and $u > 0$, and $\|T(u)x\|_2 \leq M \|x\|_2$ for x in X_2 and $u > 0$, by the inverse mapping theorem.

(b) Let $x = \sum a_i y_i$ be in X_1 .

$$\begin{aligned} \|T(u)x\|_2 &= \left\| \sum_{k_i \leq u} a_i y_i \right\|_2 \\ &\leq u \left\| \sum_{k_i \leq u} a_i k_i^{-1} y_i \right\|_2 \\ &\leq u \sum_{k_i \leq u} |a_i| \\ &\leq uM \|x\|_1 \text{ (number of } k_i\text{'s } \leq u). \end{aligned}$$

For $\epsilon > 0$ there exists N such that for all $n \geq N, k_n^{-1} \leq \epsilon$, since $\{k_n^{-1} y_n: n \geq 1\}$ is bounded in X_2 and, hence, compact in X_1 . Therefore, there is a continuous function $g: R^+ \rightarrow R^+$ with

$$g(u) \geq (\text{the number of } k_i\text{'s } \leq u).$$

Hence,

$$\|T(u)x\|_2 \leq M u g(u) \|x\|_1.$$

(c) Let $x = \sum a_n k_n^{-1} y_n$ be in X_2 . The unit ball in X_2 is compact in X_1 . Hence, for $u > 0$ there exists N such that

$$\left\| \sum_{n \geq N} a_n (\|x\|_2 k_n)^{-1} y_n \right\|_1 \leq u^{-1}.$$

We take $N(u)$ with this property. Then

$$\begin{aligned} \|T(N(u))x - x\|_1 &= \left\| \sum_{k_n \geq N(u)} a_n k_n^{-1} y_n \right\|_1 \\ &\leq u^{-1} \|x\|_2. \end{aligned}$$

For $u > 0$ we put $S(u) = T(N(u))$.

By (a) $\|S(u)\| \leq M$ in $L(X_1, X_1)$ and $L(X_2, X_2)$.

By (b) $\|S(u)x\|_2 \leq MN(u)g(N(u)) \|x\|_1$ for x in X_1 .

By (c) $\|S(u)x - x\|_1 \leq u^{-1} \|x\|_2$ for x in X_2 .

PROPOSITION 8.4. *Suppose X_1 and X_2 are separable Hilbert spaces. Then there exists a smoothing with $M = 1$ and $h(t) = t$, if i is compact.*

Proof. Let x be in X_2 and y be in X_1 .

$$(x, y)_1 \leq \|x\|_1 \|y\|_1 \leq \|i\| \|x\|_2 \|y\|_1 .$$

Hence, there is an element $C(y)$ of X_2 with $(x, y)_1 = (x, Cy)_2$. It is straightforward to show C is in $L(X_1, X_2)$. C is positive definite in X_1 since X_2 is dense in X_1 , and self adjoint. Hence, $C: X_1 \rightarrow X_1$ has a positive definite self adjoint square root A . There is a complete orthonormal basis $\{e_i: i \geq 1\}$ of X_1 such that $Ae_i = h_i e_i$ and the sequence h_i decreases to 0.

We have $\|Ax\|_2 = \|x\|_1$ for x in X_1 , and $\{h_i e_i: i \geq 1\}$ is a complete orthonormal basis for X_2 . For $x = \sum a_i e_i$ in X_1 , we put

$$S(u)x = \sum_{h_i u \geq 1} a_i e_i .$$

$S(u)$ is a bounded linear operator $X_1 \rightarrow X_2$ for u in R^+ .

- (a) $S(u)$ is a projection with norm 1 in both X_1 and X_2 .
- (b) Let $x = \sum a_i e_i$ be in X_1 .

$$\begin{aligned} \|S(u)x\|_2 &\leq \left\| \sum_{h_i u \geq 1} u a_i h_i e_i \right\|_2 \\ &= u \left\| \sum_{h_i u \geq 1} a_i e_i \right\|_1 \\ &\leq u \|x\|_1 . \end{aligned}$$

- (c) Let $x = \sum a_i h_i e_i$ be in X_2 .

$$\begin{aligned} \|S(u)x - x\|_1 &= \left\| \sum_{h_i u \geq 1} a_i h_i e_i \right\|_1 \\ &\leq \left\| \sum_{h_i u \geq 1} u^{-1} a_i e_i \right\|_1 \\ &= u^{-1} \left\| \sum_{h_i u \geq 1} a_i h_i e_i \right\|_2 \\ &\leq u^{-1} \|x\|_2 . \end{aligned}$$

THEOREM 8.5. *Let X_2 be a separable Hilbert space with the Hilbert space X_3 compactly included in X_2 and dense in X_2 , and X_1 the dual of X_3 under the pairing $(\cdot, \cdot)_2$. Suppose $f: X_2 \rightarrow X_1$ is C^1 , with $f(0) = 0$ and the graph of f closed in $X_1 \rightarrow X_1$. Suppose there exists a locally Lipschitzian function L :*

$$B_1 \rightarrow \{T \text{ in } L(X_2, X_1) \cap L(X_3, X_2): \|T\|_{L(X_2, X_1)} \leq R\}$$

where $B_1 = \{x \text{ in } X_2: \|x\|_1 \leq 1\}$ such that for x in B_1 and v in X_3 ,

$f'_x L(x)v = v$. If $\|y\|_3 \leq (2R)^{-1}$, then there is x in X_1 with $f(x) = y$.

Proof. There exists a complete orthonormal basis $\{e_i\}$ of X_1 and a sequence h_i decreasing to 0 such that $\{h_i^{-1}e_i\}$ and $\{h_i^{-2}e_i\}$ are complete orthonormal bases of X_2 and X_3 . Let $S_t = S(e^{t/2})$ where this is the smoothing of Proposition 8.4. S_t is the projection on the subspace spanned by $\{e_i: h_i \geq e^{-t/2}\}$. Consider a continuous solution, differentiable on (t_n, t_{n+1}) , where $t_n = \sup(\log h_n^{-2}, 0)$, to

$$(a) \quad \begin{aligned} \frac{du}{dt}(t) &= -Lu(t)S_t(fu(t) - y) \\ u(0) &= 0. \end{aligned}$$

As long as $u(t)$ is in B_1 , a solution exists in X_2 by the locally Lipschitzian nature of the r.h.s. of (a).

Writing $g(x) = f(x) - y$ for x in X_2 , we have from (a) for t in (t_n, t_{n+1}) ,

$$\begin{aligned} \frac{d}{dt}gu(t) &= f'_{u(t)} \frac{du}{dt}(t) \\ &= -f'_{u(t)}Lu(t)S_tgu(t) \\ &= -S_tgu(t). \end{aligned}$$

Hence,

$$\frac{d}{dt}S_tgu(t) = -S_tgu(t).$$

Hence,

$$\frac{d}{dt} \|S_tgu(t)\|_1^2 = -2 \|S_tgu(t)\|_1^2.$$

$S_tgu(t)$ is not continuous in t , so that we cannot proceed immediately as in Theorem 8.1. However, we do have

$$\|S_tgu(t)\|_1^2 = e^{-2(t-t_n)} \|S_{t_n}gu(t_n)\|_1^2.$$

Letting $y = \sum y_i e_i$, it follows that

$$\begin{aligned} \|S_tgu(t)\|_1^2 &= \sum_{h_i \geq e^{-t/2}} e^{-2(t-t_i)} y_i^2 \\ &= e^{-2t} \sum_{h_i \geq e^{-t/2}} h_i^{-4} y_i^2 \\ &= e^{-2t} \|y\|_3^2. \end{aligned}$$

Hence,

$$\|S_tgu(t)\|_1 = e^{-t} \|y\|_3.$$

From (a),

$$\begin{aligned} \left\| \frac{du}{dt}(t) \right\|_1 &\leq R \|S_t gu(t)\|_2 \\ &= R \|S_t(S_t gu(t))\|_2 \\ &\leq R e^{t/2} \|S_t gu(t)\|_1. \end{aligned}$$

by property (b) of a smoothing

$$\leq R e^{-t/2} \|y\|_3.$$

Hence, $\|u(t)\|_1 \leq 2R \|y\|_3$ as long as $u(t)$ is defined, hence, $u(t)$ is in B_1 , hence, $u(t)$ is defined for all t in R^+ .

Since $u(t)$ is Cauchy in X_1 by the bound on the derivative, there is x in X_1 with $u(t) \rightarrow x$ as $t \rightarrow \infty$.

From $dgu(t)/dt = -S_t gu(t)$ and the continuity in t of $gu(t)$,

$$gu(t) = -y - \int_0^t S_s gu(s) ds.$$

Hence,

$$S_t gu(t) = -S_t y - \int_0^t S_s gu(s) ds.$$

Hence,

$$S_t gu(t) - gu(t) = y - S_t y.$$

Now $y - S_t y \rightarrow 0$ as $t \rightarrow \infty$ in X_1 by property (c) of a smoothing. Also, $S_t gu(t) \rightarrow 0$ as $t \rightarrow \infty$ in X_1 . Hence, $gu(t) \rightarrow 0$ as $t \rightarrow \infty$ in X_1 . Since the graph of g is closed in $X_1 \times X_1$, $g(x) = 0$, and $f(x) = y$.

THEOREM 8.6 *Suppose X is the nuclear Frechet space obtained as $\cap \{H_n: 1 \leq n\}$ where H_i are Hilbert spaces and there exists $C: H_1 \rightarrow H_1$ such that for all n , C is an isometry $H_n \rightarrow H_{n+1}$ and self adjoint and compact $H_n \rightarrow H_n$. Suppose $f: X \rightarrow X$ is differentiable from $\|\cdot\|_n$ to $\|\cdot\|_{n-1}$ for all large n . Suppose there exists $L: X \rightarrow L(X, X)$ and constants M and R with $f'_x L(x) = I$ for x in X .*

$$\|L(x)v\|_n \leq R \|v\|_{n+1} \text{ for } v \text{ in } X \text{ and all large } n.$$

$\|(L(x) - L(y))v\|_n \leq M \|x - y\|_n \|v\|_{n+1}$ for x, y , and v in X and all large n . Then f is surjective.

Proof. As in Proposition 8.4 we take S_t the projection onto the subspace spanned by eigenvectors of C with eigenvalues $\geq e^{-t/2}$. By translation we have only to show that 0 is in $R(f)$. We consider a curve u in X with $u(0) = 0$ and on the intervals where S_t is constant, $du(t)/dt = -Lu(t)S_t f u(t)$. Such a curve exists by the Lipschitzian nature of the r.h.s. in each $\|\cdot\|_n$ for n large. As in Theorem 8.5,

$$\left\| \frac{du}{dt}(t) \right\|_n \leq R e^{-t/2} \|f(0)\|_{n+2},$$

hence, $u(t)$ converges in H_n for large n . Hence, $u(t)$ converges in X to an element x of X . By continuity, $f(x) = 0$.

The existence of such an $L(x)$ for x in X seems necessary since we cannot get f'_x surjective by requiring it to be near I . In fact, if X is a nuclear space, it has the approximation property and, hence, there is a sequence of finite dimensional elements of $L(X, X)$ convergent to I in any S topology making $L(X, X)$ a topological vector space (see Schaefer [18], page 79). The relation between the H_n seems necessary to get one smoothing for all the H_n , which does not in general exist. If f was differentiable $\|\cdot\|_{n+k}$ to $\|\cdot\|_n$ for some k , the surjectivity of f follows from the theorem by considering

$$X = \cap \{H_{nk}: n \geq 1\}.$$

THEOREM 8.7. *Suppose X_0 is a Banach space and $A: D(A) \rightarrow X_0$ is a closed densely defined linear operator such that $(I - t^{-1}(A - bI))^{-1}$ is in $L(X_0, X_0)$ for $t \geq t_0$ with norm $\leq M$. Suppose $X_1 = D(A)$ and $X_2 = D(A^2)$ with $\|x\|_1 = \|x\|_0 + \|Ax\|_0$ and $\|x\|_2 = \|x\|_1 + \|Ax\|_1$. Suppose $f: X_1 \rightarrow X_0$ is C^1 , with the graph of f closed in $X_0 \times X_0$. Let L be locally Lipschitzian $B_1 \rightarrow \{T$ in $L(X_2, X_1)$ and $L(X_1, X_0): \|T\|_{L(X_1, X_0)} \leq R\}$ where*

$$B_1 = \{x \text{ in } X_1: \|x\|_0 \leq 8RM^2(1 + M(1 + M)(1 + b))\}.$$

If $\|f(0)\|_2 \leq e^{-1/(4M^2)}$, then there is x in X_0 with $f(x) = 0$.

Proof. By Proposition 8.3, we have a smoothing for X_0 and X_1 and X_1 and X_2 also, since $(I - t^{-1}(A - bI))^{-1}$ is in $L(X_1, X_1)$ for $t \geq t_0$ with norm $\leq M$, where A here is the restriction of A to $(A + b)^{-1}X_1$. Set $S_t = S(e^{(M^2+2)t})$ and take $d \leq 1$ with $d \leq (4M^4e)^{-1}$. Consider a solution of

$$\begin{aligned} \text{(a)} \quad \frac{du}{dt}(t) &= -Lu(t)S_t^2fu(t-d) & t \geq t_0 \\ u(t) &= 0 & t \leq t_0. \end{aligned}$$

A strong solution exists in X_1 by the locally Lipschitzian nature of the r.h.s. as long as $u(t)$ is in B_1 . Take $K = 2(1 + M(1 + M)(1 + b))$ and assume, given t , that for $s \leq t - d$ we have

$$\begin{aligned} \text{(V)} \quad & \|fu(s)\|_2 \leq e^{M^2s}, \quad \text{and} \\ \text{(U)} \quad & \|fu(s)\|_1 \leq Ke^{-s/2}. \end{aligned}$$

$dfu(t)/dt = -S_i^2 fu(t-d)$, hence,

$$\begin{aligned} \|fu(t)\|_2 &\leq \|f(0)\|_2 + M^2 \int_{t_0}^t \|fu(s-d)\|_2 ds \\ &\leq \|fu(0)\|_2 + M^2 \int_0^t e^{M^2(s-d)} ds \end{aligned}$$

by assumption (V). But

$$\|f(0)\|_2 \leq e^{-1/(4M^2)} \leq e^{-M^2 d}.$$

Hence,

$$\begin{aligned} \|fu(t)\|_2 &\leq e^{-M^2 d} + e^{-M^2 d}(e^{M^2 t} - 1) \\ &\leq e^{M^2 t}. \end{aligned}$$

Hence, (V) holds for all t such that $u(t)$ is defined.

$$\frac{d}{dt} fu(t) = -fu(t) + (I - S_i^2)fu(t) + S_i^2 \int_{t-d}^t \frac{d}{ds} fu(s) ds.$$

Since $t \rightarrow \|fu(t)\|_1^2$ is absolutely continuous, it is differentiable except on a set of measure zero, and for any duality map J for X_1 ,

$$\frac{d}{dt} \|fu(t)\|_1^2 \leq 2 \left(Jfu(t), \frac{d}{dt} fu(t) \right)_1.$$

Hence,

$$\begin{aligned} \frac{d}{dt} \|fu(t)\|_1^2 + 2(Jfu(t), fu(t))_1 &\leq 2(Jfu(t), (I - S_i^2)fu(t)) \\ &\quad + 2 \left(Jfu(t), S_i^2 \int_{t-d}^t \frac{d}{ds} fu(s) ds \right). \end{aligned}$$

Dividing by $\|fu(t)\|_1$,

$$\frac{d}{dt} \|fu(t)\|_1 + \|fu(t)\|_1 \leq \|(I - S_i^2)fu(t)\|_1 + \left\| S_i^2 \int_{t-d}^t \frac{d}{ds} fu(s) ds \right\|_1.$$

Multiplying by e^t ,

$$\begin{aligned} \frac{d}{dt} (e^t \|fu(t)\|_1) &\leq e^t \|(I - S_i)(I + S_i)fu(t)\|_1 \\ &\quad + e^t S_i^2 \int_{t-d}^t -S_s^2 fu(s-d) ds \Big|_1. \end{aligned}$$

Now

$$\begin{aligned} \|(I - S_i)(I + S_i)fu(t)\|_1 &\leq M(1+b)e^{-(M^2+2)t} \|(I + S_i)fu(t)\|_2 \\ &\leq M(1+b)(1+M)e^{-(M^2+2)t} \|fu(t)\|_2 \\ &\leq M(1+b)(1+M)e^{-2t} \quad \text{by (V)}. \end{aligned}$$

Also,

$$\begin{aligned} \left\| S_t^2 \int_{t-d}^t -S_s^2 fu(s-d) ds \right\|_1 &\leq M^4 \int_{t-d}^t \|fu(s-d)\|_1 ds \\ &\leq KM^4 de^{-(t-2d)/2} \quad \text{by (U)}. \end{aligned}$$

Hence,

$$\frac{d}{dt} (e^t \|fu(t)\|_1) \leq M(1+M)(1+b)e^{-t} + KM^4 de^d d^{1/2}.$$

Hence,

$$\begin{aligned} e^t \|fu(t)\|_1 &\leq \|f(0)\|_1 + M(1+M)(1+b) + 2KM^4 de^d e^{t/2} \\ &\leq 2^{-1}K + 2^{-1}Ke^{t/2}. \end{aligned}$$

Hence, $\|fu(t)\|_1 \leq Ke^{-t/2}$. Hence, (U) holds for all t such that $u(t)$ is defined.

From (a),

$$\begin{aligned} \left\| \frac{du}{dt}(t) \right\|_0 &\leq R \|S_t^2 fu(t-d)\|_1 \\ &\leq RM^2 \|fu(t-d)\|_1 \\ &\leq RM^2 Ke^{d/2} e^{-t/2}. \end{aligned}$$

Therefore, $\|u(t)\|_0 \leq 2RM^2 Ke^{d/2}$ as long as $u(t)$ is defined. Hence, $u(t)$ is defined for all t in R^+ , and is Cauchy in X_0 , hence, converges to an element x of X_0 . Also, $fu(t)$ converges in X_0 to 0. Since the graph of f is closed, $f(x) = 0$.

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