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COMPACT FUNCTORS IN CATEGORIES OF NON-ARCHIMEDEAN BANACH SPACES

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Let K be a complete, non-archimedean, non-trivially valued field. Let B be the category of all non-archimedean Banach spaces over K satisfying the "condition (N)" with morphisms continuous linear transformations f, $|f| \leqq 1$. In this paper, we first characterize all compact functors $F: B \rightarrow B$ as functors which take finite dimensional spaces to finite dimensional spaces. We then show that in case K is maximally complete the Mityagin-Shvarts imbedding theorem for duals of functors holds true for functors in B. Finally, using the above results we show that the dual of a compact functor is itself compact.

The present investigation originated in an attempt to apply the Mityagin-Shvarts theory to functors in categories of non-archimedean Banach spaces. In fact, the first and the last results mentioned above are closely related to some problems proposed by Mityagin and Shvarts for functors in categories of ordinary Banach spaces in [4]. Several of these original problems have been solved by a student of mine, Kenneth L. Pothoven, in his thesis [5].

2. **Preliminaries.** Let K be a complete, non-archimedean, non-trivially valued (i.e., the value group is not trivial) field. We denote by R the valuation ring of K (i.e., the set of all $x \in K$ such that $|x| \leqq 1$). A nonarchimedean Banach space over K is a complete normed vector space over K such that the norm satisfies the ultrametric inequality:

$$|x + y| \leqq \sup(|x|, |y|) \text{ for } x, y \in X .$$

In this paper we shall assume that all the non-archimedean Banach spaces satisfy the following condition [6].

(N) For each $x \in X$, $|x|$ belongs to the closure of the value group of K.

For the pair (X, Y) of non-archimedean Banach spaces, let $L(X, Y)$ denote the non-archimedean Banach spaces of all continuous linear maps from X to Y with the norm $|f| = \sup\{|f(x)|: x \in X \text{ and } |x| \leqq 1\}$ (See [6, p. 71]). We let X' denote the dual space $L(X, K)$ and for $f: X \rightarrow Y$ in $\boldsymbol{B}$, we let $f': Y' \rightarrow X'$ denote its dual.

Now we denote by $\boldsymbol{B}$ the category whose objects are non-archimedean Banach spaces over K (satisfying condition (N)), and whose morphism sets are $\boldsymbol{B}(X, Y) = \{f: f \in L(X, Y), |f| \leqq 1\}$. Clearly, each

$\boldsymbol{B}(X, Y)$ is an R-module.

All (covariant) functors F: $\boldsymbol{B} \to \boldsymbol{B}$ are assumed to satisfy the following additional conditions:

(1) For each pair (X, Y) of objects in $\boldsymbol{B}$, the induced map F_{XY}: $\boldsymbol{B}(X, Y) \to \boldsymbol{B}(F(X), F(Y))$ is R-linear, and

(2) For each $f \in \boldsymbol{B}(X, Y)$, $|F_{XY}(f)| \leqq |f|$. Such functors are called functors *in* the category $\boldsymbol{B}$.

A functor G: $\boldsymbol{B} \to \boldsymbol{B}$ is a *subfunctor* of F: $\boldsymbol{B} \to \boldsymbol{B}$ if for each X in $\boldsymbol{B}$, $G(X)$ is a closed subspace of $F(X)$, and for each f: $X \to Y$ in $\boldsymbol{B}$, $G(f)$: $G(X) \to G(Y)$ is the restriction of $F(f)$: $F(X) \to F(Y)$ to $G(X)$.

Natural transformations t: $F \to G$ where F and G are functors in $\boldsymbol{B}$, are assumed to satisfy, in addition to the usual naturality condition, the following conditions:

(1) For each X in B, t_X: $F(X) \to G(X)$ is K-linear.

(2) $|t| = \sup\{|t_X|: X \text{ in } \boldsymbol{B}\} < \infty$.

Two functors F and G are *isometric* if there exist natural transformations t: $F \to G$ and u: $G \to F$ such that $u \cdot t = 1_F$ and $t \cdot u = 1_G$ and for each X in $\boldsymbol{B}$, t_X and u_X are isometries. A functor F is *isometrically embedded* in G if there is a natural transformation establishing an isometry of the functor F and a subfunctor of G.

For functors F and G in the category $\boldsymbol{B}$, we denote by $[F, G]$ the class of all natural transformations from F to G. Note that if $[F, G]$ is a set, $[F, G]$ has a natural structure of a non-archimedean Banach space with norm defined as in (2) above.

For each A in $\boldsymbol{B}$, we define the functor Ω_A by:

(1) $\Omega_A(X) = L(A, X)$, for X in $\boldsymbol{B}$

(2) If $f \in \boldsymbol{B}(X, Y)$, then $\Omega_A(f)$: $\Omega_A(X) \to \Omega_A(Y)$ is the morphism $(\Omega_A(f))(g) = f \cdot g$, for $g \in \Omega_A(X)$.

For each A in $\boldsymbol{B}$, we define the functor Σ_A by:

(1) $\Sigma_A(X) = A \hat{\otimes} X$ (See [6, p. 73])

(2) If $f \in B(X, Y)$, then $\Sigma_A(f)$: $\Sigma_A(X) \to \Sigma_A(Y)$ is the morphism $1_A \hat{\otimes} f$.

Note that for any $h \in L(A, B)$, there corresponds a natural transformation Σ_h: $\Sigma_A \to \Sigma_B$ defined by:

For each X in $\boldsymbol{B}$, Σ_h: $\Sigma_A \to \Sigma_B$ is equal to $h \hat{\otimes} 1_X \in L(\Sigma_A X, \Sigma_B X)$.

3. Compact Functors.

DEFINITION. A functor F: $\boldsymbol{B} \to \boldsymbol{B}$ is *compact* (resp. *of finite rank*), if whenever f: $X \to Y$ in $\boldsymbol{B}$ is compact (resp. of finite rank), then $F(f)$: $F(X) \to (F(Y)$ is compact (resp. of finite rank).

Here "compact" means "complètement continus" in [6, p. 72], and "f is of finite rank" means $\dim f(X) < \infty$.

LEMMA 1. *For X in $\boldsymbol{B}$, 1_X: $X \to X$ is compact* $\Leftrightarrow$ $\dim (X) < \infty$.

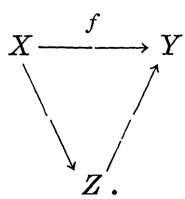
Proof. The assertion follows immediately from [1, Result 2, p.298] by letting $u = 1_X$ there.

THEOREM 1. *Let F: $\boldsymbol{B} \to \boldsymbol{B}$ be a functor. The following are equivalent*:

(i) *F is compact,*
(ii) *F takes finite dimensional spaces to finite dimensional spaces,*
(iii) *F is of finite rank.*

Proof. (i) $\Rightarrow$ (ii). Let F be a compact functor. Let X be a finite dimensional space in $\boldsymbol{B}$. By Lemma 1, 1_X: $X \to X$ is compact. Hence, $1_{F(X)} = F(1_X)$: $F(X) \to F(X)$ is compact. By Lemma 1 again, we see that $F(X)$ is finite dimensional.

(ii) $\Rightarrow$ (iii). We first note that a morphism f: $X \to Y$ is of finite rank if and only if f factors through a finite dimensional space Z as in



Clearly, if F takes finite dimensional spaces to finite dimensional spaces and if f is of finite rank then $F(f)$ is of finite rank.

(iii) $\Rightarrow$ (i). Let F: $\boldsymbol{B} \to \boldsymbol{B}$ be a functor of finite rank. Let f: $X \to Y$ be a compact morphism in $\boldsymbol{B}$. By the definition of compactness of morphisms and by the ultrametric inequality, there exists a sequence of morphisms f_n: $X \to Y$ in $\boldsymbol{B}$ of finite rank converging, in the norm, to f. The morphisms $F(f_n)$: $F(X) \to F(Y)$ are of finite rank and $|F(f_n) - F(f)| \leqq |f_n - f|$ for $n = 1, 2, \cdots$. Hence $F(f)$ is compact.

The following corollaries are immediate consequences of Theorem 1.

COROLLARY. *The following are equivalent*:

(i) *A in $\boldsymbol{B}$ is finite dimensional,*
(ii) *The functor Σ_A is of finite rank,*
(iii) *The functor Σ_A is compact,*
(iv) *The functor Ω_A is of finite rank,*
(v) *The functor Ω_A is compact.*

DEFINITION. A functor F: $\boldsymbol{B} \to \boldsymbol{B}$ is of *null type* (type N in [4,

p. 75]) if $F(K) = 0$.

COROLLARY. *If a functor F is of null type then it is compact.*

4. Duals of compact functors. In addition to all the conditions that are imposed on K in § 2, we shall require throughout this section that the scalar field K is *maximally complete*. We will continue to use the same letter $\boldsymbol{B}$ to designate the category of all non-archimedean Banach spaces over K satisfying this additional assumption.

LEMMA 2. *For each X in $\boldsymbol{B}$, the natural morphism $a_X: X \to X''$ is an isometric embedding.*

Proof. For $x \in X$, $a_X(x)$ is defined by the equation $(a_X(x))(x') = x'(x)$ for all $x' \in X'$. Since X satisfies condition (N), for an element $x \in X$ such that $x \neq 0$, we can find a sequence v_n $(n = 1, 2, \cdots)$ of real numbers in the value group of K such that $v_n \leqq |x|$ $(n = 1, 2, \cdots)$ and $v_n \to |x|$. Let $\alpha_n \in K$ be chosen so that $|\alpha_n| = v_n$. Let Y be the (closed) subspace of X generated by x. Define f_n: $Y \to K$ by

$$f_n(\kappa x) = \kappa \alpha_n$$

for $\kappa \in K$. Clearly, $|f_n| \leqq 1$. Since K is maximally complete, we can extend each f_n to some g_n: $X \to K$ such that $|f_n| = |g_n|$ ([2]). Now,

$$\begin{aligned} |a_X(x)| &= \sup\{|x'(x)|: |x'| \leqq 1 \text{ and } x' \in X'\} \\ &\geqq \sup\{|g_n(x)|: n = 1, 2, \cdots\} = \sup\{v_n: n = 1, 2, \cdots\} = |x|. \end{aligned}$$

On the other hand, clearly we have $|a_X(x)| \leqq |x|$. Hence a_X is an isometric embedding.

DEFINITION. The *dual functor* DF of a functor F: $\boldsymbol{B} \to \boldsymbol{B}$ is defined by:

(1) For each A in B, $DF(A) = [F, \Sigma_A]$,

(2) For each morphism e: $A \to B$ in $\boldsymbol{B}$, $DF(e)$: $DF(A) \to DF(B)$ is defined by the equation $(DF(e))(t) = \Sigma_e \cdot t$.

It will become evident in the course of the proof of Theorem 2 that $DF(A)$ is actually a non-archimedean Banach space.

For a functor F: $\boldsymbol{B} \to \boldsymbol{B}$, we define the functor F^U: $\boldsymbol{B} \to \boldsymbol{B}$ by:

(1) For each A in $\boldsymbol{B}$, $F^U(A) = (F(A'))'$

(2) For each morphism f: $A \to B$ in $\boldsymbol{B}$, $F^U(f)$ is equal to $(F(f'))'$.

THEOREM 2. *For any functor F: $\boldsymbol{B} \to \boldsymbol{B}$, the dual functor DF is isometrically embedded in F^U.*

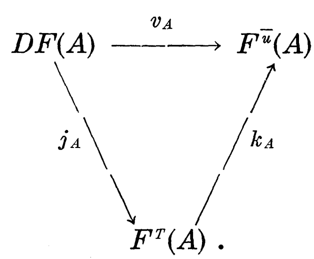
Proof. (Mityagain-Shvarts-Linton). Let F: $\boldsymbol{B} \to \boldsymbol{B}$ be a functor. For each A in $\boldsymbol{B}$, we define the morphism in $\boldsymbol{B}$, v_A: $DF(A) \to (F(A'))'$, by: $v_A(t) = Tr \cdot t'_A$ for $t \in DF(A) = [F, \Sigma_A]$. Here Tr is the trace map. (Clearly, $|v_A| \leqq 1$). We would like to show that, for each A, this morphism v_A: $DF(A) \to F^U(A)$ is an isometric embedding. To this end, we introduce the functor Σ''_A by setting $\Sigma''_A(X) = (A \hat{\otimes} X)''$ for X in $\boldsymbol{B}$, and $\Sigma''_A(f) = (1_A \hat{\otimes} f)''$ for f: $X \to Y$ in $\boldsymbol{B}$. Let

$$F^T(A) = [F, \Sigma''_A] \text{ .}$$

By a proof similar to that of [3, Lemma (4.10), p. 339], we show easily that $F^T(A)$ is a set (hence a non-archimedean Banach space). By Lemma 2, $A \hat{\otimes} X$ is isometrically embedded in $(A \hat{\otimes} X)''$. This means that the functor Σ_A is isometrically embedded in Σ''_A. The natural transformation $\Sigma_A \to \Sigma''_A$ gives rise to an isometric embedding

$$j_A\text{: } DF(A) \to F^T(A) \text{ .}$$

The theorem follows immediately from the existence of a morphism k_A in $\boldsymbol{B}$ making the following diagram commutative



This part of the proof, however, follows exactly the same line of argument as in [3, p. 340-41]. So we shall refrain from repeating the argument here.

COROLLARY. *If the functor F: $\boldsymbol{B} \to \boldsymbol{B}$ is of null type, then so is its dual.*

THEOREM 3. *Let F: $\boldsymbol{B} \to \boldsymbol{B}$ be a functor. Then:*

(i) *If F is compact, then so is its dual DF.*

(ii) *If F takes finite dimensional spaces to finite dimensional spaces, then so does its dual DF.*

(iii) *If F is of finite rank, then so is its dual DF.*

Proof. In view of Theorem 1, it is sufficient to prove (ii).

Suppose F takes finite dimensional spaces to finite dimensional spaces. Let X be a finite dimensional space in $\boldsymbol{B}$. Then X' is finite dimensional. By the assumption, $F(X')$ is finite dimensional. Hence $(F(X'))' = F^U(X)$ is finite dimensional. By Theorem 2, DF is isometrically embedded in F^U. Obviously, $DF(X)$ is finite dimensional. This completes the proof.

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