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A DISCONJUGACY CRITERION FOR HIGHER ORDER LINEAR VECTOR DIFFERENTIAL EQUATIONS

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For a higher order linear quasi-differential equation which is non-self-adjoint there is presented a disconjugacy criterion that is a consequence of the disconjugacy of an associated self-adjoint quasi-differential equation. In particular, there is considered the specific form of this criterion for a higher order differential equation of the canonical form which has been presented by the author, Transactions of the American Mathematical Society, 85 (1957), 446–461.

1. **Introduction.** For self-adjoint Hamiltonian differential systems which satisfy a condition of definiteness that in the case of accessory systems for variational problems is the strengthened Legendre or Clebsch condition, it is well-known, (see, for example, Bliss [1, Secs. 89, 90], Morse [5; 6, Ch. IV], Reid [7; 9; 11, Sec. VII. 5]), that the condition of disconjugacy is equivalent to the positive definiteness of the associated (Dirichlet) hermitian functional. In turn, for non-self-adjoint differential systems one may derive a sufficient condition for disconjugacy as a consequence of the disconjugacy of certain associated self-adjoint systems. An example of this procedure involving a linear homogeneous vector differential equation of the second order is given in Reid [7, Sec. 5]; see also, Hartman and Wintner [3]. The purpose of the present paper is to present corresponding results for more sophisticated differential systems of higher order.

Matrix notation is used throughout; in particular, one column matrices are called vectors. The $n \times n$ identity matrix is denoted by E_n, or merely by E when there is no ambiguity, and 0 is used indiscriminately for the zero matrix of any dimensions. The conjugate transpose of a matrix M is denoted by M^*. The symbols $M \geqq N$, $\{M > N\}$, are used to signify that M and N are hermitian matrices of the same dimensions and $M - N$ is a nonnegative, {positive}, definite matrix. A matrix function is termed continuous, integrable, etc., when each element of the matrix possesses the specified property.

If a matrix function $M(t)$ is a.c., (absolutely continuous), on a compact interval $[a, b]$, then $M'(t)$ signifies the matrix of derivatives at values where these derivatives exist, and zero elsewhere. Similarly, if $M(t)$ is (Lebesgue) integrable on $[a, b]$, then $\int_a^b M(t)dt$ denotes the matrix of integrals of respective elements of $M(t)$. For a given interval $[a, b]$, the symbols $\mathfrak{C}_{pq}[a, b]$, $\mathfrak{C}^n_{pq}[a, b]$, $\mathfrak{L}_{pq}[a, b]$, $\mathfrak{L}^k_{pq}[a, b]$, $\mathfrak{L}^\infty_{pq}[a, b]$, $\mathfrak{A}_{pq}[a, b]$, $\mathfrak{A}^n_{pq}[a, b]$ are used to denote the class of $p \times q$ matrix functions

$M(t) = [M_{\alpha\beta}(t)]$, $(\alpha = 1, \cdots, p; \beta = 1, \cdots, q)$ which on $[a, b]$ are respectively continuous, continuous and possessing continuous derivatives of the first n orders, (Lebesgue) integrable, (Lebesgue) measurable and $|M_{\alpha\beta}(t)|^k$ integrable, measurable and essentially bounded, a.c., of class $\mathfrak{C}^{n-1}_{pq}[a, b]$ with $M^{[n-1]}(t) \in \mathfrak{A}_{pq}[a, b]$. For brevity, the double subscript pq is reduced to merely p for the p-dimensional vector case specified by $p, q = 1$, and both subscripts are omitted in the scalar case $p = 1, q = 1$. For $n \geqq 1$, the subclass of vector functions $y \in \mathfrak{A}^n_p[a, b]$ for which $y^{[n]}(t) \in \mathfrak{L}^2_p[a, b]$ is denoted by $\mathfrak{A}^{n,2}_p[a, b]$. Also for $n \geqq 1$ the subclasses of vector functions y belonging to $\mathfrak{C}^n_p[a, b]$, $\mathfrak{A}^n_p[a, b]$, $\mathfrak{A}^{n,2}_p[a, b]$ for which $y^{[\alpha-1]}(a) = 0 = y^{[\alpha-1]}(b)$, $(\alpha = 1, \cdots, n)$, are denoted by $\mathfrak{C}^n_{p,0}[a, b]$, $\mathfrak{A}^n_{p,0}[a, b]$, $\mathfrak{A}^{n,2}_{p,0}[a, b]$, respectively. If matrix functions $M(t)$ and $N(t)$ are equal a.e. (almost everywhere) on their interval of definition we write simply $M(t) = N(t)$.

2. Preliminary results. Let $F_{ij}(t) = [F_{\sigma\tau;ij}(t)]$, $(i, j = 0, 1, \cdots, n)$, be $r \times r$ matrix functions defined on an interval I on the real line, and satisfying the following hypothesis.

$F_{nn}(t)$ is nonsingular for $t \in I$, and for arbitrary compact subintervals $[a, b] \subset I$, and $\alpha, \beta = 0, 1, \cdots, n-1$ we have:
$(\mathfrak{H})$

(a) *F_{nn}, F^{-1}_{nn}, $F_{\alpha\beta}$, $F^{-1}_{nn}F_{n\beta}$ and $F_{\alpha n}F^{-1}_{nn}$ belong to $L^\infty_{rr}[a, b]$;*

(b) *$F_{n\beta}$ and $F_{\alpha n}$ belong to $L^2_{rr}[a, b]$.*

The $(n + 1)r \times (n + 1)r$ matrix which for $i, j = 0, 1, \cdots, n$ and $\sigma, \tau = 1, \cdots, r$ has the element in the $(ir + \sigma)$th row and $(jr + \tau)$th column equal to $F_{\sigma\tau;ij}(t)$ will be denoted by $F(t)$, and for $k = 0, 1, \cdots, n$ the $r \times (n + 1)r$ matrix whose element in the σth row and $(jr + \tau)$th column is $F_{\sigma\tau;kj}(t)$ will be denoted by merely $F_k(t)$. If $[a, b] \subset I$ we shall denote by $\mathfrak{D}[a, b]$ the linear vector space of r-dimensional vector functions $y \in \mathfrak{A}^{n,2}_r[a, b]$, and by $\mathfrak{D}_0[a, b]$ the subspace consisting of those $y \in \mathfrak{D}[a, b]$ with $y^{[\alpha]}(a) = 0 = y^{[\alpha]}(b)$, $(\alpha = 0, 1, \cdots, n - 1)$. Also, if $y \in \mathfrak{D}[a, b]$ we shall denote by $\hat{y}$ the $(n + 1)r$-dimensional vector function with $\hat{y}_{jr+\tau}(t) = y^{[j]}_\tau(t)$, $(j = 0, 1, \cdots, n; \tau = 1, \cdots, r)$.

If $[a, b] \subset I$ and $y \in \mathfrak{D}[a, b]$, $z \in \mathfrak{D}[a, b]$ then the integral

$$J[y, z \mid a, b] = \int_a^b \hat{z}^*(t)F(t)\hat{y}(t)dt \tag{2.1}$$

is well defined, and is a sesquilinear form on $\mathfrak{D}[a, b] \times \mathfrak{D}[a, b]$.

LEMMA 2.1. *If $y \in \mathfrak{D}[a, b]$, then*

$$J[y, z \mid a, b] = 0, \text{ for } z \in \mathfrak{D}_0[a, b] \tag{2.1'}$$

if and only if y is a solution on $[a, b]$ of the vector quasi-differential *equation*

$$\mathfrak{L}[y\colon F](t) \equiv F_0(t)\hat{y}(t) - \{F_1(t)\hat{y}(t) - \{\cdots - \{F_n(t)\hat{y}(t)\}' \cdots\}'\}' = 0\,. \tag{2.2}$$

In conformity with usual terminology, (see, for example, Bradley [2], Reid [9, Sec. 4]), an r-dimensional vector function $y(t)$ is a solution of (2.2) if $y \in \mathfrak{D}[a, b]$ and the r-dimensional vector functions $v_k(t) = (v_{\sigma k}(t))$, $(\sigma = 1, \cdots, r;\ k = 1, \cdots, n)$, defined recursively as

$$\begin{aligned} v_n(t) &= F_n(t)\hat{y}(t) \\ v_{n-p}(t) &= F_{n-p}(t)\hat{y}(t) - v'_{n-p+1}(t),\ p = 1, \cdots, n-1\,, \end{aligned} \tag{2.3}$$

all belong to $\mathfrak{A}_r[a, b]$ and on $[a, b]$,

$$\mathscr{L}[y\colon F](t) \equiv F_0(t)\hat{y}(t) - v'_1(t) = 0\,. \tag{2.4}$$

The result of Lemma 2.1 follows by the classical proof of the fundamental lemma of the calculus of variations, (see, for example, Bliss [1, Sec. 5] for simplest instance; Reid [11, Probs. III. 2: 1–8] for more general cases). Indeed, if for an integrable vector function $w(t)$ on $[a, b]$ we introduce $I[w](t)$ for $\int_a^t w(s)ds$, and for $y \in \mathfrak{D}[a, b]$ we set

$$\begin{aligned} w_1(t) &= F_0(t)\hat{y}(t) \\ w_{1+p}(t) &= F_p(t)\hat{y}(t) - I[w_p](t), \qquad p = 1, \cdots, n-1\,, \end{aligned} \tag{2.5}$$

then upon suitable integration by parts condition (2.1) becomes

$$\int_a^b z^{*[n]}(s)\{F_n(s)\hat{y}(s) - I[w_n](s)\}ds = 0 \ \text{ for } z \in \mathfrak{D}_0[a, b]\,. \tag{2.6}$$

By the more familiar form of the fundamental lemma we obtain the existence of a vector polynomial $P_{n-1}(t)$ of degree at most $n-1$ such that on $[a, b]$ we have

$$F_n(t)\hat{y}(t) - I[w_n](t) = P_{n-1}(t)\,. \tag{2.7}$$

Relation (2.7) clearly implies that $v_n(t) = I[w_n](t) + P_{n-1}(t)$ is a vector function of class $\mathfrak{A}_r[a, b]$ such that $v_n = F_n\hat{y}$ and

$$\begin{aligned} v'_n(t) &= w_n(t) + P'_{n-1}(t) \\ &= F_{n-1}(t)\hat{y}(t) - I[w_{n-1}](t) + P'_{n-1}(t)\,. \end{aligned}$$

Then $v_{n-1}(t) = I[w_{n-1}](t) - P'_{n-1}(t)$ is a vector function of class $\mathfrak{A}_r[a, b]$ such that $v_{n-1}(t) = F_{n-1}(t)\hat{y}(t) - v'_n(t)$, and iteration of this procedure leads successively to vector functions $v_{n-p}(t) = I[w_{n-p}](t) + (-1)^p P^{[p]}_{n-1}(t)$ of class $\mathfrak{A}_r[a, b]$ and satisfying the equations (2.3). In particular, $v_1(t) = I[w_1](t) + (-1)^{n-1}P^{[n-1]}_{n-1}(t)$ is a vector function of class $\mathfrak{A}_r[a, b]$ satisfying $v_1(t) = F_1(t)\hat{y}(t) - v'_2(t)$. Since $P^{[n-1]}_{n-1}(t)$ is constant it then follows that $0 = w_1(t) - v_1'(t) = F_0(t)\hat{y}(t) - v_1'(t)$, which is the equation (2.2).

Conversely, if $v_1(t), \cdots, v_n(t)$ are vector functions of class $\mathfrak{A}_r[a, b]$ satisfying with a vector function $y \in \mathfrak{D}[a, b]$ the system of equations (2.3), (2.4), then

$$\begin{aligned}\hat{z}^* F \hat{y} &= z^* v_1' + \sum_{j=1}^{n-1} z^{*[j]}[v_j + v_{j+1}'] + z^{*[n]} v_n \\ &= \{\sum_{\alpha=0}^{n-1} z^{*[\alpha]} v_{\alpha+1}\}'\end{aligned}$$

and consequently (2.1) holds.

For a vector function $y \in \mathfrak{D}[a, b]$, let the r-dimensional vector functions $u_1(t), \cdots, u_n(t)$ be defined as

$$u_k(t) = y^{[k-1]}(t) = (u_{\sigma;k}(t)), \qquad (k = 1, \cdots, n) . \tag{2.8}$$

Finally, let $u(t)$ and $v(t)$ denote the nr-dimensional vector functions $(u_\rho(t)), (v_\rho(t)), (\rho = 1, \cdots, nr)$, with

$$\begin{aligned} &u_{ir+\sigma}(t) = y_\sigma^{[i]}(t) = u_{\sigma;i+1}(t) , \\ &v_{ir+\sigma}(t) = v_{\sigma;i+1}(t), \qquad (i = 0, 1, \cdots, n-1; \sigma = 1, \cdots, r) . \end{aligned} \tag{2.9}$$

The above quasi-differential equation (2.2), or the associated system (2.3), (2.4), may then be written in the matrix form

$$\begin{aligned} &\mathscr{L}_1[u; v](t) \equiv -v'(t) + C(t)u(t) - D(t)v(t) = 0 , \\ &\mathscr{L}_2[u; v](t) \equiv u'(t) - A(t)u(t) - B(t)v(t) = 0 , \end{aligned} \tag{2.10}$$

where $A(t), B(t), C(t), D(t)$ are $(nr) \times (nr)$ matrix functions which will be written as partitioned matrices in $r \times r$ matrices as $A(t) = [A_{hk}(t)]$, $B(t) = [B_{hk}(t)], C(t) = [C_{hk}(t)], D(t) = [D_{hk}(t)], (h, k = 1, \cdots, n)$, with

$$\begin{aligned} &\text{(a)} \quad \begin{aligned} &A_{hk}(t) = \delta_{k,h+1}E_r, (h = 1, \cdots, n-1, k = 1, \cdots, n) \\ &A_{nk}(t) = -F_{nn}^{-1}(t)F_{n,k-1}(t), k = 1, \cdots, n ; \end{aligned} \\ &\text{(b)} \quad B_{hk}(t) = \delta_{hn}\delta_{nk}F_{nn}^{-1}(t), (h, k = 1, \cdots, n) ; \\ &\text{(c)} \quad C_{hk}(t) = F_{h-1,k-1}(t) - F_{h-1,n}(t)F_{nn}^{-1}(t)F_{n,k-1}(t), (h, k = 1, \cdots, n) ; \\ &\text{(d)} \quad \begin{aligned} &D_{hk}(t) = \delta_{h,k+1}E_r, (k = 1, \cdots, n-1, h = 1, \cdots, n) , \\ &D_{hn}(t) = -F_{n-1,n}(t)F_{nn}^{-1}(t), (h = 1, \cdots, n) . \end{aligned} \end{aligned} \tag{2.11}$$

It is to be noted that whenever hypothesis ($\mathfrak{H}$) is satisfied the differential system (2.10) in $(u; v)$ is identically normal; that is, if $u(t) \equiv 0, v(t)$ is a solution of (2.10) on a nondegenerate subinterval I_0 of I then $u(t) \equiv 0$, $v(t) \equiv 0$ throughout I. Indeed, if $u(t) \equiv 0$, $v(t)$ is a solution of (2.10) on I_0, then from the equation $\mathscr{L}_2[u, v](t) = 0$ it follows that $v_n(t) \equiv 0$ on I_0. In turn, from $\mathscr{L}_1[u, v](t) = 0$ it follows that $-v'_{h+1} + v_h = 0$, $(h = 1, \cdots, n-1)$, and consequently also $v_h(t) \equiv 0$ on I_0 for $h = 1, \cdots, n-1$. From the condition $u(t) \equiv 0$, $v(t) \equiv 0$ on I_0 it then follows that $u(t) \equiv 0$, $v(t) \equiv 0$ on I, thus establishing the identical normality of (2.10) on I.

Two distinct points t_1 and t_2 on I are said to be (*mutually*) *conjugate* with respect to (2.2), or with respect to (2.10), if there exists a solution $(u(t); v(t))$ of this latter system with $u(t) \not\equiv 0$ on the subinterval with endpoints t_1 and t_2, while $u(t_1) = 0 = u(t_2)$. Since $u_h(t) = y^{[h-1]}(t)$, $(h = 1, \cdots, n)$, this condition states that $t = t_1$ and $t = t_2$ are zeros of the vector function $y(t)$ of order greater than or equal to n. Moreover, if $t_1 \in I$ and $U(t)$, $V(t)$ are $(nr) \times (nr)$ matrix functions whose column vectors are solutions of (2.10), and satisfying the initial matrix conditions

$$U(t_1) = 0,\ V(t_1) = E_{nr}\ ,$$

then a value $t_2 \neq t_1$ is conjugate to t_1 if and only if $U(t_2)$ is singular. If $U(t_2)$ has rank $nr - q$, so that there are q linearly independent solutions $(u^{(\rho)}(t); v^{(\rho)}(t))$, $(\rho = 1, \cdots, q)$, of (2.10) satisfying $u^{(\rho)}(t_1) = 0 = u^{(\rho)}(t_2)$, then t_2 is said to be a *conjugate point to* t_1 *of order* q.

If I_0 is a nondegenerate subinterval of I such that no two distinct points of I_0 are conjugate with respect to (2.2), or (2.10), then this quasi-differential equation or differential system is said to be *disconjugate* or *non-oscillatory* on I_0.

Finally, it is to be noted that $y \in \mathfrak{D}[a, b]$ if and only if the (nr)-dimensional vector function

$$\eta(t) = (\eta_\rho(t)),\quad \text{with } \eta_{ir+\sigma}(t) = y_\sigma^{[i]}(t)\ , \qquad (\sigma = 1, \cdots, r;\ i = 0, 1, \cdots, n-1)\ , \tag{2.12}$$

has an associated (nr)-dimensional vector function $\zeta(t) = (\zeta_\rho(t)) \in \mathfrak{L}^2_{nn}[a, b]$ such that $\mathscr{L}_2[\eta, \zeta](t) = 0$ on $[a, b]$. In view of the form of $B(t)$, clearly only the last r components of $\zeta(t)$ are uniquely determined, with values

$$\zeta_{(n-1)r+\sigma}(t) = \sum_{\tau=1}^{r} F_{\sigma\tau;nn}(t) y_\tau^{[n]}(t),\ (\sigma = 1, \cdots, r)\ . \tag{2.13}$$

3. **Self-adjoint systems.** The quasi-differential system (2.2), or the equivalent first order system (2.10), is *self-adjoint* when the coefficient matrix function satisfies in addition to $(\mathfrak{H})$ the further condition

$$(\mathfrak{H}_1) \qquad F(t) \text{ is hermitian for } t \in I\ .$$

The hermitian character of $F(t)$ is equivalent to the condition that the component $r \times r$ matrix functions F_{ij} are such that $[F_{ij}(t)]^* = F_{ji}(t)$ for $t \in I$. In particular, the diagonal component matrix functions $F_{ii}(t)$ are hermitian on I. It follows readily that under hypotheses ($\mathfrak{H}$) and ($\mathfrak{H}_1$) the coefficient matrices of (2.10) are such that

$$(\mathfrak{H}_1') \qquad A(t) = D^*(t), B(t) = B^*(t), C(t) = C^*(t) ,$$

and (2.10) is of the canonical form of a linear Hamiltonian system for which one has a generalization of the Sturmian theory for real scalar linear homogeneous differential equations of the second order, (see, in particular, references [5]—[11] of the Bibliography).

Corresponding to the class $\mathfrak{D}[a, b]$ we shall denote by $\boldsymbol{D}[a, b]$ the linear vector space of (nr)-dimensional vector functions $\eta(t)$ which are of class $\mathfrak{A}_{nr}[a, b]$, and for which there are corresponding (nr)-dimensional vector functions $\zeta(t) \in \mathfrak{L}^2_{nr}[a, b]$ such that $\mathscr{L}_2[\eta, \zeta](t) = 0$ on this interval. The subspace of $\boldsymbol{D}[a, b]$ on which $\eta(a) = 0 = \eta(b)$ will be denoted by $\boldsymbol{D}_0[a, b]$. The fact that a $\zeta(t) \in \mathfrak{L}^2_{nr}[a, b]$ is thus associated with $\eta(t) \in \mathfrak{A}_{nr}[a, b]$ is denoted by the respective symbols $\eta \in \boldsymbol{D}[a, b]\colon \zeta$ and $\eta \in \boldsymbol{D}_0[a, b]\colon \zeta$.

When hypotheses ($\mathfrak{H}$) and ($\mathfrak{H}_1$) hold, and $y^{(p)}(t) \in \mathfrak{D}[a, b]$, $(p = 1, 2)$, let $\eta^{(p)}(t) = (\eta^{(p)}_\rho(t))$, $(p = 1, 2)$, be defined by corresponding equations (2.12), and $\zeta^{(p)}(t) = (\zeta^{(p)}_\rho(t))$ associated vector functions of class $\mathfrak{L}^2_{nn}[a, b]$ whose last r components are specified by equations corresponding to (2.13). The functional $J[y^{(1)}, y^{(2)} \mid a, b]$ defined by (2.1) is then expressible in terms of $\eta^{(p)}(t), \zeta^{(p)}(t)$ as

$$(3.1) \qquad J[\eta^{(1)}, \eta^{(2)} \mid a, b] = \int_a^b \{\zeta^{(2)*}B\zeta^{(1)} + \eta^{(2)*}C\eta^{(1)}\}dt ,$$

with the defining relations now equivalent to the condition that $\eta(t) = \eta^{(p)}(t)$, $\zeta(t) = \zeta^{(p)}(t)$, $(p=1, 2)$ satisfy the differential equation of restraint

$$(3.2) \qquad \mathscr{L}_2[\eta, \zeta](t) = \eta'(t) - A(t)\eta(t) - B(t)\zeta(t) = 0 .$$

As pointed out at the end of the preceding section, if $\eta \in \boldsymbol{D}[a, b]\colon \zeta$ the vector function ζ corresponding to a given η is not uniquely determined; however, the vector function $B\zeta$ is uniquely determined. Consequently if $\eta^{(p)} \in \boldsymbol{D}[a, b]$, $(p = 1, 2)$, then the value of the integral in (3.1) is independent of the particular corresponding $\zeta^{(p)}$, so that this integral does indeed define a functional of $\eta^{(1)}, \eta^{(2)}$. Moreover, in view of the hermitian character of the coefficient matrix functions B and C, $J[\eta^{(1)}, \eta^{(2)} \mid a, b]$ is an hermitian functional on $\boldsymbol{D}[a, b] \times \boldsymbol{D}[a, b]$. In particular, $J[\eta \mid a, b] = J[\eta, \eta \mid a, b]$ given as

$$(3.3) \qquad J[\eta \mid a, b] = \int_a^b \{\zeta^*B\zeta + \eta^*C\eta\}dt$$

is a real-valued functional on $D[a, b]$.

For a system (2.10) which satisfies hypotheses ($\mathfrak{H}$) and ($\mathfrak{H}_1$) it follows readily that if $y^{(p)} = (u^{(p)}; v^{(p)})$, $(p = 1, 2)$, are solutions of this system then the function

$$(u^{(1)}, v^{(1)} \mid u^{(2)}, v^{(2)})(t) = v_2^*(t)u_1(t) - u_2^*(t)v_1(t)$$

is constant on I. If two solutions of this system are such that this constant is zero, these solutions are said to be (*mutually*) *conjoined*. If $Y(t) = (U(t); V(t))$ is a $(2nr) \times q$ matrix whose column vectors are linearly independent solutions of (2.10) which are mutually conjoined, then these solutions form a basis for a *conjoined family of solutions of dimension* q, consisting of these solution of (2.10) which are linear combinations of the column vector functions. In general, (see, for example, Reid [7, Sec. 2; 11, Sec. VII. 2]), the maximal dimension of a conjoined family of solutions of (2.10) is nr, and a given conjoined family of dimension less than nr is contained in a conjoined family of dimension nr.

If $[a, b]$ is a nondegenerate compact subinterval of I, then the symbol $\mathfrak{H}_+[a, b]$ will signify the condition that the functional $J[y \mid a, b]$ is positive definite on $\mathfrak{D}_0[a, b]$; that is, for $y \in \mathfrak{D}_0[a, b]$ we have $J[y \mid a, b] \geqq 0$, with the equality sign holding only if $y(t) = 0$ on $[a, b]$. This condition may be equally well stated as the nonnegativeness of the functional (3.3) on the vector space $D_0[a, b]$, with $J[\eta \mid a, b] = 0$ for an $\eta \in D_0[a, b]: \zeta$ only if $\eta(t) = 0$ and $B(t)\zeta(t) = 0$ on $[a, b]$.

From the basic result for canonical Hamiltonian systems concerning disconjugacy on a compact interval, (see, for example, Reid [10, Theorem 5.1] or Reid [11, Sec. VII. 4]), we have the following criterion.

THEOREM 3.1. *If hypotheses* ($\mathfrak{H}$) *and* ($\mathfrak{H}_1$) *are satisfied, and* $[a, b]$ *is a nondegenerate compact subinterval of* I, *then* $\mathfrak{H}_+[a, b]$ *holds if and only if* $F_{nn}(t) > 0$ *for* t a.e. *on* $[a, b]$, *together with one of the following conditions*:

(i) (2.10) *is disconjugate on* $[a, b]$;

(ii) *there exists a conjoined family of solutions* $Y(t) = (U(t); V(t))$ *of* (2.10) *of dimension* nr *with* $U(t)$ *nonsingular on* $[a, b]$.

4. A disconjugacy criterion for (2.2). Suppose that hypothesis ($\mathfrak{H}$) is satisfied by the coefficient matrix function $F(t)$ of (2.2) on an interval I, and that $[a, b]$ is a nondegenerate subinterval of I such that $t = a$ and $t = b$ are mutually conjugate with respect to the equation (2.2). Let $y(t)$ be a solution of (2.2) such that $y(t) \not\equiv 0$ on $[a, b]$, and $y^{[\alpha]}(a) = 0 = y^{[\alpha]}(b)$, $(\alpha = 0, 1, \cdots, n-1)$. Then $y \in \mathfrak{D}_0[a, b]$, and in view of Lemma 2.1 we have that

$$(4.1)\qquad 0 = J[y, y \mid a, b] = \int_a^b \hat{y}^*(t)F(t)\hat{y}(t)dt\,.$$

From this relation it follows that $\Re e F(t) = \frac{1}{2}\{F(t) + F^*(t)\}$ and $\Im m F(t) = \frac{1}{2}\sqrt{-1}\{F^*(t) - F(t)\}$ are hermitian matrix functions. If λ_0, λ_1 are real constants then

$$(4.2)\qquad F(t; \lambda) = \lambda_0 \Re e F(t) + \lambda_1 \Im m F(t)$$

is an hermitian matrix function such that the given solution $y(t)$ of (2.2) satisfies the condition

$$(4.3)\qquad \int_a^b \hat{y}^*(t)F(t; \lambda)\hat{y}(t)dt = 0\,.$$

Now if $F(t; \lambda)$ has the partitioned representation $[F_{ij}(t; \lambda)]$, $(i, j = 0, 1, \cdots, n)$ in terms of $r \times r$ matrix functions, and $F(t; \lambda)$ satisfies hypothesis ($\mathfrak{H}$) with $F_{nn}(t; \lambda) > 0$ for t a.e. on $[a, b]$, then the conclusion (i) of Theorem 3.1 applied to the self-adjoint matrix differential equation $\mathfrak{L}[y: F(\cdot\,; \lambda)](t) = 0$ implies that this equation fails to be disconjugate on $[a, b]$. Consequently, we have the following result, corresponding to that of § 5 of Reid [7] for a second order linear homogeneous matrix differential equation. The reader is also referred to Hartman and Wintner [3] for a similar treatment of disconjugacy criteria for second order vector differential systems. For a consideration of non-self-adjoint differential equations of even order by a method which is similar in basic idea, but different in specific detail, see Kreith [4].

THEOREM 4.1. *Suppose that hypothesis* ($\mathfrak{H}$) *is satisfied by the coefficient matrix function* $F(t)$ *of* (2.2) *on an interval* I, *and for a given nondegenerate subinterval* $[a, b]$ *of* I *there exist real constants* λ_0, λ_1 *such that on* $[a, b]$ *the matrix function* $F(t; \lambda) = [F_{ij}(t; \lambda)]$, $(i, j = 0, 1, \cdots, n)$, *of* (4.2) *satisfies hypothesis* ($\mathfrak{H}$) *and* $F_{nn}(t; \lambda) > 0$ *for* t a.e. *on* $[a, b]$. *Then whenever the self-adjoint quasi-differential equation* $\mathfrak{L}[y: F(\cdot\,; \lambda)](t) = 0$ *is disconjugate on* $[a, b]$, *the system* (2.2) *is also disconjugate on* $[a, b]$.

It is to be emphasized that in the above theorem the constant multipliers λ_0, λ_1 may depend upon the subinterval $[a, b]$, and that any criterion of disconjugacy for the associated self-adjoint equation $\mathfrak{L}[y: F(\cdot\,; \lambda)](t) = 0$ yields a sufficient condition for disconjugacy of the original equation (2.2). In particular, the results of Reid [9, Sec. 4] for scalar quasi-differential equations of even order, and their analogues for vector equations, provide sufficient conditions for (2.2) to be disconjugate on a non-compact interval (t_1, ∞).

5. A special canonical form. Attention will be directed now to a linear differential expression of order m in the r-dimensional vector function $y(t) = (y_\sigma(t))$ of the form

$$\mathscr{L}[y](t) = \sum_{\mu=0}^{m} P_\mu(t) y^{[\mu]}(t) \tag{5.1}$$

where the $r \times r$ coefficient matrix functions $P_\mu(t) \equiv [P_{\sigma\tau;\mu}(t)]$ are supposed to be of class $\mathfrak{L}_{rr}[a, b]$ for arbitrary compact subintervals $[a, b]$ of a given interval I on the real line. It is to be emphasized that in the discussion leading to the result of Theorem 5.1 we do not require the leading coefficient matrix $P_m(t)$ to be nonsingular, or even to be nonzero. The purpose of this section is to present for vector differential operators of the form (5.1) an analogue of the results of Reid [8] for linear scalar differential equations, and to note the particular form of the disconjugacy criterion of § 4 for the involved canonical form.

For a given compact subinterval $[a, b]$ of I, let T_0 denote a corresponding differential operator with domain $\mathfrak{C}^m_{r,0}]a, b]$ and value $T_0 y = \mathscr{L}[y]$. If $\mathfrak{D}^*$ denotes the totality of r-dimensional vector functions $z \in \mathfrak{L}_{rr}[a, b]$ with $P^*_\mu(t)z(t) \in \mathfrak{L}_{rr}[a, b]$, $(\mu = 0, 1, \cdots, m)$, and for which there exists a corresponding $f_z \in \mathfrak{L}_r[a, b]$ such that

$$\int_a^b z^* \mathscr{L}[y] dt = \int_a^b f_z^* y dt, \text{ for } y \in C^m_{r,,0}[a, b] , \tag{5.2}$$

then the operator T_0^* with domain $\mathfrak{D}^*$ and value $T_0^* z = f_z$ is termed the adjoint of T_0. In particular, if $P_\mu \in \mathfrak{C}^\mu_{rr}[a, b]$ and $P_m(t)$ is nonsingular for $t \in [a, b]$, then by classical results, (see, for example, Reid [11, Sec. III. 9]) we have that $\mathfrak{D}^* = \mathfrak{A}^m_r[a, b]$, and for $z \in \mathfrak{A}^m_r[a, b]$ the value of $T_0^* z$ is given by the Lagrange adjoint $\sum_{\mu=0}^m (-1)^\mu \{P^* z\}^{[\mu]}$. Of special importance is the Hilbert space case that occurs when $P_\mu \in \mathfrak{L}^2_{rr}[a, b]$, $(\mu = 0, 1, \cdots, m)$, and analogous to the above defined T_0 one considers the operator with values $\mathscr{L}[y]$ on the domain of functions $y \in \mathfrak{A}^m_{r,0}[a, b]$ such that $\mathscr{L}[y] \in \mathfrak{L}^2_r[a, b]$.

Of particular significance for the present considerations are differential expressions $\mathscr{L}[y] = \Lambda_q[y; P]$ where P is an $r \times r$ matrix function, and

$$\begin{aligned} &\Lambda_0[y; P](t) = P(t)y(t),\ \Lambda_{2p}[y; P](t) = \{P(t)y^{[p]}(t)\}^{[p]} , \\ &\Lambda_{2p-1}[y; P](t) = \{P(t)y^{[p-1]}(t)\}^{[p]} + \{P(t)y^{[p]}(t)\}^{[p-1]} , \quad (p = 1, 2, \cdots), \end{aligned} \tag{5.3}$$

with the understanding that in the definition of Λ_{2p} and Λ_{2p-1} the involved matrix function P is of class $\mathfrak{A}^p_r[a, b]$. If for (5.1) we have $\mathscr{L}[y] = \Lambda_m[y;P]$, $(m \geqq 1)$, then the fact that $\mathfrak{A}^m_r[a, b] \subset \mathfrak{D}^*$ and $T_0^* z = \Lambda_m[z; (-1)^m P^*]$ for $z \in \mathfrak{A}^m_r[a, b]$ is a direct consequence of the well-known equation

$$z^*\Lambda_m[y; P] - (-1)^m\{\Lambda_m[z; P^*]\}^*y = \{K_n[y, z; P]\}'$$

for arbitrary y, z of $\mathfrak{A}_r^m[a, b]$, where $K_n[y, z; P]$ is the so-called bilinear concomitant of the form $\sum_{\mu,\nu=1}^m z^{*[\nu-1]}(t)K_{\nu\mu}(t; P)y^{[\mu-1]}(t)$.

Let $e^{(k)}$ denote the r-dimensional unit vector $e^{(k)} = (\delta_{hk}), (h = 1, \cdots, r)$, and designate by $g_\lambda(t), (\lambda = 0, 1, \cdots)$ the particular scalar polynomials $g_0(t) \equiv 1$, $g_\lambda(t) = t^\lambda/\lambda!$, $(\lambda = 1, 2, \cdots)$. Moreover, let k_j equal $j/2$ or $(j + 1)/2$ according as j is even or odd. Corresponding to Theorem 3.2 of Reid [8], we now have the following representation theorem.

THEOREM 5.1. *Suppose that* $\mathscr{L}[y]$ *is given by* (5.1) *with* $P_\mu \in \mathfrak{L}_{rr}[a, b]$, $(\mu = 0, 1, \cdots, m)$, *and the differential operator* T_0 *is defined as specified above. If for* $h = 1, \cdots, r$ *and* $\lambda = 0, 1, \cdots, k_m - 1$ *the vector functions* $g_\lambda(t)e^{(h)}$ *belong to* $\mathfrak{D}^*$, *then there exist matrix functions* $\Pi_\mu(t) \in \mathfrak{A}_r^{k_\mu}[a, b]$, $(\mu = 0, 1, \cdots, m)$, *such that*

$$\mathscr{L}[y](t) = \sum_{\mu=0}^m \Lambda_\mu[y; \Pi_\mu](t) \text{ for } y \in \mathfrak{A}_r^m[a, b] ; \tag{5.4}$$

also $\mathfrak{A}_r^m[a, b] \subset \mathfrak{D}^*$ *and*

$$(T_0^*z)(t) = \mathscr{L}^*[z](t) = \sum_{\mu=0}^m \Lambda_\mu[z; (-1)^\mu \Pi_\mu^*](t), \text{ for } z \in \mathfrak{A}_r^m[a, b] .$$

Moreover, $\Pi_\mu \in \mathfrak{A}_r^{k_\mu,2}[a, b]$, $(\mu = 0, 1, \cdots, m)$, *if and only if*

$$T_0^*\{g_\lambda e^{(h)}\} \in \mathfrak{L}_r^2[a, b], (h = 1, \cdots, r; \lambda = 0, 1, \cdots, k_m - 1) ,$$

and $P_\mu \in \mathfrak{L}_{rr}^2[a, b]$, $(\mu = 0, 1, \cdots, m - k_m)$.

The result of the above theorem is a direct consequence of Theorem 3.2 of Reid [8] applied to the associated scalar differential operators

$$\mathscr{L}_{hk}[u](t) = \sum_{\mu=0}^m \{e^{(h)*}P_\mu(t)e^{(k)}\}u^{[\mu]}, (h, k = 1, \cdots, r) ,$$

and expressing in matrix form the scalar results thus obtained.

If for a differential expression (5.1) with $m = 2n$ we have that $\mathscr{L}[y]$ is given in a corresponding form (5.4) then the differential equation $\mathscr{L}[y](t) = 0$ is of the form (2.2) with the $(n + 1)r \times (n + 1)r$ matrix function $F(t)$ expressible in partitioned form $[F_{ij}(t)]$ with F_{ij}, $(i, j = 0, 1, \cdots, n)$, the $r \times r$ matrix functions specified for $i, j = 0, 1, \cdots, n$ as

$$\begin{aligned} F_{ij}(t) &= 0, \text{ if } |i-j| > 1 ; \\ F_{ij}(t) &= (-1)^i \Pi_{i+j}(t), \text{ if } |i-j| \leqq 1. \end{aligned} \tag{5.5}$$

For such a matrix function $F(t)$ we have that $\Re e\, F(t) = G(t) \equiv [G_{jk}(t)]$, $(j, k = 0, 1, \cdots, n)$, where each G_{jk} is an $r \times r$ matrix function specified for $j, k = 0, 1, \cdots, n$ as

$$\begin{aligned} G_{jk}(t) &= 0, \text{ if } |j-k| > 1 ; \\ G_{jj}(t) &= (-1)^j \Re e\, \Pi_{2j}(t) ; \\ G_{j,j+1}(t) &= \sqrt{-1}(-1)^j \Im m\, \Pi_{2j+1}(t) ; \\ G_{j,j-1}(t) &= \sqrt{-1}(-1)^j \Im m\, \Pi_{2j-1}(t) . \end{aligned} \tag{5.6}$$

Correspondingly, $\Im m\, F(t) = H(t) = [H_{jk}(t)]$, $(j, k = 0, 1, \cdots, n)$, where each H_{jk} is an $r \times r$ matrix function specified for $j, k = 0, 1, \cdots, n$ as

$$\begin{aligned} H_{jk}(t) &= 0, \text{ if } |j-k| > 1 ; \\ H_{jj}(t) &= (-1)^j \Im m\, \Pi_{2j}(t) ; \\ H_{j,j+1}(t) &= \sqrt{-1}(-1)^{j+1} \Re e\, \Pi_{2j+1}(t) ; \\ H_{j,j-1}(t) &= \sqrt{-1}(-1)^{j+1} \Re e\, \Pi_{2j-1}(t) . \end{aligned} \tag{5.7}$$

As an application of the result of Theorem 4.1 with multipliers $\lambda_0 = 1$, $\lambda_1 = 0$, or $\lambda_0 = -1$, $\lambda_1 = 0$, one has the following special criterion for disconjugacy of a differential equation (2.2).

THEOREM 5.2. *Suppose that* (5.1) *with* $m = 2n$ *is expressible in the form* (5.4) *with coefficient matrices* $\Pi_0(t), \cdots, \Pi_{2n}(t)$ *satisfying the conditions given in Theorem* 5.1, *while* $\Im m\, \Pi_{2j-1}(t) = 0$, $j = 1, \cdots, n$, *and on a given nondegenerate compact subinterval* $[a, b]$ *of* I *we have either* $\Re e\, \Pi_{2n}(t) > 0$ *or* $\Re e\, \Pi_{2n}(t) < 0$. *If the associated self-adjoint differential system*

$$\mathscr{L}_1[y](t) = \sum_{j=0}^{n} \Lambda_{2j}[y; \Re e\, \Pi_{2j}](t) = 0 \tag{5.8}$$

is disconjugate on $[a, b]$ *then the differential equation* (5.4) *is also disconjugate on this subinterval.*

In particular, the functions $\Im m\, \Pi_{2j-1}(t)$, $(j = 1, \cdots, n)$ are all zero in the scalar case when $r = 1$, and the coefficients of (5.1) are real-valued.

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