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PLANAR SURFACES IN KNOT MANIFOLDS

HOWARD LAMBERT

Let K be a knot manifold, that is the 3-sphere S^3 minus an open regular neighborhood of a polygonal simple closed curve in S^3. Whether K can be embedded in S^3 differently or in a homotopy 3-sphere different from S^3 (if such really exist) leads in a natural way to the question of which planar surfaces can be embedded in K. Geometric conditions are imposed on the embedded planar surfaces which are sufficient to imply that K is not knotted, that is K is homeomorphic to a disk cross S^1.

1. **Introduction and definitions.** In this paper we consider some geometric problems motivated by the so called "Property P" [3] of a knot manifold K. In particular, we will investigate whether there is a continuous map f of a planar surface S (compact, submanifold of E^2) into K such that $f(\text{Int } S) \subset \text{Int } K$, $f \mid BdS \subset BdK$ and f is 1-1 on each component $\Delta_1, \cdots, \Delta_n$ of BdS (each Δ_i is a simple closed curve (scc)). We are interested in the cases of either *I*. f is 1-1 and no $f(\Delta_i)$ is contained in a disk on BdK or *II*. S is connected, $f(\Delta_1)$ is parallel to $K's$ longitude and each $f(\Delta_i)$, $2 \leqq i \leqq n$, is parallel to a fixed exotic homotopy killer of K (definitions below). For example, if $\pi_1(K) \neq Z$, Case II holds and the homotopy killer of K is exotic, then we would have a counter-example to "Property P". Conversely, if we had a K violating "Property P", then there exists $f: S \rightarrow K$ as in Case II and each $f(\Delta_i)$, $2 \leqq i \leqq n$, is parallel to an exotic homotopy killer of K. In Theorem 1 we develop a geometric condition which is sufficient to imply K is unknotted and in Theorem 2 we develop a related geometric condition which is sufficient to imply K has "Property P".

Everything here is taken to be polyhedral. Definitions for such terminology as "properly embedded" and "boundary-irreducible" may be found in [17]. A knot manifold K is a submanifold of S^3 such that $Cl(S^3\text{-}K)$ is a solid torus $T = S^1 \times D^2$. On BdK, but not separating BdK, there exists a unique (up to isotopy on BdK) scc homologous to zero (Mod Z) in K, called K's longitude. A meridian of K is $x \times BdD^2, x \in S^1$, and we call it $K's$ ordinary homotopy killer. Any other scc on BdK which kills $\pi_1(K)$ (by attaching a 3-cell along this scc) will be called an exotic homotopy killer. An exotic homotopy killer is of the form $m(l)^n$, where m is the meridian of K, l is the longitude of K and $n \neq 0$. If K has no exotic homotopy killer, then K is said to have "Property P". Some results on "Property P" have been obtained by R. Bing and J. Martin [3], A. C. Connor [4], F.

Gonzales [9], J. Hempel [12] and J. Simon [15]. Results about the existence of surfaces (singular or not) in 3-manifolds have been obtained by W. R. Alford [2], C. Feustel [5], C. Feustel and N. Max [6], W. Heil [11], J. Hempel and W. Jaco [13], H. Lambert [14], J. Simon [16], and F. Waldhausen [18] among others.

2. Results for Case I. Suppose $f: S \to K$ as in Case I (since f is a homeomorphism, identify S with $f(S)$) and that each Δ_i is not parallel to K's ordinary homotopy killer. Let X_n be the 3-manifold obtained by adding $T(= \mathrm{Cl}(S^3\text{-}K))$ to a regular neighborhood, $S \times [0, 1]$, of S in K (see Figure 1 for a picture of an X_3 with S connected).

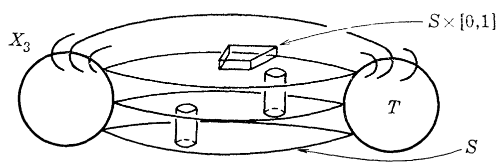

FIGURE 1

Recall from the first paragraph that n is the number of boundary components of S and picture X_n as being obtained by attaching $BdS \times [0, 1]$ to n disjoint annuli $A_1, \cdots, A_n$ on BdT.

LEMMA 1. *X_n is boundary-irreducible.*

Proof. Assume S is connected, as the proof is similar if not. Suppose BdX_n is compressible, i.e., there exists a properly embedded disk D in $X_n(BdD \subset BdX_n$ and $\mathrm{Int}\, D \subset \mathrm{Int}\, X_n)$ such that BdD does not bound a disk in BdX_n. Put D in general position relative to $\bigcup_{i=1}^n A_i$. After removing simple closed curves of $D \cap \bigcup_{i=1}^n A_i$ which bound disks in $\bigcup_{i=1}^n A_i$, it follows that there exists a subdisk D' of D such that either 1. $D' = D$ and $D' \cap (\bigcup_{i=1}^n A_i) = \varnothing$, 2. $BdD' \subset A_i$ and $\mathrm{Int}\, D' \cap (\bigcup_{i=1}^n A_i) = \varnothing$ or 3. BdD' consists of two arcs, one in BdX_n and the other in A_i, and $\mathrm{Int}\, D' \cap (\bigcup_{i=1}^n A_i) = \varnothing$. In Case 1, if $D \subset S \times [0, 1]$, then it follows by Proposition 3.1 of [17] that BdD bounds a disk in BdX_n, contradiction. If $D \subset T$, then either each $f(\Delta_i)$ is parallel to K's ordinary homotopy killer, contradiction, or BdD bounds a disk in BdX_n, contradiction. Case 2 cannot occur since the center line of each A_i is not homologous to zero in either $S \times [0, 1]$ or T. In Case 3 if $D' \subset S \times [0, 1]$, the arc $BdD' \cap A_i$ intersects one boundary component of A_i and, by using Proposition 3.1 of [17], the number of components of $D \cap (\bigcup_{i=1}^n A_i)$ can be reduced. Similarly, in Case 3 for $D' \subset T$ it follows that the number of components of $D \cap (\bigcup_{i=1}^n A_i)$ can be reduced (assume $n > 1$, since X_1 is a 3-cell). All three cases now imply D could not have existed and therefore X_n is boundary-irreducible.

Suppose M is a 3-manifold. If D is a disk properly embedded in M such that BdD does not bound a disk on BdM, then we say M has a handle D. More generally, if S is a connected planar 2-manifold properly embedded in M such that 1. n, the number of boundary components $\Delta_1, \cdots, \Delta_n$ of S, is odd and 2. there exists an annulus $A = S^1 \times [1, n]$ on BdM such that each $\Delta_i = S^1 \times \mathrm{i}, 1 \leqq i \leqq n$, then call A handle-like in M.

LEMMA 2. *Suppose M is a 3-manifold with a handle D and a handle-like annulus A. Then M has a handle D_0 such that $D_0 \cap A = \emptyset$ and A is handle-like in M-D_0.*

Proof. The case $n = 1$ is easy. Suppose then that $n \geqq 3$ (and n odd) but that $BdD \cap A = \emptyset$ (we may need to pull BdD off A by an isotopy in BdM to achieve this). If S is in general position relative to D, we may choose a subdisk D' of D such that $BdD' \subset S$ and $\mathrm{Int}\, D' \cap S = \emptyset$. Now cut S at BdD' and fill in the resulting two holes by disks close to but on opposite sides of D' to obtain two planar surfaces, at least one of which, S', has an odd number of boundary components ($BdS' \subset BdS$) and $S' \cap D$ has fewer components than $S \cap D$. Repeating this argument a finite number of times yields $D_0(= D)$ in this special case.

Now suppose $BdD \cap A (\neq \emptyset)$ consists of arcs, each connecting one boundary component of A to its other, and that $D \cap S$ consists of arcs only (simple closed curves may be removed as in the special case). Note that each arc of $D \cap S$ starts and ends in $BdD \cap A$ and that n such arcs start at each arc of $BdD \cap A$. If an arc of $D \cap S$ starts and ends on the same arc of $BdD \cap A$, then there exists a subdisk D' of D such that $D' \cap A$ is an arc on BdD', the complementary arc of BdD' is contained in $D \cap S$ and $\mathrm{Int}\, D' \cap S = \emptyset$. Now cut S at $BdD' \cap S$ and attach two disks close to but on opposite sides of D'. The resulting S' then contains one boundary component which bounds a disk in A. Fill in this boundary component to obtain S'' such that S'' is planar, $BdS'' \subset BdS$ and S'' has n-2 boundary components.

If no arc of $D \cap S$ has both its end points in the same arc of $BdD \cap A$, then, in D, there are two adjacent arcs Q_1, Q_2 of $BdD \cap A$ (relative to BdD) such that $Q_1 \times (n + 1)/2 (= Q_1 \cap \Delta(n+1)/2)$ is connected to $Q_2 \times (n + 1)/2$ by an arc γ_0 of $D \cap S$. Since S is orientable and γ_0 has both ends in the same boundary component of S, namely $\Delta(n + 1)/2$, γ_0 does not separate $Q_1 \times 1$ from $Q_2 \times 1$ in D. Hence there is an arc of $D \cap S$ with both ends in Δ_1 (or Δ_n). Since all arcs of $D \cap S$ with one end point in $\Delta_1 \cup \Delta_n$ have both end points in $\Delta_1 \cup \Delta_n$, we may ignore all these arcs and repeat the above argument $((n + 1)/2) - 2$

times more to conclude that for each boundary component Δ_i of S there exists an arc of $S \cap D$ with both endpoints in Δ_i. Since S is planar, one of these arcs together with an arc on BdS bounds a disk D' in S such that $\text{Int } D' \cap D = \emptyset$. Now cut D at $BdD' \cap D$ and attach two disks close to but on opposite sides of D' to obtain two disks properly embedded in M and at least one of them is a handle of M which intersects A in fewer arcs than D does. Applying the various cases above a finite number of times yields the desired handle D_0.

It follows as a corollary to Lemma 2 that if M is a cube with handles, then $n = 1$, i.e., the center line of A bounds a disk in M.

THEOREM 1. *Suppose $f: S \to K$ as in Case I, that $f(S)$ has at least two components S_1, S_2 such that each has an odd number of boundary components and that there exists an annulus on BdK whose boundary separates BdS_1, from BdS_2 in $BdK(= S^1 \times S^1)$. Then K is unknotted* (homeomorphic to $T = \text{Cl}(S^3 - K)$).

Proof. Since S_1 and S_2 have an odd number of boundary components and no boundary component of $f(S)$ is contained in a disk on BdK, it follows that each boundary component of $f(S)$ is parallel to K's longitude. Let A_1, A_2 be disjoint annuli in BdK, parallel to K's longitude, such that $BdS_1 \subset A_1$ and $BdS_2 \subset A_2$. Let U_1, U_2 be disjoint regular neighborhoods of $S_1 \cup A_1$, $S_2 \cup A_2$ in K, respectively. Then U_1 is homeomorphic to an X_n of Lemma 1; hence it is boundary-irreducible. Similarly U_2 is boundary-irreducible and by [7] it follows that there is a properly embedded disk D in $\text{Cl}(S^3 - U_1 \cup U_2)$ such that BdD does not bound a disk in $Bd(\text{Cl}(S^3 - U_1 \cup U_2)) = BdU_1 \cup BdU_2$. Suppose, without loss of generality, that $BdD \subset BdU_1$. Since $D \cap U_2 = \emptyset$, it follows that we may cut D and fill in on the two annuli components of $\text{Cl}(BdK - U_1 \cup U_2)$ so as to assume $D \cap T = \emptyset$ (note that obtaining $D \cap T = \emptyset$ involves assuming K is knotted). Now add to U_1 a regular neighborhood of D in $\text{Cl}(S^3 - U_1 \cup U_2)$ to obtain a new 3-manifold U_1' (if BdD separates BdU_1 also add the component of $\text{Cl}(S - U_1) - D$ not containing U_2 to U_1'). Note that the genus of BdU_1' is less than the genus of BdU_1. Repeat these steps on U_1', U_2. But now it is possible that U_1' is not boundary-irreducible. If $D \subset U_1'$, Lemma 2 says we may assume $D \cap S_1 = \emptyset$ and cut out an open regular neighborhood of D in U_1' to obtain the new U_1''. Again the genus of BdU_1'' is less than the genus of BdU_1'. Continuing, we eventually conclude that there is a 3-cell B in K such that $B \cap BdK$ is either A_1 or A_2 and hence K is unknotted.

3. Results for Case II. Suppose $f: S \to K$ as in Case II and, in addition, assume each $f(\Delta_i)$, $2 \leqq i \leqq n$, is parallel to a fixed exotic homotopy killer of K. We may also assume that f is in general position, that is the singularities of f on S consist of pairwise disjoint arcs with endpoints in BdS and f sews these arcs together in pairs, each pair forming a single arc in the image (see W. Haken's [10] to see how to eliminate branch points and triple points at the expense of increasing n). There are two types of such arcs of singularities, Type α where the arc runs from Δ_1 to some Δ_i, $i \neq 1$, and Type β where the arc has both endpoints in Δ_1 and its associated arc runs from Δ_i to Δ_j, $i, j > 1$ and $i \neq j$. In [10], Haken shows that we can always make every arc of Type α. Unfortunately, from the point of view of studying "Property P", Type α arcs seem to be particularly intractible. If all arcs are of Type β, then K corresponds to being like a ribbon knot [8, p. 172] relative to its exotic homotopy killer. It is a very particular case of Type β arcs we wish to look at. Suppose S contains a pair of arcs β_1, β_2 of Type β sewed together by f where $Bd\beta_1 \subset \Delta_1$ and one of the two components of $S - \beta_1$ contains no other arc of singularity but β_2. Denote the closure of this component of $S - \beta_1$ by Γ (Γ is a disk with 2 holes, see Figure 2 for a picture of $f(\Gamma) \cup T$).

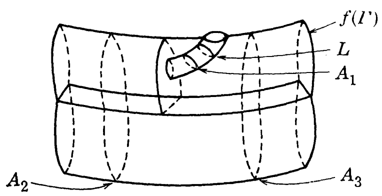

FIGURE 2

THEOREM 2. *Suppose* 1. $f: S \to K$ *as in Case II,* 2. S *contains two* (disjoint) Γ*'s,* Γ_1 *and* Γ_2*, and* 3. n*, the number of boundary components of* S*, is minimal with respect to property* 1. *Then* K *is unknotted.*

Proof. First assume $n > 1$, since $n = 1$ implies, by Dehn's Lemma, that K is unknotted. Let A_1, A_2 be disjoint annuli on BdK such that $f(\Gamma_1) \cap BdK \subset A_1$ and $f(\Gamma_2) \cap BdK \subset A_2$. Let U_1, U_2 be disjoint regular neighborhoods of $A_1 \cup f(\Gamma_1)$, $A_2 \cup f(\Gamma_2)$ in K, respectively. We claim both U_1 and U_2 are homeomorphic to an X_3 of Lemma 1. (To see this we have indicated in Figure 2 where the three annuli A_1, A_2 and A_3 of Lemma 1 would be located in U_1.) By Lemma 1, U_1, U_2 are boundary irreducible and we follow the technique used in the proof of Theorem 1 to conclude that there is a disk D properly embedded in $\mathrm{Cl}(S^3 - U_1 \cup U_2)$ such that $BdD \subset U_1$ (or U_2) and $D \cap T = \varnothing$. As in the proof of Theorem 1, we add a regular neighborhood of D to U_1 to obtain U_1'. Now BdU_1' is a torus, $S^1 \times S^1$. By [1], the closure of one complementary domain of $S^1 \times S^1$ in S^3 is a solid torus T'. If $f(\Gamma_1) \subset T'$, then the scc

L of Figure 2 can be shrunk to a point in T' since homology and homotopy are the same in T'. (To see that L is homologous to zero Mod Z, note that L bounds an orientable surface in $f(\Gamma_1) \cup A_1$.) Suppose $f(\Gamma_2) \subset T'$. Then $T' - (\text{Int } T \cup \text{Int } A_1)$ is a solid torus, $L \sim 0$ Mod Z in $f(\Gamma_2) \cup A_2 \subset T' - (\text{Int } T \cup \text{Int } A_1)$ and hence the L of $f(\Gamma_2)$ can be shrunk to a point in $T' - \text{Int } T$. In either case, by using the singular disk that L bounds, it follows that there is an $f' \colon S' \to K$ as in Case II with $n' < n$, contradicting property 3 of the hypothesis. Then $n = 1$ and K is unknotted.

4. A question. Suppose $f \colon S \to K$ as in Case II, each $f(\Delta_i)$, $2 \leqq i \leqq n$, is parallel to a fixed exotic homotopy killer of K and each arc of singularity in S is of Type β. We can say in general that there exist disjoint Γ_1, Γ_2 in S as before but now Γ_1, Γ_2 contain holes whose boundaries go parallel to the exotic homotopy killer under f. It does not seem likely that K is knotted if Γ_1, Γ_2 exist, but the author could not show this. We conclude then with the following

Question. If K does not have "Property P" and all singularities of the resulting $f \colon S \to K$ are of Type β, then is K unknotted?

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UNIVERSITY OF IOWA