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ON CERTAIN POSET AND SEMILATTICE HOMOMORPHISMS

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In this paper a coordinatizing semigroup is used to define and characterize certain homomorphisms on a bounded poset or semilattice. These homomorphisms are determined by their kernels and in the semilattice case the ideals which occur as such kernels are characterized.

1. Introduction. In [4] B. J. Thorne characterized certain congruence relations on a bounded lattice by looking at AP homomorphisms on a coordinatizing Baer semigroup. We intend to carry out a similar procedure for bounded posets and semilattices. It will turn out that one of our semilattice results gives Thorne's central result as a corollary.

Our notation will be that of [4]. If S is a semigroup with 0 and $A \subseteqq S$ we define $L(A) = \{x \in S; xa = 0 \text{ for all } a \in A\}$, $R(A) = \{x \in S; ax = 0 \text{ for all } a \in A\}$, $LR(A) = L(R(A))$, $RL(A) = R(L(A))$, and so forth. If $x \in S$ we write $L(\{x\}) = L(x)$ and $R(\{x\}) = R(x)$. We define $\mathscr{L}(S) = \{L(x); x \in S\}$ and $\mathscr{R}(S) = \{R(x); x \in S\}$ and say that S *coordinatizes* a poset P in case $P \cong \mathscr{L}(S)$ when $\mathscr{L}(S)$ is partially ordered by set inclusion.

The coordinatization machinery which we will use is developed in [2]. The following is a summary of the relevant material.

DEFINITION 1.1. A semigroup S with 0 and 1 will be called a *pre-Baer semigroup* in case, for each $x \in S$, there exist elements $x^r, x^l \in S$ such that $LR(x) = L(x^r)$ and $RL(x) = R(x^l)$.

Recall that a map ϕ of a poset P into itself is *residuated* if the inverse image of a principal ideal is again a principal ideal or, equivalently, if ϕ is isotone and there is another isotone map ϕ^+ (called a *residual* map) of P into itself such that $x\phi^+\phi \leqq x \leqq x\phi\phi^+$ for all $x \in P$.

LEMMA 1.2. *If S is a pre-Baer semigroup and $z \in S$, then $\phi_z: \mathscr{L}(S) \to \mathscr{L}(S)$ given by $LR(x)\phi_z = LR(xz)$ is residuated with $\phi_z^+: \mathscr{L}(S) \to \mathscr{L}(S)$ given by $L(x)\phi_z^+ = L(zx)$ as its residual.*

If P is a bounded poset we use $S(P)$ to denote the semigroup of residuated maps on P.

THEOREM 1.3. *Every bounded poset can be coordinatized by a pre-*

Baer semigroup. In particular, if P is a bounded poset, then $S(P)$ is a pre-Baer semigroup which coordinatizes P. If S is any other pre-Baer semigroup which coordinatizes P, then $z \mapsto \phi_z$ is a homomorphism, with kernel 0, of S into $S(P)$ and the image of S in $S(P)$ is a pre-Baer semigroup which coordinatizes P.

DEFINITION 1.4. A pre-Baer semigroup S is a *right Baer semigroup* in case for each $x \in S$ there exists an idempotent $x^r \in S$ such that $R(x) = x^r S$, i.e., such that $xy = 0 \Leftrightarrow y = x^r y$. S is a *left Baer semigroup* in case for each $x \in S$ there exists an idempotent $x^l \in S$ such that $L(x) = Sx^l$.

THEOREM 1.5. *Every right* (resp., left) *Baer semigroup coordinatizes a bounded join* (resp., meet) *semilattice. Conversely, every bounded join* (resp., meet) *semilattice can be coordinatized by a right* (resp., left) *Baer semigroup. In particular, if P is a bounded join* (resp., meet) *semilattice, then $S(P)$ is a right* (resp., left) *Baer semigroup which coordinatizes P. If S is any other right* (resp., left) *Baer semi-group which coordinatizes P then the image of S in $S(P)$ under the homomorphism $z \mapsto \phi_z$ is a right* (resp., left) *Baer semigroup.*

REMARK. If S is a right Baer semigroup the join operation in $\mathscr{L}(S)$ is given by $LR(x) \vee LR(y) = L(y^r(xy^r)^l) = LR(x)\phi_{y^r}\phi^+_{y^r}$. If S is a left Baer semigroup the meet operation in $\mathscr{L}(S)$ is given by $L(x) \cap L(y) = LR((y^l x)^l y^l) = L(x)\phi^+_{x^l}\phi_{y^l}$.

2. Homomorphisms preserving r and l.

DEFINITION 2.1. A homomorphism ϕ of a pre-Baer semigroup S onto a semigroup T is called *r-preserving* in case, for each $x \in S$, $LR(x\phi) = L(x^r\phi)$ for some choice of x^r. (Recall x^r is such that $LR(x) = L(x^r)$.) ϕ is *l-preserving* in case, for each $x \in S$, $RL(x\phi) = R(x^l\phi)$ for some choice of x^l. (Recall x^l is such that $RL(x) = R(x^l)$.) Notice that if ϕ is r-and l- preserving, then T is a pre-Baer semigroup.

LEMMA 2.2. *Let ϕ be a homomorphism of a pre-Baer semigroup S onto a semigroup T.*

(i) *If ϕ is r-preserving, then Φ: $\mathscr{L}(S) \to \mathscr{L}(T)$ given by $LR(x)\Phi = LR(x\phi)$ is well defined and isotone.*

(ii) *If ϕ is l-preserving, then Φ: $\mathscr{L}(S) \to \mathscr{L}(T)$ given by $L(x)\Phi = L(x\phi)$ is well defined and isotone.*

Proof. (i). Suppose that ϕ is r-preserving and that $LR(x) \subseteqq LR(y)$. Choose y^r so that $LR(y\phi) = L(y^r\phi)$. Then we have $LR(x) \subseteqq LR(y) \Rightarrow x \in LR(x) \subseteqq L(y^r) \Rightarrow xy^r = 0 \Rightarrow x\phi y^r\phi = 0 \Rightarrow x\phi \in L(y^r\phi) = LR(y\phi) \Rightarrow LR(x\phi) \subseteqq$

$LR(y\phi)$. This shows that Φ is well defined and isotone. Finally, $LR(x)\Phi = L(x^r\phi) \in \mathscr{L}(T)$.

(ii). Suppose that ϕ is l-preserving and that $L(x) \subseteqq L(y)$. Choose x^l so that $RL(x\phi) = R(x^l\phi)$. Then we have $L(x) \subseteqq L(y) \Rightarrow RL(y) \subseteqq RL(x) \Rightarrow y \in RL(y) \subseteqq R(x^l) \Rightarrow x^l y = 0 \Rightarrow x^l\phi y\phi = 0 \Rightarrow y\phi \in R(x^l\phi) = RL(x\phi) \Rightarrow RL(y\phi) \subseteqq RL(x\phi) \Rightarrow L(x\phi) \subseteqq L(y\phi)$. This makes Φ well defined and isotone.

REMARK. Notice that, in part (i) of the lemma, $L(x)\Phi = LR(x^l)\Phi = LR(x^l\phi)$. Hence $L(x)\Phi = L(x\phi)$ for all $x \in S$ iff ϕ is l-preserving. Similarly, in part (ii), $LR(x)\Phi = L(x^r\phi)$ and it is clear that $LR(x)\Phi = LR(x\phi)$ for all $x \in S$ iff ϕ is r-preserving. If ϕ is r-and l-preserving, then the mappings in parts (i) and (ii) of the lemma coincide.

If S is a pre-Baer semigroup and $\phi: S \twoheadrightarrow T$ (i.e., from S onto T) an r-preserving homomorphism, then the map defined in part (i) of Lemma 2.2 induces an equivalence relation $\equiv$ on $\mathscr{L}(S)$ by the rule $LR(x) \equiv LR(y)$ iff $LR(x)\Phi = LR(y)\Phi$ iff $LR(x\phi) = LR(y\phi)$. It is this equivalence relation we wish to examine.

DEFINITION 2.3. If S is a pre-Baer semigroup and $\phi: S \twoheadrightarrow T$ an r-preserving homomorphism, then the equivalence relation on $\mathscr{L}(S)$ just described will be called the *equivalence relation on $\mathscr{L}(S)$ induced by ϕ.*

DEFINITION 2.4. An equivalence relation $\equiv$ on $\mathscr{L}(S)$ where S is a pre-Baer semigroup is *S-compatible* in case $LR(x) \equiv LR(y) \Rightarrow LR(x)\phi_z \equiv LR(y)\phi_z$ for all $z \in S$. It is *S^+-compatible* in case $LR(x) \equiv LR(y) \Rightarrow LR(x)\phi_z^+ \equiv LR(y)\phi_z^+$ for all $z \in S$.

DEFINITION 2.5. An equivalence relation $\equiv$ on a poset P is *ordered* if $P/\equiv$ is partially ordered by the rule $[x] \leqq [y] \Leftrightarrow$ there exist elements $x_1 \in [x]$ and $y_1 \in [y]$ such that $x_1 \leqq y_1$.

REMARK. Congruence relations on lattices and semilattices are ordered.

LEMMA 2.6. *If $\equiv$ is an equivalence relation on $\mathscr{L}(S)$, S a pre-Baer semigroup, and $\mathscr{L}(S)/\equiv$ is partially ordered in such a way that $LR(x) \subseteqq LR(y) \Rightarrow [LR(x)] \leqq [LR(y)]$, then the following are equivalent.*

(a) $[LR(x)\phi_{zr}] = [0] \Rightarrow [LR(x)] \leqq [0\phi_{zr}^+]$, *for all* $x \in S$.

(b) $[LR(x)] = [0] \Rightarrow [LR(x)\phi_{zr}^+] = [0\phi_{zr}^+]$, *for all* $x \in S$.

(b′) $LR(x) \equiv 0 \Rightarrow LR(x)\phi_{zr}^+ \equiv 0\phi_{zr}^+ = LR(z)$, *for all* $x \in S$.

Proof. (b) $\Leftrightarrow$ (b′). This is only a difference in notation.

(a) $\Rightarrow$ (b). Suppose $[LR(x)] = [0]$. Since $LR(x)\phi_{zr}^+\phi_{zr} \subseteqq LR(x)$, we have $[LR(x)\phi_{z^r}^+\phi_{z^r}] = [0]$. Now by (a), $[LR(x)\phi_{z^r}^+] \leqq [0\phi_{z^r}^+]$. The reverse inequality holds since $0\phi_{z^r}^+ \subseteqq LR(x)\phi_{z^r}^+$.

(b) $\Rightarrow$ (a). If $[LR(x)\phi_{z^r}] = [0]$, we have by (b) that $[LR(x)\phi_{z^r}\phi_{z^r}^+] = [0\phi_{z^r}^+]$. Now $LR(x) \subseteqq LR(x)\phi_{z^r}\phi_{z^r}^+$ gives $[LR(x)] \leqq [LR(x)\phi_{z^r}\phi_{z^r}^+] = [0\phi_{z^r}^+]$.

THEOREM 2.7. *If S is a pre-Baer semigroup and $\phi: S \twoheadrightarrow T$ an r-preserving homomorphism, the equivalence relation $\equiv$ on $\mathscr{L}(S)$ induced by ϕ has the following properties:*

(i) *For each $z \in S$, z^r can be chosen so that $LR(x) \equiv 0 \Rightarrow LR(x)\phi_{z^r}^+ \equiv 0\phi_{z^r}^+$ for all $x \in S$.*

(ii) *$\equiv$ is ordered.*

(iii) *$\equiv$ is S-compatible.*

In part (i) any z^r such that $L(z^r\phi) = LR(z\phi)$ suffices.

Proof. Recall that $LR(x) \equiv LR(y) \Leftrightarrow LR(x\phi) = LR(y\phi)$.

(i). $\mathscr{L}(S)/\equiv$ is partially ordered by $[LR(x)] \leqq [LR(y)] \Leftrightarrow LR(x\phi) \subseteqq LR(y\phi)$. Choose z^r so that $L(z^r\phi) = LR(z\phi)$. Since $LR(x) \subseteqq LR(y) \Rightarrow LR(x\phi) \subseteqq LR(y\phi)$ by Lemma 2.2, we can apply Lemma 2.6. Since $LR(x)\phi_{z^r} \equiv 0 \Rightarrow LR(xz^r\phi) = 0 \Rightarrow x\phi z^r\phi = 0 \Rightarrow x\phi \in L(z^r\phi) \Rightarrow LR(x\phi) \subseteqq LR(z\phi)$ for all $x \in S$, part (a) of Lemma 2.6 is satisfied and part (b) is what we are trying to prove.

(ii). It will suffice to show that $LR(x\phi) \subseteqq LR(y\phi) \Rightarrow$ there exists $y_1 \in S$ such that $LR(x) \subseteqq LR(y_1)$ and $LR(y_1\phi) = LR(y\phi)$. If $LR(x\phi) \subseteqq LR(y\phi) = L(y^r\phi)$, we have $x\phi y^r\phi = 0 \Rightarrow LR(xy^r\phi) = 0 \Rightarrow LR(xy^r) = 0$. By (i), $LR(xy^r)\phi_{y^r}^+ \equiv 0\phi_{y^r}^+ = LR(y)$. Since $LR(xy^r)\phi_{y^r}^+ = L(y^r(xy^r)^r) = LR((y^r(xy^r)^r)^l)$, this says that $LR((y^r(xy^r)^r)^l\phi) = LR(y\phi)$. Letting $y_1 = (y^r(xy^r)^r)^l$ finishes the proof since $x \in L(y^r(xy^r)^r) \Rightarrow LR(x) \subseteqq L(y^r(xy^r)^r) = LR((y^r(xy^r)^r)^l) = LR(y_1)$.

(iii). $LR(x) \equiv LR(y) \Rightarrow LR(x\phi) = LR(y\phi) \Rightarrow LR(x\phi z\phi) = LR(y\phi z\phi) \Rightarrow LR(xz\phi) = LR(yz\phi) \Rightarrow LR(x)\phi_z \equiv LR(y)\phi_z$.

The equivalence relation in Theorem 2.7 has another nice property. It is determined by its kernel.

THEOREM 2.8. *Let $\equiv$ be the equivalence relation of Theorem 2.7. The following are equivalent.*

(a) *$LR(x) \equiv LR(y)$.*

(b) *If $L(x^r\phi) = LR(x\phi)$ and $L(y^r\phi) = LR(y\phi)$, then $LR(x)\phi_{y^r} \equiv 0$ and $LR(y)\phi_{x^r} \equiv 0$.*

Proof. (a) $\Rightarrow$ (b). Since $\equiv$ is S-compatible, $LR(x) \equiv LR(y) \Rightarrow LR(x)\phi_{y^r} \equiv LR(y)\phi_{y^r} = 0$. Similarly $LR(y)\phi_{x^r} \equiv 0$.

(b) $\Rightarrow$ (a). Part (b) of Lemma 2.6 is satisfied by Theorem 2.7, so by part (a) of Lemma 2.6, $LR(x)\phi_{y^r} \equiv 0 \Rightarrow [LR(x)] \leqq [0\phi_{y^r}^+] = [LR(y)]$. Similarly $LR(y)\phi_{x^r} \equiv 0 \Rightarrow [LR(y)] \leqq [LR(x)]$. Thus $[LR(x)] = [LR(y)]$.

We now wish to show that any equivalence relation on $\mathscr{L}(S)$ having the three properties of Theorem 2.7 is induced on $\mathscr{L}(S)$ by some r-preserving homomorphism.

LEMMA 2.9. *Let S be a pre-Baer semigroup and let $\equiv$ be an S-compatible equivalence relation on $\mathscr{L}(S)$. For each $z \in S$ define Φ_z: $\mathscr{L}(S)/\equiv \to \mathscr{L}(S)/\equiv$ by $[LR(x)]\Phi_z = [LR(x)\phi_z] = [LR(xz)]$. Φ_z is well defined because of S-compatibility. Let S' denote the semigroup generated by $\{\Phi_z; z \in S\}$ under composition. The map $z \mapsto \Phi_z$ is a homomorphism of S onto S' and if $\equiv$ also possesses properties* (i) *and* (ii) *of Theorem* 2.7, *this homomorphism is r-preserving.*

Proof. It is a clear that $z \mapsto \Phi_z$ is a homomorphism of S onto S'. Let $z \in S$ and choose z^r to satisfy part (i) of Theorem 2.7. $\Phi_z\Phi_{z^r} = 0$ since $zz^r = 0$ so we have $LR(\Phi_z) \subseteqq L(\Phi_{z^r})$. To show that $L(\Phi_{z^r}) \subseteqq LR(\Phi_z)$ we suppose that $\Phi_x \in L(\Phi_{z^r})$ and show that $\Phi_y \in R(\Phi_z)$ implies $\Phi_x\Phi_y = 0$. Since $\Phi_{xz^r} = 0$ we have $[LR(1)]\Phi_{xz^r} = [LR(xz^r)] = 0$ and by Lemma 2.6, which applies since we are assuming part (i) of Theorem 2.7, $[LR(x)] \leqq [LR(z)]$. Since $\equiv$ is ordered, the elements of S' are isotone maps and we have $[LR(xy)] = [LR(x)]\Phi_y \leqq [LR(z)]\Phi_y = [LR(zy)] = [LR(1)]\Phi_{zy} = [0]$. Now $[LR(1)]\Phi_{xy} = [0]$ implies $\Phi_{xy} = \Phi_x\Phi_y = 0$.

REMARK. If an S-compatible equivalence relation $\equiv$ possesses properties (i) and (ii) of Theorem 2.7, and if we denote the kernel of $z \mapsto \Phi_z$ by I, then $z \mapsto \Phi_z$ is the homomorphism studied by R. S. Pierce in [3]. To prove this we must show that $\Phi_x = \Phi_y \Leftrightarrow axb \in I$ iff $ayb \in I$. Suppose $\Phi_x = \Phi_y$. Then $axb \in I \Leftrightarrow \Phi_{axb} = \Phi_a\Phi_x\Phi_b = 0 \Leftrightarrow \Phi_{ayb} = \Phi_a\Phi_y\Phi_b = 0 \Leftrightarrow ayb \in I$. Now suppose $axb \in I$ iff $ayb \in I$. Then $\Phi_{zxw} = 0$ iff $\Phi_{zyw} = 0 \Rightarrow [LR(zxw)] = [0]$ iff $[LR(zyw)] = [0] \Rightarrow [LR(zx)\phi_w] = [0]$ iff $[LR(zy)\Phi_w] = [0]$. Setting $w = (zx)^r$, where $(zx)^r$ is chosen as in part (i) of Theorem 2.7, and using part (a) of Lemma 2.6 we have $[LR(zy)] \leqq [L((zx)^r)] = [LR(zx)]$. Similarly we have $[LR(zx)] \leqq [LR(zy)]$. Thus $[LR(zx)] = [LR(zy)]$ for all $z \in S$, but this just says that $\Phi_x = \Phi_y$.

THEOREM 2.10. *Let S be a pre-Baer semigroup and let $\equiv$ be an equivalence relation on $\mathscr{L}(S)$ which possesses properties* (i), (ii), *and* (iii) *of Theorem* 2.7. *Then $\equiv$ is induced on $\mathscr{L}(S)$ by the r-preserving homomorphism $z \mapsto \Phi_z$ described in Lemma* 2.9. *Furthermore, $z \mapsto \Phi_z$ is the largest r-preserving homomorphism* (considered as a congruence relation on S) *which induces* $\equiv$.

Proof. Consider the r-preserving homomorphism $z \mapsto \Phi_z$ of Lemma 2.9. We wish to show that $LR(\Phi_x) = LR(\Phi_y)$ iff $LR(x) \equiv LR(y)$. Let $LR(\Phi_x) = LR(\Phi_y)$ and choose y^r as in part (i) of Theorem 2.7. Then

$R(\Phi_x) = R(\Phi_y)$ and we have $\Phi_x\Phi_{y^r} = 0$ since $\Phi_y\Phi_{y^r} = 0$. $\Phi_{xy^r} = 0$ means $[LR(x^r y)] = [0]$ and by Lemma 2.6 $[LR(x)] \leqq [LR(y)]$. Similarly we get $[LR(y)] \leqq [LR(x)]$ and thus $LR(x) \equiv LR(y)$. Conversely, suppose $LR(x) \equiv LR(y)$. Choose x^r and y^r such that $L(\Phi_{x^r}) = LR(\Phi_x)$ and $L(\Phi_{y^r}) \equiv LR(\Phi_y)$. By S-compatibility we have $LR(xy^r) \equiv LR(yy^r) = 0$ and $LR(yx^r) \equiv LR(xx^r) = 0$. This means $\Phi_{xy^r} = \Phi_{yx^r} = 0$. Now $\Phi_x \in L(\Phi_{y^r}) = LR(\Phi_y)$ gives $LR(\Phi_x) \subseteqq LR(\Phi_y)$ and $\Phi_y \in L(\Phi_{x^r}) = LR(\Phi_x)$ gives $LR(\Phi_y) \subseteqq LR(\Phi_x)$.

Finally, suppose ϕ is another r-preserving homomorphism which induces $\equiv$. Then $x\phi = y\phi \Rightarrow zx\phi = zy\phi$ for all $z \in S \Rightarrow LR(zx\phi) = LR(zy\phi)$ for all $z \in S \Rightarrow LR(z)\phi_x \equiv LR(z)\phi_y$ for all $z \in S \Rightarrow \Phi_x = \Phi_y$.

REMARK. The r-preserving homomorphisms which induce $\equiv$ all have the same kernel since, if ϕ is such a homomorphism, $x\phi = 0 \Leftrightarrow LR(x\phi) = 0 \Leftrightarrow LR(x) \equiv 0$.

THEOREM 2.11. *Let S be a pre-Baer semigroup and $\phi: S \twoheadrightarrow T$ an r-preserving homomorphism. Let $\Phi: \mathscr{L}(S) \to \mathscr{L}(T)$ be the map described in Lemma 2.2 (i), i.e., $LR(x)\Phi = LR(x\phi)$. The following are equivalent.*

(a) $\ker \phi \in \mathscr{L}(S)$.

(b) $\ker \Phi$ *is a principal ideal.*

(c) $\Phi: \mathscr{L}(S) \to \mathscr{L}(T)$ *is residuated.*

Proof. (a) $\Leftrightarrow$ (b). This follows from the observation that $x \in \ker \phi \Leftrightarrow x\phi = 0 \Leftrightarrow LR(x\phi) = 0 \Leftrightarrow LR(x) \in \ker \Phi$.

(c) $\Rightarrow$ (b). This is clear.

(a) $\Rightarrow$ (c). Suppose $\ker \phi = LR(w)$. Define $\Phi^+: \mathscr{L}(T) \to \mathscr{L}(S)$ by $L(x\phi)\Phi^+ = L(xw^r)$. Φ^+ is well defined and isotone since when $L(x\phi) \subseteqq L(y\phi)$ we have $z \in L(xw^r) \Rightarrow zxw^r = 0 \Rightarrow zx \in \ker \phi \Rightarrow z\phi x\phi = 0 \Rightarrow z\phi \in L(x\phi) \subseteqq L(y\phi) \Rightarrow z\phi y\phi = 0 \Rightarrow zy \in \ker \phi \Rightarrow zyw^r = 0 \Rightarrow z \in L(yw^r)$, which says that $L(xw^r) \subseteqq L(yw^r)$. Choose x^r so that $L(x^r\phi) = LR(x\phi)$. Now since $x \in L(x^r w^r)$ we have $LR(x) \subseteqq L(x^r w^r) = L(x^r\phi)\Phi^+ = LR(x\phi)\Phi^+ = LR(x)\Phi\Phi^+$. Now all that remains is to show that $L(x\phi)\Phi^+\Phi \subseteqq L(x\phi)$. Since $(xw^r)^l xw^r = 0 \Rightarrow (xw^r)^l x \in \ker \phi \Rightarrow (xw^r)^l \phi x\phi = 0 \Rightarrow (xw^r)^l\phi \in L(x\phi)$ we have $L(x\phi)\Phi^+\Phi = L(xw^r)\Phi = LR((xw^r)^l\phi) \subseteqq L(x\phi)$.

If S is a pre-Baer semigroup and $z \in S$, notice that $\mathscr{R}(S)$ is dual isomorphic to $\mathscr{L}(S)$ and the residuated map on $\mathscr{R}(S)$ given by $RL(x) \mapsto RL(zx)$, considered as a map on $\mathscr{L}(S)$, is ϕ_z^+. (See Lemma 1.2.) Bearing this in mind and applying left-right duality to the results obtained thus far, we find that every l-preserving homomorphism on a pre-Baer semigroup S induces on $\mathscr{L}(S)$ an ordered S^+-compatible equivalence relation $\equiv$ with the property that, for each $z \in S$, z^l can be chosen so that $LR(x) \equiv 1 \Rightarrow LR(x)\phi_{y^l} \equiv 1\phi_{z^l}$ for all $x \in S$. Furthermore, every such equivalence relation on $\mathscr{L}(S)$ is induced by some l-preserving homomorphism on S. We now have

THEOREM 2.12. *Let ϕ be an r-and l-preserving homomorphism on a pre-Baer semigroup S. The ordered equivalence relation on $\mathscr{L}(S)$ induced by ϕ is S- and S^+-compatible. Furthermore, every S- and S^+-compatible ordered equivalence relation on $\mathscr{L}(S)$ is induced by some r- and l-preserving homomorphism on S.*

Proof. This follows from previous results and the remarks preceding the theorem if we make the following observation: If an ordered equivalence relation $\equiv$ on $\mathscr{L}(S)$ is S- and S^+-compatible, then $\Phi_z: \mathscr{L}(S)/\equiv \Rightarrow \mathscr{L}(S)/\equiv$ given by $[LR(x)]\Phi_z = [LR(x)\phi_z]$ is residuated with residual $\Phi_z^+: \mathscr{L}(S)/\equiv \Rightarrow \mathscr{L}(S)/\equiv$ given by $[LR(x)]\Phi_z^+ = [LR(x)\Phi_z^+]$. Since residuated maps uniquely determine their residuals and vice versa, the r-preserving homomorphism $z \mapsto \Phi_z$ (considered as a congruence on S) coincides with the l-preserving congruence on S associated with the anti-homomorphism $z \mapsto \Phi_z^+$.

3. RAP and LAP homomorphisms.

DEFINITION 3.1. If S is a right Baer semigroup, a semigroup homomorphism $\phi: S \twoheadrightarrow T$ is *right annihilator preserving* or *RAP* in case $R(x\phi) = R(x)\phi$. Notice that $R(x)\phi = (x^r\phi)T$. Dually, if S is a left Baer semigroup, ϕ is *left annihilator preserving* or *LAP* in case $L(x\phi) = L(x)\phi$. Finally, ϕ is *annihilator preserving* or *AP* if it is both *RAP* and *LAP*.

REMARK. Any *RAP* homomorphism is r-preserving since $LR(x\phi) = L((x^r\phi)T) = L(x^r\phi)$. Dually, any *LAP* homomorphism is l-preserving.

LEMMA 3.2. *In a right Baer semigroup S we have*
(i) $LR(x) \vee LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} = LR(y) \vee LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r}$.
(ii) $LR(zy) \vee LR(xy) = LR(zx^ry) \vee LR(xy)$.
(iii) $LR(x) \vee LR(y) \vee LR(xy^r) = LR(y) \vee LR(xy^r)$.

Proof. It is shown in [2] that, in a right Baer semigroup S, $R(x) \cap R(y) \in \mathscr{R}(S)$ and that the join operation in $\mathscr{L}(S)$ is given by $LR(x) \vee LR(y) = L(R(x) \cap R(y))$.

(i). It is enough to show that $R(x) \cap R(xy^r) \cap R(yx^r) = R(y) \cap R(xy^r) \cap R(yx^r)$. If $z \in R(x) \cap R(xy^r) \cap R(yx^r)$, then $z = x^rz$ and $yz = yx^rz = 0$ so $z \in R(y) \cap R(xy^r) \cap R(yx^r)$. The other inclusion follows by symmetry.

(ii). It is enough to show $R(zy) \cap R(xy) = R(zx^ry) \cap R(xy)$. This follows from the observation that if $xyw = 0$, then $yw = x^r yw$ so that $zyw = 0 \Leftrightarrow zx^r yw = 0$.

(iii). It is enough to show that $R(y) \cap R(xy) \subseteq R(x) \cap R(y) \cap R(xy^r)$. If $yw = 0$, then $w = y^r w$ so that $xy^r w = 0 \Rightarrow xw = 0$.

LEMMA 3.3. *If S is a right Baer semigroup and $\equiv$ is an S-compatible equivalence relation on $\mathscr{L}(S)$, the following are equivalent.*

(a) $\equiv$ *is a join congruence.*

(b) $LR(x) \vee LR(z) = LR(y) \vee LR(z)$, $LR(z) \equiv 0 \Rightarrow LR(x) \equiv LR(y)$.

Proof. (a) $\Rightarrow$ (b). Since $LR(z) \equiv 0$, we have $LR(x) = LR(x) \vee 0 \equiv LR(x) \vee LR(z) = LR(y) \vee LR(z) \equiv LR(y) \vee 0 = LR(y)$.

(b) $\Rightarrow$ (a). Suppose $LR(x) \equiv LR(y)$. If $LR(z) \in \mathscr{L}(S)$, we have, using Lemma 3.2, that $LR(x) \vee LR(z) \vee LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} = LR(y) \vee LR(z) \vee LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r}$. To show that $LR(x) \vee LR(z) \equiv LR(y) \vee LR(z)$ it will suffice, by (b), to show $LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} \equiv 0$. Since $\equiv$ is S-compatible we have $LR(x)\phi_{y^r} \equiv LR(y)\phi_{y^r} = 0 = LR(x)\phi_{x^r} \equiv LR(y)\phi_{x^r}$. Using (b), $LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} \vee LR(y)\phi_{x^r} = LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r}$ and $LR(y)\phi_{x^r} \equiv 0 \Rightarrow LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} \equiv LR(x)\phi_{y^r} \equiv 0$.

THEOREM 3.4. *Let S be a right Baer semigroup and $\phi: S \twoheadrightarrow T$ an RAP homomorphism. Then the equivalence relation $\equiv$ induced on $\mathscr{L}(S)$ by ϕ* (recall $LR(x) \equiv LR(y)$ iff $LR(x\phi) = LR(y\phi)$) *is an S-compatible join congruence.*

Proof. S-compatibility was proven in Theorem 2.7. By Lemma 3.3 it is sufficient to show that $LR(x) \vee LR(z) = LR(y) \vee LR(z)$ and $LR(z\phi) = 0 \Rightarrow LR(x\phi) = LR(y\phi)$. Now $LR(z\phi) = 0$ means that $R(z\phi) = (z^r\phi)T = T$, so $1\phi = z^r\phi 1\phi = z^r\phi$. Since $LR(xz^r) = (LR(x) \vee LR(z))\phi_{z^r} = (LR(y) \vee LR(z))\phi_{z^r} = LR(yz^r)$, we have $LR(x\phi) = LR(xz^r\phi) = LR(yz^r\phi) = LR(y\phi)$.

An S-compatible join congruence is determined by its kernel in the following manner.

THEOREM 3.5. *If S is a right Baer semigroup and $\equiv$ is an S-compatible join congruence on $\mathscr{L}(S)$, the following are equivalent.*

(a) $LR(x) \equiv LR(y)$.

(b) $LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} \equiv 0$.

(c) *There is an* $LR(z) \equiv 0$ *such that*
$LR(x) \vee LR(z) = LR(y) \vee LR(z)$.

Proof. (a) $\Rightarrow$ (b). If $LR(x) \equiv LR(y)$, then $LR(x)\phi_{y^r} \equiv LR(y)\phi_{y^r} = 0 = LR(x)\phi_{x^r} \equiv LR(y)\phi_{x^r}$ and hence $LR(x)\phi_{y^r} \vee LR(y)\phi_{x^r} \equiv 0$.

(b) $\Rightarrow$ (c). Follows from part (i) of Lemma 3.2.
(c) $\Rightarrow$ (a). Follows from Lemma 3.3.

COROLLARY 3.6. *An S-compatible join congruence* $\equiv$ *has the property that, for each* $z \in S$, *any choice of* z^r *gives* $LR(x) \equiv 0 \Rightarrow LR(x)\phi^+_{z^r} \equiv 0\phi^+_{z^r}$ *for all* $x \in S$.

Proof. Since a join congruence is ordered, it is sufficient by Lemma 2.6 to show that $LR(xz^r) \equiv 0 \Rightarrow [LR(x)] \leqq [LR(z)]$. Since by part (iii) of Lemma 3.2 we have $LR(x) \vee LR(z) \vee LR(xz^r) = LR(z) \vee LR(xz^r)$, it follows from the theorem that when $LR(xz^r) \equiv 0$, $LR(x) \vee LR(z) \equiv LR(z)$. Since $\equiv$ is a join congruence, this says that $[LR(x)] \leqq [LR(z)]$.

THEOREM 3.7. *If S is a right Baer semigroup and* $\equiv$ *is an S-compatible join congruence on* $\mathscr{L}(S)$, *then the homomorphism* $z \mapsto \Phi_z$ *described in Lemma 2.9 is RAP.*

Proof. We wish to show that $R(\Phi_x) = \Phi_{x^r}S'$ or, in other words, that $\Phi_x\Phi_y = 0 \Leftrightarrow \Phi_y = \Phi_{x^r}\Phi_y$. Notice that $\Phi_x\Phi_y = 0 \Leftrightarrow [1]\Phi_x\Phi_y = [0] \Leftrightarrow [LR(xy)] = [0] \Leftrightarrow LR(xy) \equiv 0$ and that $\Phi_y = \Phi_{x^r}\Phi_y \Leftrightarrow LR(zy) \equiv LR(zx^ry)$ for all $z \in S$. Since it is clear that $\Phi_y = \Phi_{x^r}\Phi_y \Rightarrow \Phi_x\Phi_y = 0$, we will be done if we can show that $LR(xy) \equiv 0 \Rightarrow LR(zy) \equiv LR(zx^ry)$ for all $z \in S$. Since $LR(zy) \vee LR(xy) = LR(zx^ry) \vee LR(xy)$ by part (ii) of Lemma 3.2, $LR(xy) \equiv 0$ implies by Theorem 3.5 that $LR(zy) \equiv LR(zx^ry)$ for all $z \in S$.

COROLLARY 3.8. *If S is a right Baer semigroup, then any S-compatible join congruence* $\equiv$ *on* $\mathscr{L}(S)$ *is induced by an RAP homomorphism on S.*

Proof. Since, by Corollary 3.6, $\equiv$ has property (i) of Theorem 2.7, the proof of Theorem 2.10 applies and says that $\equiv$ is induced on $\mathscr{L}(S)$ by the homomorphism $z \mapsto \Phi_z$ on S. By Theorem 3.7, $z \mapsto \Phi_z$ is *RAP*.

COROLLARY 3.9. *If S is a right Baer semigroup, then every S- and* S^+*-compatible join congruence on* $\mathscr{L}(S)$ *is induced by an RAP and l-preserving homomorphism on S.*

Proof. This follows from Corollary 3.8 and from Theorem 2.12 and its proof.

COROLLARY 3.10. *If S is a left Baer semigroup, then any* S^+*-*

compatible meet congruence on $\mathscr{L}(S)$ is induced by an LAP homomorphism on S.

Proof. This is the dual of Corollary 3.8. (See the remarks preceding Theorem 2.12.)

COROLLARY 3.11 (Thorne). *If S is a Baer semigroup, then every S- and S^+-compatible congruence on $\mathscr{L}(S)$ is induced by an AP homomorphism on S.*

Proof. This follows from Corollaries 3.8 and 3.10 and from Theorem 2.12 and its proof.

4. Kernels of S-compatible join congruences.

THEOREM 4.1. *Let I be an ideal of a join semilattice $L = \mathscr{L}(S)$, S a right Baer semigroup. The following are equivalent.*

(a) *I is the kernel of an S-compatible join congruence.*

(b) *$I\phi_z \subseteqq I$ for each $z \in S$.*

Proof. (a) $\Rightarrow$ (b). If $LR(x) \in I$, then $LR(x) \equiv 0$ and by S-compatibility $LR(x)\phi_z \equiv 0\phi_z = 0$, i.e., $LR(x)\phi_z \in I$.

(b) $\Rightarrow$ (a). Suppose $I\phi_z \subseteqq I$ for each $z \in S$. Define $LR(x) \equiv LR(y)$ iff $LR(x) \vee LR(w) = LR(y) \vee LR(w)$ for some $LR(w) \in I$. It is easy to see that $\equiv$ is a join congruence. If $LR(x) \equiv LR(y)$, then $LR(x) \vee LR(w) = LR(y) \vee LR(w)$ with $LR(w) \in I$ and since ϕ_z, being a residuated map, preserves join we have $LR(x)\phi_z \vee LP(w)\phi_z = LR(y)\phi_z \vee LR(w)\phi_z$. Since $LR(w)\phi_z \in I$ it follows that $LR(x)\phi_z \equiv LR(y)\phi_z$. Clearly $\equiv$ has I as its kernel.

LEMMA 4.2. *In any semigroup S with 0, if $R(w)$ is a two-sided ideal, for some $w \in S$, then $LR(w)$ is a two-sided ideal. Hence, if S is a pre-Baer semigroup, $LR(w)$ is two-sided if and only if $R(w)$ is two-sided.*

Proof. Suppose $R(w)$ is two-sided. $LR(w)$ is already a left ideal so we must show that it is a right ideal. Let $x \in LR(w)$, $y \in S$, and $z \in R(w)$. We need $xyz = 0$. But $yz \in R(w)$ since $R(w)$ is two-sided and hence $xyz = 0$. The second assertion follows from the first and its dual.

Theorem 4.1 characterized kernels of S-compatible join congruences. We now look at principal ideals which occur as kernels of S-compatible join congruences.

THEOREM 4.3. *Let S be a right Baer semigroup. The following are equivalent.*

(a) *$[0, LR(w)]$ is the kernel of an S-compatible join congruence on $\mathscr{L}(S)$.*

(b) *$LR(w)$ is the kernel of an RAP homomorphism on S.*

(c) *$LR(w)\phi_x \subseteqq LR(w)$ for all $x \in S$.*

(d) *$xw^r = w^r x w^r$ for all $x \in S$ and for any choice of w^r.*

(e) *$LR(w)$ is a two-sided ideal.*

(f) *$R(w)$ is a two-sided ideal.*

Proof. (a) $\Leftrightarrow$ (b). Since every RAP homomorphism ϕ on S induces an S-compatible join congruence $\equiv$ on $\mathscr{L}(S)$ by the rule $LR(x) \equiv LR(y)$ iff $LR(x\phi) = LR(y\phi)$ and since every S-compatible join congruence arises in this manner for some ϕ, it suffices to notice that $x \in \ker \phi \Leftrightarrow x\phi = 0 \Leftrightarrow LR(x\phi) = 0 \Leftrightarrow LR(x) \equiv 0$.

(a) $\Leftrightarrow$ (c). Use Theorem 4.1.

(e) $\Leftrightarrow$ (f). Use Lemma 3.2.

(d) $\Leftrightarrow$ (f). This follows from the dual of Theorem 1 of [1].

(b) $\Rightarrow$ (e). This is obvious.

(d) $\Rightarrow$ (b). $x \mapsto xw^r$ is a homomorphism of S onto Sw^r and it is RAP since $yw^r \in R(xw^r) \Leftrightarrow xw^r yw^r = 0 \Leftrightarrow yw^r = w^r yw^r = x^r w^r yw^r \Leftrightarrow yw^r \in (x^r w^r)(Sw^r) = (R(x))w^r$.

REMARK. By Theorem 2.11, the kernel of an S-compatible join congruence $\equiv$ is a principal ideal if and only if $\equiv$ is residuated in the sense that the canonical join homomorphism taking $\mathscr{L}(S)$ onto $\mathscr{L}(S)/\equiv$ is a residuated map.

In light of Theorem 4.1 we make the following definition.

DEFINITION 4.4. An ideal I of a join semilattice $L = \mathscr{L}(S)$, S a right Baer semigroup, is called *S-compatible* in case $I\phi_z \subseteqq I$ for all $z \in S$.

THEOREM 4.5. *Let S be a right Baer semigroup and let $L = \mathscr{L}(S)$. The set $I_S(L)$ of S-compatible ideals of L forms a subcomplete sublattice of $I(L)$, the lattice of ideals of L. $I_S(L)$ is isomorphic to the lattice of S-compatible join congruences on $\mathscr{L}(S)$.*

Proof. If $\{I_i\}$ is a family of S-compatible ideals of $\mathscr{L}(S)$ it is clear that $\bigcap_i\{I_i\}$ is an S-compatible ideal. Suppose $LR(x) \in \bigvee_i\{I_i\}$. Then there exist

$$LR(y_1) \in I_{i_1},\ LR(y_2) \in I_{i_2},\ \cdots,\ LR(y_n) \in I_{i_n}$$

such that

$$LR(x) \subseteqq LR(y_1) \vee LR(y_2) \vee \cdots \vee LR(y_n).$$

Hence

$$\begin{aligned} LR(x)\phi_z &\subseteqq (LR(y_1) \vee LR(y_2) \vee \cdots \vee LR(y_n))\phi_z \\ &= LR(y_1)\phi_z \vee LR(y_2)\phi_z \vee \cdots \vee LR(y_n)\phi_z \end{aligned}$$

and since $LR(y_k)\phi_z \subseteqq I_{i_k}$ $(k = 1, 2, \cdots, n)$ we have $LR(x)\phi_z \in \bigvee_i\{I_i\}$. Thus $\bigvee_i\{I_i\}$ is S-compatible and we have proven the first part of the theorem. Now, if $I \in I_S(L)$ let Θ_I denote the unique S-compatible join congruence with kernel I. In light of Theorem 3.5 it is clear that $I \subseteqq J \Leftrightarrow \Theta_I \subseteqq \Theta_J$.

THEOREM 4.6. *Let S be a right Baer semigroup in which, for each $x \in S$, $LR(x^l) = LR(x^l x^l)$ for some choice of x^l. Then $I_S(L)$ is distributive and obeys the following infinite distributive law:*

$$I \cap (\bigvee_i\{J_i\}) = \bigvee_i\{I \cap J_i\} .$$

Proof. It will suffice to show $I \cap (\bigvee_i\{J_i\}) \subseteqq \bigvee_i\{I \cap J_i\}$. Suppose $L(x) = LR(x^l) \in I$ and $LR(x^l) \in \bigvee_i\{J_i\}$. Then $LR(x^l) \subseteqq LR(y_1) \vee LR(y_2) \vee \cdots \vee LR(y_n)$ where $LR(y_k) \in J_{i_k}$ $(k = 1, 2, \cdots, n)$. Now $LR(x^l) = LR(x^l)\phi_{x^l} \subseteqq LR(y_1)\phi_{x^l} \vee LR(y_2)\phi_{x^l} \vee \cdots \vee LR(y_n)\phi_{x^l}$. For $k = 1, 2, \cdots, n$ we have $LR(y_k)\phi_{x^l} \in J_{i_k}$ by S-compatibility and $LR(y_h)\phi_{x^l} = LR(y_k x_l) \subseteqq LR(x^l) \in I$. Thus $LR(y_k)\phi_{x^l} \in I \cap J_{i_k}$ for $k = 1, 2, \cdots, n$. Thus $LR(x^l) \in \bigvee_i\{I \cap J_i\}$.

REMARK. Theorem 4.6 applies, in particular, when S is a Baer semigroup. In that case x^l is taken to be an idempotent generating $L(x)$. The $LR(x^l) = LR(x^l x^l)$ condition could also be taken care of by requiring, in the definition of pre-Baer semigroup, that x^r and x^l be idempotents. (It is pointed out in [2] that all our results involving pre-Baer semigroups remain valid if x^r and x^l are required to be idempotents.)

THEOREM 4.7. *Let S be a right Baer semigroup in which, for each $x \in S$, $LR(x^l) = LR(x^l x^l)$ for some choice of x^l. Let $L = \mathscr{L}(S)$. $I_S(L)$ is pseudo complemented since it is complete and obeys the infinite distributive law of Theorem 4.6. If $I \in I_S(L)$, its pseudo complement I^* is given by $I^* = \{LR(x); LR(x) \subseteqq L(J)\}$, where J is the kernel of any RAP homomorphism which induces the S-compatible join congruence with kernel I, i.e., $y \in J \Leftrightarrow LR(y) \in I$.*

Proof. I^* is an ideal since $LR(x), LR(y) \subseteqq L(J) \Rightarrow J \subseteqq R(x) \cap$

$R(y) \Rightarrow LR(x) \vee LR(y) = L(R(x) \cap R(y)) \subseteqq L(J)$. Suppose $LR(x) \in I^*$ and $y \in S$. Then $z \in J \Rightarrow yz \in J \Rightarrow xyz = 0 \Rightarrow xy \in L(J) \Rightarrow LR(xy) \subseteqq L(J) \Rightarrow LR(x)\phi_y = LR(xy) \in I^*$. Thus I^* is S-compatible. Now suppose $L(x) \in I \cap I^*$. Then $L(x) = LR(x^l) \in I \Rightarrow x^l \in J$ and $LR(x^l) \in I^* \Rightarrow x^l \in LR(x^l) \subseteqq L(J)$. Thus $x^l x^l = 0$ and $L(x) = LR(x^l) = LR(x^l x^l) = 0$. Therefore $I \cap I^* = 0$. Finally, suppose $I \cap K = 0$, with $K \in I_S(L)$. Let $LR(x) \in K$, $y \in J$. Then $LR(y) \in I \Rightarrow LR(xy) \subseteqq LR(y) \in I$ and $LR(x) \in K \Rightarrow LR(x)\phi_y = LR(xy) \in K$. Thus $LR(xy) \in I \cap K = 0 \Rightarrow xy = 0 \Rightarrow x \in L(J) \Rightarrow LR(x) \subseteqq L(J) \Rightarrow LR(x) \in I^*$. Therefore $K \subseteqq I^*$.

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