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A GENERALIZATION OF A THEOREM OF F. RIESZ

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In this paper, the concept of bounded slope variation, that of the derivative of a function with respect to an increasing function, and the Lane integral are used to obtain a generalization of a theorem of Frédéric Riesz.

In [3], R. E. Lane defined an integral which is an extension of the Stieltjes mean sigma integral defined by H. L. Smith [5]. If each of f and g is a real-valued function whose domain includes $[a, b]$ and $D = \{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$, then $S_D(f, g)$ denotes the sum

$$\sum_{i=1}^{n} \frac{1}{2}[f(x_i) + f(x_{i-1})][g(x_i) - g(x_{i-1})] .$$

The concepts of singular graph, exceptional number and summability set are as in [3]. If each of f and g is a real-valued function whose domain includes $[a, b]$ and if there exists a summability set G for f and g in $[a, b]$, then the Lane integral $\int_a^b fdg$ is the refinement limit

$$\operatorname*{limit}_{D\subset G} S_D(f, g) .$$

In case the entire interval $[a, b]$ is a summability set for f and g in $[a, b]$, the Lane integral $\int_a^b fdg$ is the Stieltjes mean sigma integral $M\int_a^b fdg$.

By Theorem 4.1 of [2], if f is quasicontinuous on $[a, b]$ and g is of bounded variation on $[a, b]$, then $\int_a^b fdg$ exists. (A function f is said to be *quasicontinuous* at $(c, f(c))$ if both $f(c+)$ and $f(c-)$ exist.)

DEFINITION 1. The statement that f has *bounded slope variation with respect to m over* $[a, b]$ means that f is a function whose domain includes $[a, b]$, m is a real-valued increasing function on $[a, b]$, and there exists a nonnegative number B such that if $\{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$ with $n > 1$, then

$$\sum_{i=1}^{n-1} \left| \frac{f(x_{i+1}) - f(x_i)}{m(x_{i+1}) - m(x_i)} - \frac{f(x_i) - f(x_{i-1})}{m(x_i) - m(x_{i-1})} \right| \leqq B .$$

The least such number B is called the slope variation of f with respect to m over $[a, b]$ and is denoted by $V_a^b(df/dm)$. [Note: $V_a^a(df/dm) = 0$.]

The above sum is nondecreasing with respect to refinements.

In [4], F. Riesz proved that a necessary and sufficient condition that a function F defined on the interval $[a, b]$ be the integral of a function of bounded variation on $[a, b]$ is that F have bounded slope variation with respect to I over $[a, b]$, where I is the function defined, for each x, by $I(x) = x$. In this paper, Riesz's result will be generalized using the Lane integral instead of the Riemann integral.

By Lemma 3.3 of [6], if f has bounded slope variation with respect to m over $[a, b]$ and $a \leqq c < b$, then

$$D_m^+ f(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{m(x) - m(c)}$$

exists and if $a < c \leqq b$,

$$D_m^- f(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{m(x) - m(c)}$$

exists.

LEMMA 1. *If f has bounded slope variation with respect to m over $[a, b]$, c is a number in $[a, b]$, and m is continuous on the right (left) at $(c, m(c))$, then f is continuous on the right (left) at $(c, f(c))$.*

Proof. Let ε denote a positive number and let c be a number in $[a, b]$. Suppose m is continuous on the right at $(c, m(c))$. Then $a \leqq c < b$ and $D_m^+ f(c)$ exists. Therefore there exists a positive number δ_1 such that if $c < x < c + \delta_1$, then

$$\left| \frac{f(x) - f(c)}{m(x) - m(c)} - D_m^+ f(c) \right| < 1$$

from which it follows that

$$| f(x) - f(c) | < [| D_m^+ f(c) | + 1] \, | m(x) - m(c) | \, .$$

Since m is continuous on the right at $(c, m(c))$, there exists a positive number δ_2 such that if $c < x < c + \delta_2$, then $| m(x) - m(c) | < \varepsilon / [| D_m^+ f(c) | + 1]$. Let $\delta = \min. [\delta_1, \delta_2]$. Then if $c < x < c + \delta$,

$$\begin{aligned} | f(x) - f(c) | &< [| D_m^+ f(c) | + 1] \, | m(x) - m(c) | \\ &< [| D_m^+ f(c) | + 1] \cdot \varepsilon / [| D_m^+ f(c) | + 1] \\ &= \varepsilon \, . \end{aligned}$$

Therefore f is continuous on the right at $(c, f(c))$.

If m is continuous on the left at $(c, m(c))$, a similar argument will show that f is continuous on the left at $(c, f(c))$.

DEFINITION 2. Suppose m is an increasing function on $[a, b]$, f is a function whose domain includes $[a, b]$ and c is a number in $[a, b]$. The statement that f has a *derivative with respect to* m at the point $(c, f(c))$ means that

$$D_m f(c) = \lim_{x \to c} \frac{f(x) - f(c)}{m(x) - m(c)}$$

exists.

THEOREM 1. *If f has bounded slope variation with respect to m over $[a, b]$, then $D_m f(x)$ exists for each x in $[a, b] - E$, where E is a countable set.*

Proof. Since f has bounded slope variation with respect to m over $[a, b]$, $D_m^+ f(x)$ exists for each x in $[a, b)$ and $D_m^- f(x)$ exists for each x in $(a, b]$. Let E_1 denote the set of all numbers x in $[a, b]$ such that $D_m^- f(x) < D_m^+ f(x)$ and let E_2 denote the set of all number x in $[a, b]$ such that $D_m^- f(x) > D_m^+ f(x)$. Let all rational numbers be arranged in a sequence $r_1, r_2, r_3, \cdots$. Then if c is a number in E_1 there is a smallest positive integer k such that

$$D_m^- f(c) < r_k < D_m^+ f(c) .$$

There is a smallest positive integer h such that $r_h < c$ and

$$\frac{f(x) - f(c)}{m(x) - m(c)} < r_k$$

for $r_h < x < c$ and a smallest positive integer n such that $r_n > c$ and

$$\frac{f(x) - f(c)}{m(x) - m(c)} > r_k$$

for $c < x < r_n$. These two inequalities together give

$$f(x) - f(c) > r_k[m(x) - m(c)] \tag{1}$$

for $r_h < x < r_n$, $x \neq c$. Thus to every number c in E_1 there corresponds a unique triad (h, k, n) of positive integers. Suppose some two numbers x_1 and x_2 of E_1 correspond to the same triad (h, k, n). Then, on putting $c = x_1$ and $x = x_2$ in (1), we have

$$f(x_2) - f(x_1) > r_k[m(x_2) - m(x_1)]$$

and, on putting $c = x_2$ and $x = x_1$,

$$f(x_1) - f(x_2) > r_k[m(x_1) - m(x_2)]$$

or

$$f(x_2) - f(x_1) < r_k[m(x_2) - m(x_1)] .$$

This involves a contradiction. Therefore no two numbers of E_1 correspond to the same triad. Since the set of triads of positive integers is countable, it follows that E_1 is countable. A similar argument will show that E_2 is countable. Therefore $E = E_1 \cup E_2$ is countable.

THEOREM 2. *If the function m is increasing on $[a, b]$, each of the functions f and g is continuous on $[a, b]$ and $D_m f(x) = D_m g(x)$ for each x in $[a, b] - H$, where H is a countable set, then $f(x) = g(x) - g(a) + f(a)$ for each x in $[a, b]$.*

Proof. Let F be the function defined, for each x in $[a, b]$, by $F(x) = f(x) - g(x)$. Then F is continuous on $[a, b]$ and $D_m F(x) = 0$ for each x in $[a, b] - H$. Let ε denote a positive number and let c be a number in $(a, b]$. Let $H \cap [a, c] = \{p_1, p_2, \cdots, p_n, \cdots\}$. Since F is continuous on $[a, b]$, for each positive integer n there exists a positive number δ_n such that if x is in $(p_n - \delta_n, p_n + \delta_n) \cap [a, c]$, then

$$| F(x) - F(p_n) | < \varepsilon/2^{n+2} .$$

Let $h_n = (p_n - \delta_n, p_n + \delta_n)$. It follows that if x_1 and x_2 are numbers in $h_n \cap [a, c]$, then

$$| F(x_1) - F(x_2) | < \varepsilon/2^{n+1} .$$

For each n, choose some particular h_n satisfying the above conditions. Now consider any number t in $[a, c] - H \cap [a, c]$. Then $D_m F(t) = 0$. If t is in (a, c), there is a positive number δ_t such that $(t - \delta_t, t + \delta_t)$ is a subset of (a, c) and if x is in $(t - \delta_t, t + \delta_t)$ and $x \neq t$, then

$$\left| \frac{F(x) - F(t)}{m(x) - m(t)} \right| < \frac{\varepsilon}{12[m(c) - m(a)]}$$

or

$$| F(x) - F(t) | < \frac{\varepsilon \, | m(x) - m(t) |}{12[m(c) - m(a)]} < \frac{\varepsilon \cdot V(t)}{12[m(c) - m(a)]}$$

where $V(t)$ is the variation of m over $[t - \delta_t, t + \delta_t]$. If $t = a$, there exists a positive number δ_a such that if $x \neq a$ and x is in $(a - \delta_a, a + \delta_a) \cap [a, c]$, then

$$| F(x) - F(a) | < \frac{\varepsilon \cdot V(a)}{12[m(c) - m(a)]}$$

where $V(a)$ is the variation of m over $[a, a + \delta_a]$. If $t = c$, there exists a positive number δ_c such that if $x \neq c$ and x is in $(c - \delta_c, c + \delta_c) \cap [a, c]$, then

$$|F(x) - F(c)| < \frac{\varepsilon \cdot V(c)}{12[m(c) - m(a)]}$$

where $V(c)$ is the variation of m over $[c - \delta_c, c]$. It follows that if t is in $[a, c] - H \cap [a, c]$ and x_1 and x_2 are numbers in $(t - \delta_t, t + \delta_t) \cap [a, c]$, then

$$|F(x_1) - F(x_2)| < \frac{\varepsilon \cdot V(t)}{6[m(c) - m(a)]} .$$

Let $g_t = (t - \delta_t, t + \delta_t)$. For each t in $[a, c] - H \cap [a, c]$, choose some particular g_t satisfying the above conditions. Let G denote the collection to which g belongs if and only if either (1) for some positive integer n, $g = h_n$ or (2) for some t in $[a, c] - H \cap [a, c]$, $g = g_t$. G is a collection of open intervals covering $[a, c]$, hence there exists a finite subcollection G' of G that covers $[a, c]$. Choose a finite chain $\{R_1, R_2, \cdots, R_k\}$ of intervals of G' covering $[a, c]$ and having the property that if $R_i \cap R_j \neq \emptyset$, then $|i - j| = 1$. Let $a = x_0$, x_1 be a number in $R_1 \cap R_2$, x_2 be a number in $R_2 \cap R_3, \cdots, x_{k-1}$ be a number in $R_{k-1} \cap R_k$, and $x_k = c$. Note that if for every $i \leqq k$, R_i is g_t for some t in $[a, c] - H \cap [a, c]$ and $V_i = V(t)$ for that t, then

$$\sum_{i=1}^{k} V_i < 3[m(c) - m(a)] .$$

Now

$$F(c) - F(a) = \sum_{i=1}^{k} [F(x_i) - F(x_{i-1})] .$$

Therefore

$$\begin{aligned} |F(c) - F(a)| &\leqq \sum_{i=1}^{k} |F(x_i) - F(x_{i-1})| \\ &= \textstyle\sum_1 |F(x_i) - F(x_{i-1})| \\ &\quad + \textstyle\sum_2 |F(x_i) - F(x_{i-1})| \end{aligned}$$

where the first sum is the sum of those terms for which R_i is some h_n and the second sum is the sum of those terms for which R_i is some g_t. Now x_{i-1} and x_i are in R_i so that

$$|F(x_i) - F(x_{i-1})| < \begin{cases} \varepsilon/2^{n+1} \text{ if } R_i = h_n \\ \dfrac{\varepsilon \cdot V(t)}{6[m(c) - m(a)]} \text{ if } R_i = g_t . \end{cases}$$

Hence

$$\textstyle\sum_1 |F(x_i) - F(x_{i-1})| < \sum_{n=1}^{\infty} \varepsilon/2^{n+1} = \varepsilon/2$$

and

$$\textstyle\sum_2 |F(x_i) - F(x_{i-1})| < \frac{\varepsilon}{6[m(c) - m(a)]} \sum_{i=1}^{k} V_i$$

$$< \frac{\varepsilon \cdot 3[m(c) - m(a)]}{6[m(c) - m(a)]} = \frac{\varepsilon}{2} \, .$$

Therefore $|F(c) - F(a)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus $F(c) = F(a)$. But c was any number in $(a, b]$. Hence for each x in $[a, b]$, $F(x) = F(a)$ or $f(x) = g(x) - g(a) + f(a)$.

THEOREM 3. *In order that the function F defined on $[a, b]$ be the Lane integral of a function f of bounded variation on $[a, b]$ with respect to a continuous, increasing function m on $[a, b]$, it is necessary and sufficient that F have bounded slope variation with respect to m over $[a, b]$.*

Proof. It is easy to see that the condition is necessary. Suppose that F has bounded slope variation with respect to m over $[a, b]$. Then F is continuous on $[a, b]$. Let f be the function defined, for each x in $[a, b]$, by

$$\begin{cases} f(x) = D_m^+F(x) \text{ for each } x \text{ in } [a, b) \\ f(b) = D_m^-F(b) \, . \end{cases}$$

Then f is of bounded variation on $[a, b]$ and is therefore quasicontinuons on $[a, b]$. Moreover, $D_mF(x) = f(x)$ for each x in $[a, b] - E$, where E is a countable set. Let G be the function defined, for each x in $[a, b]$, by $G(x) = \int_a^x f dm$. Then G is continuous on $[a, b]$ and $D_mG(x) = f(x)$ at each number x in $[a, b]$ such that f is continuous at $(x, f(x))$. Since f is quasicontinuous on $[a, b]$, $D_mG(x) = f(x)$ for each x in $[a, b] - K$, where K is a countable set. Therefore $D_mF(x) = D_mG(x)$ for each x in $[a, b] - H$, where H is a subset of $E \cup K$. It follows from Theorem 2 that $F(x) = \int_a^x f dm + F(a)$ for each x in $[a, b]$. That is, F is the Lane integral of a function f of bounded variation on $[a, b]$ with respect to a continuous, increasing function m over $[a, b]$.

It should be noted that if $m = I$, then the Lane integral reduces to the Riemann integral so that Theorem 3 contains Riesz's theorem as a special case.

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