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$C\theta\theta$-GROUPS INVOLVING NO SUZUKI GROUPS

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In the terminology of G. Higman, a finite group with order divisible by 3 in which centralizers of 3-elements are 3-groups is called a $C\theta\theta$-group.

The aim of this paper is to classify simple $C\theta\theta$-groups which involve no Suzuki simple groups.

Although simple $C\theta\theta$-groups have been studied by several authors, their general classification remains an unsolved problem.

We will prove the following

THEOREM. *Let G be a simple $C\theta\theta$-group and suppose that G involves no Suzuki simple groups. Then G is isomorphic to one of the following groups*: $PSL(2, 4)$; $PSL(2, 8)$; $PSL(3, 4)$; $PSL(2, 3^n)$, $n > 1$ *and* $PSL(2, q)$, q *such that* $(q+1)/2$ *or* $(q-1)/2$ *is a power of* 3.

It follows immediately from the Theorem that the following characterization of $PSL(2, 8)$ holds:

COROLLARY 1. *Let G satisfy the assumptions of the Theorem and suppose that no element of G of order* 3 *normalizes a nontrivial* 2-*subgroup of G. Then* $G \cong PSL(2, 8)$.

The Theorem leads also to a complete classification of simple $C\theta\theta$-groups whose order is divisible by at most four distinct primes. We have

COROLLARY 2. *Let G be a simple $C\theta\theta$-group and suppose that* $|\pi(G)| = 3$. *Then G is isomorphic to one of the following groups*: $PSL(2, 4)$, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 9)$ *and* $PSL(2, 17)$,

and

COROLLARY 3. *Let G be a simple $C\theta\theta$-group and suppose that* $|\pi(G)| = 4$. *Then G is isomorphic to one of the following groups*: $PSL(3, 4)$ *and those among* $PSL(2, 3^n)$, $n > 1$ *and* $PSL(2, q)$, $q \pm 1 = 2 \cdot 3^r$, $r > 1$, *which are divisible by exactly four distinct primes.*

2. Proof of the Theorem. We will prove first the following

PROPOSITION. *Let G be a nonsolvable $C\theta\theta$-group. Then at least one of the following statements holds.*

(i) *Whenever a section K/M of G is isomorphic to a minimal simple group L, then either L is a Suzuki group or $M = \{1\}$ and L is $PSL(2, 8)$.*

(ii) *Some nontrivial 2-subgroup of G is normalized by an element of order 3.*

Proof of the Proposition. Let G be a counter-example. Then there exist subgroups K and M of G, M normal in K, such that K/M is isomorphic to a minimal simple group L which is not of Suzuki type and if $M = \{1\}$ then L is not $PSL(2, 8)$. By Thompson's theorem [5, Corollary 1] L is one of the following: $PSL(2, 2^p)$, p any prime, $PSL(2, 3^p)$, p any odd prime; $PSL(2, p)$, p any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod 5$ and $PSL(3, 3)$. Denote by Q the Sylow 3-subgroup of K. Since a Sylow 3-subgroup of a nonsolvable $C\theta\theta$-group is abelian [1, Theorem 2.9], L is not $PSL(3, 3)$ and Q is the centralizer in K of each of its nonunit elements. Suppose that there exists a normal complement S of $N_K(Q)$ in K. Since M is a maximal normal subgroup of K, it follows that either $K = N_K(Q)M$ or $K = SM$, and consequently $L = K/M$ has a normal (possibly trivial) Sylow 3-subgroup, a contradiction. Thus $N_K(Q)$ has no normal complement in K and by [2, Theorem 2.3.e] the fact that 3 divides the order of L implies that 3 does not divide the order of M. It follows then by the results of Stewart[1], [4, Propositions 3.2 and 4.2] that $M = \{1\}$ if $L = PSL(2, q)$, where $q = 3^p$ or $q = p > 5$ and M is a 2-group if $L = PSL(2, q)$, where $q = 2^p$. Since no element of order 3 in G normalizes a nontrivial 2-subgroup of G, $M = \{1\}$ in all cases and L is not $PSL(2, 8)$. It is well known that the Sylow 2-subgroups of $PSL(2, q)$, where $q = 3^p$, $p > 2$ or $q = p > 3$, p a prime, are normalized by an element of order 3. Consequently, $L = PSL(2, 2^p)$, where p is a prime exceeding 3. Since the Sylow 3-subgroups in L are the centralizers of each of their nonidentity elements, it follows that $2^p \pm 1 = 3^k$ for some k. This equation has no solution for $p > 3$ and consequently G does not exist.

We proceed with a proof of the Theorem. If case (ii) of the Proposition holds, then it follows from the results of Fletcher and Gorenstein [1, Corollary 3.2] that G is isomorphic to one of the groups mentioned in the Theorem, $PSL(2, 8)$ excluded. If case (i) of the Proposition holds, but not case (ii), then we will show that G is an N-group and it follows then from Thompson's classification theorem of

[1] The author is indebted to Dr. W. B. Stewart for communication of results prior to publication.

simple N-groups [5] that only $PSL(2, 8)$ is a $C\theta\theta$-group of the required type.

Let U be a p-subgroup of G with a nonsolvable normalizer. By our assumptions and by the Proposition $N = N_G(U)$ contains a subgroup V isomorphic to $PSL(2, 8)$. As $V \cap U = \{1\}$, VU/U is isomorphic to $PSL(2, 8)$ and consequently $U = \{1\}$. Thus G is an N-group and the proof is complete.

3. Proof of the corollaries. Since $PSL(2, 8)$ is the only group mentioned in the Theorem without an element of order 3 normalizing a nontrivial 2-subgroup, Corollary 1 immediately follows from the Theorem.

If $|\pi(G)| = 3$ then G does not involve Suzuki groups and by the Theorem and [3, Theorem 3] it is isomorphic to one of the following: $PSL(2, 4)$, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 9)$ and $PSL(2, 17)$.

Corollary 3 follows from the fact that 3 divides the order of G and 3 does not divide the order of the Suzuki groups. Consequently, as the Suzuki groups have orders divisible by at least 4 distinct primes, G does not involve them. Corollary 3 follows therefore immediately from the Theorem.

References

1. L. R. Fletcher, *Some finite simple groups*, Ph. D. Thesis, Oxford University, 1969.

2. M. Herzog, *On finite groups which contain a Frobenius subgroup*, J. Algebra, **6** (1967), 192-221.

3. ———, *On finite groups with cyclic Sylow subgroups for all odd primes*, Israel J. Math., **6** (1968), 206-216.

4. W. B. Stewart, *Groups having strongly self-centralizing 3-centralizers*: I Preliminaries, to appear.

5. J. G. Thompson, *Non-Solvable finite groups all of whose local subgroups are solvable*, Bull. Amer. Math. Soc., **74** (1968), 384-437.

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