PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 3, 1971

SUBDIRECTLY IRREDUCIBLE IDEMPOTENT SEMIGROUPS

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Subdirectly irreducible idempotent semigroups have been discussed by B.M. Schein. Using his results, a subdirectly irreducible idempotent semigroup is shown to be either a semigroup of mappings of a set into itself, with certain stated conditions, or the dual of such a semigroup, or one of these with an adjoined zero. This characterization is used to show that, with a few exceptions, the join reducible elements of the lattice of equational classes of idempotent semigroups contain only subdirectly irreducible members belonging to proper subclasses. This result gives structure theorems, special cases of which appear in the literature. An example is also given of an infinite subdirectly irreducible idempotent semigroup in the equational class of idempotent semigroups defined by $xyx = xy$.

The starting point for the study in this paper is the paper of B. M. Schein [5], in which he gives necessary conditions for an idempotent semigroup to be subdirectly irreducible. This result is exploited in § 1 to give a characterization of subdirectly irreducible idempotent semigroups (Theorem 1.5).

In § II, the description of the lattice of equational classes of idempotent semigroups given in [1], and the characterization of subdirectly irreducible idempotent semigroups given in § I are used to show that a subdirectly irreducible idempotent semigroup which satisfies certain equations also satisfies other, more restrictive equations. These results are used to prove Theorem 2.4 and its corollary. These give conditions under which a join reducible equational class contains no subdirectly irreducibles not contained in a proper subclass. The other join reducible classes are subclasses of the class defined by the equation $(xyzx = xzyx)$, and a complete list of the subdirectly irreducibles of this class is given. (There are only six of these; the cardinality of the largest is three).

Finally we give an example of an infinite subdirectly irreducible idempotent semigroup which satisfies the equation $(xyx = xy)$. This answers a question of Schein [6] and shows that all equational classes of idempotent semigroups except the subclasses of the class defined by $(xyzx = xzyx)$ contain infinite subdirectly irreducible idempotent semigroups.

1. The characterization. Let S be a semigroup. If $a, b \in S$, let $\theta(a, b)$ denote the smallest congruence which identifies a and b.

Let $\Delta = \{(s, s) \mid s \in S\}$. The semigroup S is *subdirectly irreducible* if and only if there exist $a, b \in S, a \neq b$, such that $\theta(a, b) \subseteqq \theta(c, d)$ for all $c, d \in S, c \neq d$. The congruence $\theta(a, b)$ is just the smallest congruence of S which is distinct from Δ.

The *dual* $(S^*, *)$ of a semigroup $(S, \cdot)$ is defined by $S^* = S$ and $a*b = b\cdot a$ for all $a, b \in S$. If we denote a semigroup simply by S, we denote its dual by S^*.

A *zero* of a semigroup is an element $0 \in S$ which satisfies $x0 = 0x = 0$ for all $x \in S$. A semigroup has at most one zero.

B. M. Schein has shown (Theorem 3.6 and Theorem 4.7 of [5]), that in order to characterize subdirectly irreducible idempotent semigroups, it is sufficient to consider those without zero. This result is given in the following theorem.

THEOREM 1.1 (Schein [5]). *An idempotent semigroup S which contains a zero 0 is subdirectly irreducible if and only if S—$\{0\}$ is a subsemigroup which is subdirectly irreducible and (if $|S| > 2$) contains no zero.*

For the remainder of this section we will deal only with subdirectly irreducible idempotent semigroups without zero.

THEOREM 1.2 (Schein [5], Theorem 4.7). *If S is a subdirectly irreducible idempotent semigroup* (without zero), *then S satisfies one of the following conditions.*

(1.3) $$Let\ K = \{k \mid ks = k\ for\ all\ s \in S\}\ .$$

Then K is a two-sided ideal of S, and for any $x, y \in S, xk = yk$ for all $k \in K$ implies $x = y$.

(1.3)* $$Let\ K = \{k \mid sk = k\ for\ all\ s \in S\}\ .$$

Then K is a two-sided ideal of S, and for any $x, y \in S, kx = ky$ for all $k \in K$ implies $x = y$.

It is clear that S satisfies (1.3) if and only if S^* satisfies (1.3)*. Because of this we will say that condition (1.3)* is the dual of (1.3). In a similar way the following lemma will give rise to a dual lemma (which we will label with*) and which will be obtained just as (1.3)* was obtained from (1.3), that is by replacing the operation $\cdot$ everywhere it occurs by the operation*, or more simply by assuming the original condition holds for S^* instead of S.

LEMMA 1.4. *Let R be a subdirectly irreducible idempotent semi-*

group (without zero) *which satisfies* (1.3). *Let* $\theta(a, b)$ *be the smallest congruence of* S *distinct from* Δ. *Then* $a \in K$.

Proof. Let $k \in K$. ($K = \emptyset$ is impossible by (1.3) since S has no zero and hence $|S| > 1$.) Since k is not a zero of S and $ks = k$ for all $s \in S$, there exists $s \in S$ such that $sk \neq k$. It follows that $(a, b) \in \theta(sk, k)$. Since K is a two-sided ideal, $\theta(sk, k) \subseteqq K^2$, and therefore $a \in K$.

THEOREM 1.5. *An idempotent semigroup* S (without zero) *is subdirectly irreducible if and only if* S *or* S^* *is isomorphic to a semigroup* T *which satisfies the following two conditions*:

(i) $C(X) \subseteqq T \subseteqq X^X$, *where* X^X *is the semigroup of all mappings of* X *into itself,* $C(X)$ *is the set of all constant mappings of* X, *and* T *is a subsemigroup of* X^X.

(ii) *There exists* k, $k' \in C(X)$ *such that* $\theta(k, k') \subseteqq \theta(c, d)$ *for all* $c, d \in C(X)$, $c \neq d$.

Proof. Let S satisfy (1.3). Define $\varphi: S \to K^K$ by $(\varphi(s))(k) = sk$, for all $s \in S$, $k \in K$. It is easy to check that φ is a homomorphism, and that φ is one-one. The monomorphism establishes (i) and (ii) for $\varphi(S)$, since $\varphi(K) = C(K)$. If S satisfies (1.3)*, the above argument shows that S^* is isomorphic to a semigroup T satisfying (i) and (ii).

To establish the converse, it is enough to show that if T satisfies (i) and (ii), then T is subdirectly irreducible. By (ii) it is enough to show that if $s, t \in T$, $s \neq t$, then there exist $c, d \in C(X)$, $c \neq d$, such that $\theta(c, d) \subseteqq \theta(s, t)$. Since $s \neq t$, there exists $k \in C(X)$ such that $sk \neq tk$. Since $sk, tk \in C(X)$ and $\theta(sk, tk) \subseteqq \theta(s, t)$, the proof is complete.

II. Subdirectly irreducible idempotent semigroups and equational classes.

1. Join reducible classes.

The lattice of equational classes of idempotent semigroups has been completely described in [1]. We refer the reader to that paper for any notation not explained here, and especially to Fig. 2 of [1], which gives a summary of some of the results which we use. Our first task is to show that a subdirectly irreducible semigroup which is assumed to satisfy certain equations can then be shown to satisfy more restrictive equations.

LEMMA 2.1. *Let* S *be a subdirectly irreducible idempotent semigroup which satisfies* (1.3). *If* S *satisfies* $(p = q)$, *where for some* $n \geqq 3$, $|E(p)| = n$, $p \sim {}_n q$, *and if* $n = 3$, pR_3q, *then* S *satisfies* $(p(0) = q(0))$.

Proof. Without loss of generality, $p = p(0)\bar{p}(0)\bar{F}(p)$ and $q = q(0)\bar{q}(0)\bar{F}(q)$, (by Theorem 1.8 of [1]). It follows immediately from Proposition 2.13 of [1] if $n \geqq 4$, and from the definition of R_3, if $n = 3$, that $\bar{p}(0) = \bar{q}(0)$. Let ψ be any substitution of the variables occuring in $(p = q)$ by elements of S. (As usual ψ may be thought of as a mapping of the free semigroup on suitable generators into S.) Since $p = q$ holds in S we have

$$\psi(p(0))\psi(\bar{p}(0))\psi(\bar{F}(p)) = \psi(q(0))\psi(\bar{q}(0))\psi(\bar{F}(q)) .$$

In particular if $\psi(\bar{p}(0)) = \psi(\bar{q}(0)) = k \in K$, then $\psi(p(0))k = \psi(q(0))k$. Now $\bar{p}(0) \notin E(p(0)) = E(q(0))$, and consequently the above argument shows that $\psi(p(0))k = \psi(q(0))k$ for all $k \in K$. It follows from Lemma 1.6 that $\psi(p(0)) = \psi(q(0))$ and therefore that $p(0) = q(0)$ holds in S.

LEMMA 2.2. *Let S be a subdirectly irreducible idempotent semigroup which satisfies* (1.3). *If S satisfies an equation $(p = q)$, where pR_nq, $p\mathcal{S}_nq$ for some $n \geqq 3$, then S also satisfies an equation $(p_0 = q_0)$ where if $n \geqq 4$, $p_0T_{n-1}q_0$, $p_0R^*_{n-1}q_0$, $p_0\mathcal{S}^*_{n-1}q_0$ and if $n = 3$, $I(p_0) = I(q_0)$* (and therefore $E(p_0) = E(q_0)$) *and $H^*(q_0) \neq H^*(q_0)$.*

Proof. It is enough to prove that the Lemma holds if in addition pT^*_nq. In this case we can assume without loss of generality that $|E(p)| = n$, since $(p = q)$ defines a member of the skeleton of the lattice. We will show that $(p(0) = q(0))$ can be taken for the equation $(p_0 = q_0)$.

By Lemma 2.1 we know that S satisfies $(p(0) = q(0))$. The fact that $p(0)$ and $q(0)$ satisfy the conditions of the present lemma is given implicitly in [1]. The rest of the proof consists of showing which results of [1] can be used to show that $p(0)$ and $q(0)$ satisfy these conditions.

We consider the case $n \geqq 4$ and $n = 3$ separately. If $n \geqq 4$, it follows from Proposition 2.13 of [1] that $\bar{I}(p(0)) \sim \bar{I}(q(0))$ and therefore in particular that $p(0)T_{n-1}q(0)$. Since pR_nq, it follows immediately from (2.20) of [1] that $p(0)R^*_{n-1}q(0)$. If we had $p(0)S^*_{n-1}q(0)$, then by (2.17) of [1], it would follow that $p(0) \sim q(0)$, and therefore that pS_nq. This contradiction shows that $p(0)\mathcal{S}^*_{n-1}q(0)$.

If $n = 3$ the definition of pR_3q gives immediately that $I(p(0)) = I(q(0))$. If $H^*(p(0)) = H^*(q(0))$, it follows from Proposition 2.4 of [1] that $p(0) \sim {}_3q(0)$. Since $|E(p(0))| = 2$, this implies that $p(0) \sim q(0)$, and therefore in particular that pS_3q. This contradiction establishes that $H^*(p(0)) \neq H^*(q(0))$, completing the proof of the lemma.

LEMMA 2.3. *Let S be an idempotent semigroup which satisfies*

(1.3). *If S satisfies an equation $(p = q)$ where pR_nq for some $n \geqq 3$, then S also satisfies an equation $(p_0 = q_0)$ where if $n \geqq 4$, $p_0S_{n-1}q_0$* and *$p_0R^*_{n-1}q_0$, and if $n = 3$, $E(p_0) \neq E(q_0)$, $H(p_0) = H(q_0)$, $H^*(p_0) \neq H^*(q_0)$.*

Proof. We deal with the case $n = 3$ separately since Lemma 2.1 cannot be applied in this case. It follows from results of [1] (see Fig. 2) that if S satisfies an equation $(p = q)$, where pR_3p, it satisfies $(xyzx = xzyzx)$, since this equation satisfies pR_3q and pS_3^*q. But then $xyz = xz$ for all $x, y \in S$, $z \in K$, and therefore, $xy = x$ for all $x, y \in S$. The equation $(xy = x)$ can be taken for the equation $(p_0 = q_0)$ in case $n = 3$, since it satisfies the given conditions.

In case $n \geqq 4$, we proceed in a manner similar to that used for the proof of Lemma 2.2. We may assume in addition that pS_n^*q and that $|E(p)| = n$, and apply Lemma 2.1 to show that $(p(0) = q(0))$ holds in S. It remains only to show that $p(0)$ and $q(0)$ satisfy the stated conditions.

As in the proof of Lemma 2.2 we can establish that $\bar{I}(p(0)) \sim \bar{I}(q(0))$ and therefore in particular that $p(0)S_{n-1}q(0)$. That $p(0)R^*_{n-1}q(0)$ is an immediate consequence of 2.20 of [1].

THEOREM 2.4. *Let $\mathfrak{A} = \mathfrak{A}_1 \vee \mathfrak{A}_2$ be the join of its proper subclasses $\mathfrak{A}_1$ and $\mathfrak{A}_2$, in the lattice of equational classes of idempotent semigroups. If $S \in \mathfrak{A}$ is subdirectly irreducible without zero, then $S \in \mathfrak{A}_1 \cup \mathfrak{A}_2$.*

Proof. The classes $\mathfrak{A}$ which satisfy the hypothesis of this theorem can be obtained from Fig. 2 of [1]. We will prove the theorem for each in turn. It is of course enough to show that each subdirectly irreducible $S \in \mathfrak{A}$ is contained in a proper subclass of $\mathfrak{A}$. We list the several cases and indicate the results used in each. We give a complete proof for the first example but leave the details of the other examples to the reader.

Let $\mathfrak{A}$ be defined by an equation $(p = q)$ for which pR_nq, pS_nq, pR_n^*q and assume $n \geqq 4$. If S satisfies (1.3) we can use Lemma 2.2 to show that S satisfies an equation $(p_0 = q_0)$ where $p_0T_{n-1}q_0$, $p_0R^*_{n-1}q_0$, $p_0S^*_{n-1}q_0$. It follows that $(p_0 = q_0)$ defines a proper subclass of $\mathfrak{A}$. If S satisfies (1.3)*, S^* satisfies (1.3) and we can use Lemma 2.3 for S^* (or Lemma (2.3)* for S) to show that S^* satisfies an equation $(p_0 = q_0)$ where $p_0S_{n-1}q_0$ and $p_0R_{n-1}q_0$. But then S satisfies an equation $(p_0^* = q_0^*)$ where $p_0^*S^*_{n-1}q_0^*$ and $p_0^*R_{n-1}q_0^*$, and $(p_0^* = q_0^*)$ therefore defines a proper subclass of $\mathfrak{A}$. The proof for $n = 3$ is similar.

If $\mathfrak{A}$ is defined by an equation $(p = q)$ where pR_nq, pR_n^*q, pS_nq, pS_n^*q, then if S satisfies (1.3) use Lemma 2.2, and if S satisfies (1.3)* use Lemma (2.2)*.

If $\mathfrak{A}$ is defined by an equation $(p = q)$ where pT_nq, pS_nq, pR_n^*q, pS_n^*q, then an equation $(p' = q')$ where $p'R_{n+1}q'$ also holds in $\mathfrak{A}$. If S satisfies (1.3) use Lemma 2.3 (in case $n + 1$), and if S satisfies (1.3)* use Lemma (2.2)*.

If $\mathfrak{A}$ is defined by an equation $(p = q)$ where pR_nq, pR_n^*q then if S satisfies (1.3) use Lemma 2.3, and if S satisfies (1.3)* use Lemma (2.3)*.

If $\mathfrak{A}$ is defined by an equation $(p = q)$ where $E(p) = E(q)$, $H(p) = H(q)$, $I(p) \neq I(q)$, $H^*(p) \neq H^*(q)$, then an equation $(p' = q')$ where $p'R_3^*q'$ and $p'R_3q'$ also holds in $\mathfrak{A}$. If S satisfies (1.3) use Lemma 2.3 to show that S satisfies an equation $(p_0 = q_0)$ where $E(p_0) \neq E(q_0)$, $H(p_0) = H(q_0)$, $H^*(p_0) \neq H^*(q_0)$. If S satisfies (1.3)* use Lemma (2.3)* to show that S satisfies an equation $(p_0 = q_0)$ where $E(p_0) \neq E(q_0)$, $H(p_0) \neq H(q_0)$ and $H^*(p_0) = H^*(q_0)$. Since S satisfies $(p_0 = q_0)$ and $(p = q)$ it also satisfies the equation $(x = y)$ which defines a proper subclass of $\mathfrak{A}$.

The remaining cases are just the duals of the above.

COROLLARY 2.5. *Let* $\mathfrak{A} = \mathfrak{A}_1 \vee \mathfrak{A}_2$, *where* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *are defined by equations* $(f_1 = g_1)$ *and* $(f_2 = g_2)$ *where* $E(f_1) = E(g_1)$ *and* $E(f_2) = E(g_2)$. *If* $S \in \mathfrak{A}$ *is subdirectly irreducible then* $S \in \mathfrak{A}_1 \cup \mathfrak{A}_2$. *($E(f)$ is the set of variables of f.)*

Proof. The theorem establishes the result in case S does not contain a zero. By Theorem 1.1, the subdirectly irreducible semigroups with zero are obtained by adding a zero to a semigroup without zero. It is easy to see that if $\bar{S} = S \cup \{0\}$ is a semigroup such that S is a subsemigroup, the equations $(f = g)$ true in $\bar{S}$ are exactly these true in S with $E(f) = E(g)$, and this establishes the corollary.

REMARK 1. Corollary 2.5 is equivalent to the statement that for $\mathfrak{A}$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ as given, if $S \in \mathfrak{A}$, then S is a subdirect product of $S_1 \in \mathfrak{A}_1$ and $S_2 \in \mathfrak{A}_2$. This result can of course be proved without reference to subdirectly irreducibles. In fact the solutions of the various word problems, which show that $f = g$ holds in $\mathfrak{A}$ if and only if a condition holds for $\bar{I}(f)$ and $\bar{I}(g)$, and a second condition holds for $\bar{F}(f)$ and $\bar{F}(g)$, enable us to define congruences θ_1 and θ_2, on any free algebra T in $\mathfrak{A}$, such that $\theta_1 \wedge \theta_2 = \Delta$. The splitting of non-free algebras is accomplished via the appropriate isomorphism theorem by showing that $(\theta_1 \vee \theta) \wedge (\theta_2 \vee \theta) = \theta$ for any congruence θ on T. This last result is easy to check since by definition if $f\theta_1u$ and $f\theta_2v$, then $f = uv$ in T. Taking $(f, g) \in (\theta_1 \vee \theta) \wedge (\theta_2 \vee \theta)$, and using this result enables us to show immediately that $(f, g) \in \theta$. We leave the details to the interested reader.

REMARK 2. Special cases of the above results are known ([2], [3], [4], [7]). In particular the paper of M. Petrich [3] gives the results of our next section. We include the next section, however, for completeness and because the arguments become especially simple in view of Theorem 2.4 and its Corollary.

2. Subdirectly irreducible semigroups in subclasses of the class defined by the equation $(xyzx = xzyx)$.

The hypotheses of Corollary 2.5 hold for all join reducible elements of the lattice of equational classes of idempotent semigroups except those which are proper subclasses of the class defined by $(xyzx = xzyx)$. The conclusion of Corollary 2.5 holds for the class defined by $(xyx = x)$ but fails for the class defined by $(xyz = xzy)$ and the class defined by its dual. To show this we list all subclasses of the class defined by $(xyzx = xzyx)$, together with their subdirectly irreducible elements.

In the class defined by $(x = y)$ the only (subdirectly irreducible) member is the semigroup S_0 consisting of one element. In the class defined by $(xy = x)$, the subdirectly irreducible members are S_0, and S_1, the two element semigroup which satisfies $(xy = x)$. In the class defined by $(xy = yx)$, there can be no subdirectly irreducible semigroup satisfying (1.3) or (1.3)* since (by Lemma 1.4), $|K| \geqq 2$, and for $k, k' \in K$, $k \neq k'$, $kk' = k \neq k' = k'k$. The subdirectly irreducible elements in this class are therefore just S_0 and $S_0 \cup \{0\}$. By our general results, the subdirectly irreducible elements of the class defined by $(xyz = xzy)$ are S_0, $S_0 \cup \{0\}$, S_1 and the semigroups formed from these by adding a zero if the semigroup has no zero. This gives the new semigroup $S_1 \cup \{0\}$. In the class defined by $(xyx = x)$ the only subdirectly irreducible elements are S_1, S_1^* and S_0 since $E(xyx) \neq E(x)$. Finally the subdirectly irreducible elements of the class defined by $(xyzx = xzyx)$ are just S_0, $S_0 \cup \{0\}$, S_1, $S_1 \cup \{0\}$, S_1^*, $S_1^* \cup \{0\}$, The remaining subclasses of the class defined by $(xyzx = xzyx)$ are the duals of classes already listed.

3. An infinite subdirectly irreducible idempotent semigroup satisfying $(xyx = xy)$.

In the previous section we listed several equational classes of idempotent semigroups with only finite subdirectly irreducible elements. In this section we show that every other equational class has infinite subdirectly irreducible members by exhibiting an infinite subdirectly irreducible idempotent semigroup satisfying $(xyx = xy)$. This answers a question of B. M. Schein [6].

Let S consist of the following mappings of the set of natural numbers into itself. The mappings a_i, $i \geqq 0$ are the constant mappings defined by $a_i(j) = i$ for all j. The mappings b_i, c_i, $i \geqq 2$, are defined by $b_i(i) = c_i(i) = i$, $b_i(j) = 0$, $i \neq j$, $c_i(j) = 1$, $i \neq j$. Since the a_i, b_i and c_i are idempotent, we can show that S is an idempotent semigroup by proving that it is closed with respect to composition of mappings. If one of the factors is a constant the result is a constant. If both are non-constants, they occur among the following: $b_ic_i = b_i$, $c_ib_i = c_i$ and if $i \neq j$, $b_ib_j = b_ic_j = a_0$, $c_ic_j = c_ib_j = a_1$.

In order to show that S is subdirectly irreducible it is enough (by Theorem 1.5) to show that $\theta(a_0, a_1) \subseteqq \theta(a_i, a_j)$ for all $i \neq j$. If $i \neq j$, $\theta(a_i, a_j)$ contains $(a_i = b_ia_i,\ a_0 = b_ia_j)$ and $(a_i = c_ia_i,\ a_1 = c_ia_j)$ and therefore (a_0, a_1).

We now show that S satisfies $(xyx = xy)$. If x, y or xy is a constant, or if $x = y$, the result is trivial. The only other case is $\{x, y\} = \{b_i, c_i\}$ for some i, and in that case it is easy to check that $xyx = xy$.

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Received February 23, 1971 and in revised form June 11, 1971. The results of this paper were announced in abstract 71T-A67 Notices America Math. Soc. This research was supported by the National Research Council of Canada.

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