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ON THE BRAUER GROUP OF Z

ROBERT A. MORRIS

Two dimensional Amitsur cohomology is computed for certain rings of quadratic algebraic integers. Together with computations of Picard groups, this yields information on the Brauer group $B(S/Z)$, for S quadratic algebraic integers, without resort to class field theory.

The classical Brauer group of central simple algebras over a field [10, X, Sec. 5] has been generalized to the Brauer group $B(R)$ of central separable R-algebras over a commutative ring R [2]. One can prove, using class field theory, that the Brauer group $B(Z)$ of the integers, is trivial. The proof is apparently well known but not in the literature, although it does appear in the dissertation of Fossum [9].

This paper is devoted to our attempt to establish this result using only an exact sequence of Chase and Rosenberg [7, p. 76]. We are able to show that if S is the integers of $Q(\sqrt{m})$ for $m = \pm 3, -1, 2$, or 5, the subgroup $B(S/Z)$ of $B(Z)$ consisting of elements split by S, vanishes.

In § 2 we develop some technical results on norms which we use in § 3 to show that the Amitsur cohomology group $H^2(S/Z, U)$ is zero whenever S is the ring of integers of a quadratic extension of the rationals. In § 4 we use a Mayer-Vietoris sequence of algebraic K-theory to show that the Picard group $\mathrm{Pic}(S \otimes_Z S) = 0$ for S the integers of $Q(\sqrt{m})$, $m = \pm 3, -1, 2$, or 5. In § 5 we use this result and an exact sequence of Chase and Rosenberg [7, p. 76] to show $B(S/Z) = 0$ for these rings.

Dobbs [8] has results relating $B(S/Z)$ to $H^2(S/Z, U)$ which together with the triviality of $B(Z)$ imply our results.

§ 2. **Norms.** If S is a commutative algebra over a commutative ring R, S^n denotes $S \otimes S \cdots \otimes S$, n times (here and throughout, $\otimes$ means $\otimes_R$), and $\varepsilon_i \colon S^n \to S^{n+1}$, $i = 0, \cdots, n$, is given by $x_0 \otimes \cdots \otimes x_{n-1} \to x_0 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_{n-1}$. These maps satisfy $\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i$ for $i \leqq j$. For any ring A, $U(A)$ denotes the group of units of A. All unexplained notation and terminology is an in [7].

THEOREM 2.0. *Let M/K be a galois extension of commutative rings* [6], *with group G, and let F be an additive functor on a full subcategory $\mathscr{C}$ of the category of commutative K-algebras, and suppose M and $M \otimes_K M$ lie in $\mathscr{C}$. Then for any x in $F(M)$, $y = \sum_{g \text{ in } G} Fg(x)$*

lies in $Ker(F\varepsilon_0 - F\varepsilon_1)$.

Proof. By Theorem 3.1 of [6] there are orthogonal idempotents e_g (g in G), in $M \otimes_K M$ with $\sum_g e_g = 1$ and $s \otimes 1 = \sum_g (1 \otimes g(s))e_g$. In the above notation this becomes: $\varepsilon_1(s) = \sum_g \varepsilon_0(g(s))e_g$ for all s in M.

Now $\sum_g e_g = 1$ implies that $M \otimes_K M = \Pi(M \otimes_K M)e_g$ as K-algebras. Thus, if π_g denotes the projection of the g^{th} component, we have $\pi_g \varepsilon_1 = \pi_g \varepsilon_0 g$ as maps $M \to (M \otimes_K M)e_g$. Now $y = \sum_{h \text{ in } G} Fh(x)$ is trivially invariant under Fg so we obtain $F\pi_g F\varepsilon_1(y) = F\pi_g F\varepsilon_0 Fg(y) = F\pi_g F\varepsilon_0(y)$ for each g in G. By the additivity of F, this implies $F\varepsilon_0(y) = F\varepsilon_1(y)$ as was to be shown.

Now let R be the ring of integers of an algebraic number field K. Let M be a finite galois field extension of K with group G and S its ring of integers and let $M: K = n$. For each $i \geqq 0$ there is a map $n_i: U(S^{i+1}) \to U(S^i \otimes R)$ given by $n_i(\sum x_0 \otimes \cdots \otimes x_i) = \Pi_{g \text{ in } G} \sum x_0 \otimes \cdots \otimes x_{i+1} \otimes g(x_i)$.

Now S^{i+1} is projective, hence faithfully flat as an $S^i \otimes R$ module. By [7, Lemma 3.8] $S^i \otimes R = \text{Ker}(S^{i+1} \xrightarrow{\varepsilon_0 - \varepsilon_1} S^{i+1} \otimes_S s_{\otimes R} S^{i+1})$, so applying Thm. 2.0 to $M^{i+1}/(M^i \otimes_K K)$(here $M^j = M \otimes_K M \cdots \otimes_K M$) noting that the natural map $S^n \to M^n$ is injective for all n we see that the map n_i indeed has its image in $S^i \otimes R$.

DEFINITION. The ith *norm map*, N^i: $U(S^{i+1}) \to U(S^i)$ is Cn_i where $C: S^i \otimes R \to S^i$ is the natural isomorphism. N^i is easily seen to be an abelian group map.

LEMMMA 2.1. *If* ε_j: $U(S^{i+1}) \to U(S^{i+2})$ *denote the maps defined at the beginning of the section, then* $N^{i+1}\varepsilon_j(x) = \varepsilon_j N^i(x)$ *for* $0 \leqq j < i + 1$ *and* $N^{i+1}\varepsilon_{i+1}(x) = x^n$, *where* $n = M: K$.

Proof. Clear

PROPOSITION 2.2. *If* d^i: $U(S^{i+1}) \to U(S^{i+2})$ *is the Amitsur coboundary (given by* $d^i(x) = \prod_{j=0}^{i+1} \varepsilon_j(x^{(-1)^j})$), *then* $N^{i+1}d^i(x) = [d^{i-1}N^i(x)](x^n)^{(-1)^{i+1}}$.

Proof.

$$N^{i+1}d^i(x) = N^{i+1}\prod_{j=0}^{i+1} \varepsilon_j(x^{(-1)^j}) = \prod_{j=0}^{i+1} N^{i+1}\varepsilon_j(x^{(-1)^j})$$
$$= [\prod_{j=0}^{i} \varepsilon_j N^i(x^{(-1)^j})](x^n)^{(-1)^{i+1}}$$

by Lemma 2.1. The proposition then follows from the definition of d^{i-1}.

COROLLARY 2.3. *If for x in $U(S^{i+1})$ we have $d^i(x) = 1$, then* $(x^n)^{(-1)^i} = d^{i-1}(N^i(x))$. *In particular,* $nH^i(S/R, U) = 0$ *for* $i \geqq 1$.

REMARK. The above are all closely parallel to results of Amitsur [1, Thm. 2.10] who defines a norm map via determinants whenever S/R is finitely generated and free. In the that case, our norm maps agree with Amitsur's [1, Lemma 5.2].

We are primarily interested in two-cocycles:

COROLLARY 2.4. *Let x in $U(S^3)$ have $d^2(x) = 1$. Then $N^1N^2(x)$ is in* $U(R)\cdot 1_S$.

Proof. By Corollary 2.3 with $i = 2$, $x^n = d^1(N^2(x))$ and so $N^2(x^n) = N^2d^1(N^2(x)) = [d^0N^1N^2(x)]N^2(x^n)$ by Proposition 2.2 with $i = 1$. Hence $d^0[N^1N^2(x)] = 1$ in $S \otimes S$. Since S is projective, hence faithfully flat, over R, it follows from Lemma 3.8 of [7] that $N^1N^2(x)$ is in $R\cdot 1_S$; say $N^1N^2(x) = r\cdot 1_S$. A priori r is a unit in S, but not obviously so in R. Let t be the inverse in S of $r\cdot 1_S$ and let t satisfy the integral equation $x^m + r_1x^{m-1} + \cdots + r$ in $R[x]$. So

$$0 = (t^m + r_1t^{m-1} + \cdots + r_m)r^m\cdot 1 = 1 + r_1r + \cdots + r_mr^m.$$

Hence r is a unit in R, completing the proof.

Henceforth we will suppress the superscripts on norm maps.

Finally we give a technical lemma of general application:

LEMMA 2.5. *If R is any commutative ring and S a faithfully flat R-algebra, then* ɒ *two cocycle x in $U(S^3)$ lies in $S \otimes S \otimes 1$ if and only if x is in $1 \otimes S \otimes 1$. In this case x is a coboundary.*

Proof. One implication is trivial.

If x is in $S \otimes S \otimes 1$ we may write $x = \varepsilon_2(a) = a \otimes 1$ for some a in $S \otimes S$. Then $1 = d^2(x) = \varepsilon_0(x)\varepsilon_1(x^{-1})\varepsilon_2(x)\varepsilon_3(x^{-1})$. Since $x = \varepsilon_2(a)$, it is clear that $\varepsilon_2(x) = \varepsilon_3(x)$, so that

$$1 = \varepsilon_0(x)\varepsilon_1(x^{-1}) = \varepsilon_0\varepsilon_2(a)\varepsilon_1\varepsilon_2(a^{-1}) = \varepsilon_3\varepsilon_0(a)\varepsilon_3\varepsilon_1(a^{-1}).$$

Since ε_3 is a monomorphism, we have $\varepsilon_0(a) = \varepsilon_1(a)$. As in the previous result, an application of Lemma 3.8 of [7] shows that a is in $1 \otimes S$ so that $x = a \otimes 1$ is in $1 \otimes S \otimes 1$. We must have $a = 1 \otimes u$ for some unit u of S and so $x = 1 \otimes u \otimes 1 = d^1(1 \otimes u) = d^1(a)$.

3. The cohomology of quadratic integers. In this section we use the results of the last section for explicit computations of cohomology groups. In this section $R = Z$ and S is the ring of integers of a quadratic field extension, K, of the rationals, Q. Thus $K = Q(\sqrt{m})$ for a square free integer m. The computations naturally divide themselves into the cases $m \equiv 2$ or 3 and $m \equiv 1 \pmod 4$.

THEOREM 3.0. *Let $K = Q(\sqrt{m})$ with $m \equiv 2$ or $3 \pmod 4$. If S denotes the ring of integers of K, then $H^2(S/Z, U) = 0$.*

Proof. Let $\rho = \sqrt{m}$. Then $\{1, \rho\}$ constitutes a basis of S over Z [12, Thm. 6-1-1]. For any x and y in Z, the nontrivial Q-automorphism takes $x + y\rho$ to $x - y\rho$, so that $N(x + y\rho) = (x + y\sqrt{m})(x - y\sqrt{m}) = x^2 - my^2$.

Now S^i is free over S^{i-1} (acting on the first $i - 1$ factors) with generators $1_{S^{i-1}} \otimes 1$ and $1_{S^{i-1}} \otimes \rho$, so that $N(x \otimes 1 + y \otimes \rho) = (x \otimes 1 + y \otimes \sqrt{m})(x \otimes 1 - y \otimes \sqrt{m}) = x^2 - my^2$ for x and y in S^{i-1}. For convenience, we call $x \otimes 1 - y \otimes \rho$ the *conjugate* of $x \otimes 1 + y \otimes \rho$ in S^i.

Suppose x in $U(S^3)$ is a two cocycle and let $y = N(x) = a \otimes 1 + b \otimes \rho$ with a and b in S. By Corollary 2.4, $a^2 - mb^2 = N(y) = \pm 1$ in S. We treat the two cases separately, letting $a = a_1 + a_2\rho$ and $b = b_1 + b_2\rho$ with a_i, b_i in Z.

Case 1. $N(y) = 1$. Here one easily sees that $y^{-1} = a \otimes 1 - b \otimes \rho$ the conjugate of y. Let M denote the ring homomorphism $S \otimes S \to S$ defined by $M(c \otimes d) = cd$ for c and d in S. Then the unit of S, $M(y) = a + b\rho$ has inverse $M(y^{-1}) = a - b\rho$. Explicitly

$$M(y) = a + b\rho = a_1 + mb_2 + (a_2 + b_1)\rho \tag{1}$$

and

$$M(y^{-1}) = a - b\rho = a_1 - mb_2 + (a_2 - b_1)\rho. \tag{2}$$

Now $NM(y)$ is in $U(Z)$, so is ± 1. If $NM(y) = 1$ we see that $M(y^{-1}) = M(y)^{-1}$ is the conjugate of $M(y)$, that is $M(y)^{-1} = (a_1 + mb_2) - (a_2 + b_1)\rho$. Using equation (2) we then have $b_2 = a_2 = 0$. Thus $y = N(x) = a_1 \cdot 1 \otimes 1 + b_1 \cdot 1 \otimes \rho = \varepsilon_0(c)$ where $c = a_1 + b_1\rho$ is in $U(S)$ since $y^{-1} = a \otimes 1 - b \otimes \rho = \varepsilon_0(a_1 - b_1\rho)$.

Now by Corollary 2.3 $x^2 = d^1(N(x)) = d^1(\varepsilon_0(c)) = \varepsilon_0\varepsilon_0(c)\varepsilon_1\varepsilon_0(c^{-1})\varepsilon_2\varepsilon_0(c)) = \varepsilon_1\varepsilon_0(c)\varepsilon_1\varepsilon_0(c^{-1})\varepsilon_2\varepsilon_0(c) = \varepsilon_2\varepsilon_0(c) = \varepsilon_0(c) \otimes 1 = N(x) \otimes 1$. On the other hand, if we write $x = \alpha \otimes 1 + \beta \otimes \rho$ with α and β in S^2, then $x^2 = (\alpha^2 + m\beta^2) \otimes 1 + 2\alpha\beta \otimes \rho$ and equating coefficients gives $2\alpha\beta = 0$ and $\alpha^2 + m\beta^2 = N(x) = \alpha^2 - m\beta^2$ (by the definition of N). Hence $m\beta^2 = 0$. But since the natural map of S^2 into K^2 is injective, S^2 is torsion free with no nilpotents, so $\beta = 0$. Thus $x = \alpha \otimes 1$ and so is a coboundary by Lemma 2.5.

In Case 1 there remains the possibility that $NM(y) = -1$. With the notation of the previous subcase we see that $M(y)^{-1} = -(a_1 + mb_2) + (a_2 + b_1)\rho$, the negative of the conjugate of $M(y)$. Equation (2) here leads to $a_1 = b_1 = 0$ so that $y = N(x) = a_2\rho \otimes 1 + b_2\rho \otimes \rho$. Hence $NN(x) = a_2^2\rho^2 + mb_2^2\rho^2 = a_2^2 m + m^2 b_2^2 = m(a_2^2 + mb_2^2)$. By Corollary 2.4, this must be $\pm 1_S$. Since a_2, b_2 and m are integers, this happens only if $m = \pm 1$. If $m = 1$, K is not a proper extension (and in any case m is not congruent to 2 or 3 (mod 4)). We are thus, in Case 1, reduced to considering the Gaussian integers and must consider solutions of $b_2^2 - a_2^2 = \pm 1$. Thus in this subcase, $\rho = i$. Returning to equation (1), we have $M(y) = -b_2 + a_2 i$ and we have assumed $-1 = NM(y) = b_2^2 - a_2^2 = (b_2 + a_2)(b_2 - a_2)$ in Z. The only solutions of this are $b_2 = 0$ and $a_2 = \pm 1$. Thus by Corollary 2.3, $x^2 = d^1(N(x)) = d^1(a_2 i \otimes 1) = d^1(\varepsilon_1(a_2 i)) = \varepsilon_0\varepsilon_1(a_2 i)\varepsilon_1\varepsilon_1(a_2^{-1}i^{-1})\varepsilon_2\varepsilon_1(a_2 i) = \varepsilon_0\varepsilon_1(a_2 i)$ (since $\varepsilon_1\varepsilon_1 = \varepsilon_2\varepsilon_1$) and so $x^2 = \pm 1 \otimes i \otimes 1$. But $\pm 1 \otimes i \otimes 1$ is not a square in S^3, else after applying the ring homomorphism $a \otimes b \otimes c \to abc$ of S^3 to S, we would have that $\pm i$, and hence i, is a square in S.

Case 2. $N(y) = -1$. Here $y^{-1} = -a \otimes 1 + b \otimes \rho$ and we obtain

$$M(y) = a + b\rho = a_1 + mb_2 + (a_2 + b_1)\rho \tag{3}$$

and

$$M(y^{-1}) = -a + b\rho = -a_1 + mb_2 + (b_1 - a_2)\rho. \tag{4}$$

Again $NM(y) = \pm 1$ in Z. As in Case 1, $NM(y) = 1$ implies $M(y^{-1}) = M(y)^{-1}$ is the conjugate of $M(y)$, that is, $M(y^{-1}) = (a_1 + mb_2) - (a_2 + b_1)\rho$. Comparing cefficients with (4) gives $a_1 = b_1 = 0$. By computations similar to the second subcase of Case 1, we are reduced to considering only $m = -1$, (S the Gaussian integers) and $a_2^2 + b_2^2 = 1$ in Z. This equation has the solutions $a_2 = 0$ and $b_2 = \pm 1$; $a_2 = \pm 1$, $b_2 = 0$. $b_2 = 0$ and $a_2 = \pm 1$ yields, parallel to Case 1, $x^2 = d^1(N(x)) = d^1(y) = d^1(a_2 i \otimes 1) = -a_2(i \otimes i \otimes i)$ which again cannot be a square in S^3.

In the subcase $NM(y) = 1$ there remains the possibility $a_2 = 0$, $b_2^2 = 1$. Then again by Corollary 2.3, $x^2 = d^1(N(x)) = d^1(y) = d^1(b_2 i \otimes i) = b_2(1 \otimes i \otimes i)b_2(i \otimes 1 \otimes i)b_2(i \otimes i \otimes i) = -b_2(1 \otimes 1 \otimes 1) = \pm 1 \otimes 1 \otimes 1$. That x is a coboundary then follows from Lemma 3.1 below, completing the subcase $NM(y) = 1$.

The subcase $NM(y) = -1$, by similar computations leads to $b_2 = a_2 = 0$. As in the first subcase of Case 1, an application of Corollary 2.3 and Lemma 2.5 shows that x is a coboundary, completing Case 2 and the proof, except for the following Lemma.

LEMMA 3.1. *Let S be the Gaussian integers and x in $U(S^3)$ a two cocyle. If $x^2 = \pm 1$ in S^3 then x is a coboundary.*

Proof. Consider first $x^2 = 1 \otimes 1 \otimes 1$. The following are eight solutions in S^3: $\pm 1 \otimes 1 \otimes 1$, $\pm 1 \otimes i \otimes i$, $\pm i \otimes 1 \otimes i$, and $\pm i \otimes i \otimes 1$. We claim this exhausts the solutions of the equation in S^3. To see this note that if $K = Q(i)$, then distinct solutions in S^3 are also distinct in $K \otimes_Q K \otimes_Q K$, since the natural map $S \otimes S \otimes S \rightarrow K \otimes_Q K \otimes_Q K$ is monic. Since K/Q is galois, $K \otimes_Q K \otimes_Q K$ is isomorphic to a direct product of copies of K. Comparing Q dimensions yields $K \otimes_Q K \otimes_Q K \cong K \times K \times K \times K$. Since the only solutions in K of $x^2 = 1$ are ± 1, it follows that there are exactly 16 solutions in $K \otimes_Q K \otimes_Q K$.

Let x_i denote the eight above mentioned distinct solutions which lie in S^3 and let $y = (1/2)(1 \otimes 1 \otimes 1 - i \otimes i \otimes 1 - i \otimes 1 \otimes i + 1 \otimes i \otimes i)$. Then it can be seen that $y^2 = 1$ and $\{x_i, x_i y\}$ are solutions of $x^2 = 1$ in $K \otimes_Q K \otimes_Q K$. We claim these are distinct and that the $x_i y$ do not lie in S. For both claims it suffices, since the x_i are in $U(S^3)$, to show that y cannot lie in S^3. This follows easily from the fact that $1 \otimes 1 \otimes 1$, $i \otimes i \otimes 1$, $i \otimes 1 \otimes i$ and $1 \otimes i \otimes i$ are linearly independent over Z and that $1/2$ does not lie in Z.

Thus the x_i exhaust the solutions in S^3 of $x^2 = 1$. Now among these solutions a simple computation shows that the only cocycles are $1 \otimes 1 \otimes 1$ and $-1 \otimes 1 \otimes 1$ and these are, respectively $d^1(1 \otimes 1)$ and $d^1(-1 \otimes 1)$. Similarly among the solutions of $x^2 = -1 \otimes 1 \otimes 1$ only $\pm i \otimes 1 \otimes 1$, $\pm i \otimes i \otimes i$, $\pm 1 \otimes i \otimes 1$ and $1 \otimes 1 \otimes i$ lie in S^3 (the remaining eight comprise the multiples of these by the element y given above and again cannot lie in S). The only cocycles are $\pm 1 \otimes i \otimes 1$ and these are coboundaries of $1 \otimes i$ and $i \otimes 1$ respectively. Thus the lemma, and so Theorem 3.0, is proved.

THEOREM 3.2. *Let $K = Q(\sqrt{4k+1})$. If S denotes the integers of K, then $H^2(S/Z, U) = 0$.*

Proof. Let $\rho = (1 + \sqrt{4k+1})/2$. Then $\{1, \rho\}$ is a basis of S over Z [12, Thm. 6-1-1]. The nontrivial Q-automorphism of K, since it must preserve the roots of $x^2 = 4k + 1$, takes $\sqrt{4k+1}$ to $-\sqrt{4k+1}$ and so takes $a + b\rho$ to $a + b((1 - \sqrt{4k+1})/2) = a + b(1 - \rho)$. Hence, $N(x) = a^2 - b^2\rho^2 + ab + b^2\rho$. Since $\rho^2 = \rho + k$, we have $N(x) = a^2 - b^2k + ab$.

As in the previous theorem, the structure of S^i as S^{i-1}-algebra is analogous to the ring structure on S. That is 1_{S^i} and $1_{S^{i-1}} \otimes \rho$ are a basis and $N(a \otimes 1 + b \otimes \rho) = a^2 + ab - b^2k$ for a, b in S^{i-1}.

Computations closely paralleling those of the previous theorem show that if x is a two cocycle in $U(S^3)$ then x is in $S^2 \otimes 1$ and so by Lemma 2.5 is a coboundary. As before the computation divides itself into two cases, $NN(x) = 1$ or $NN(x) = -1$. Various subcases lead either to the desired result or to an equation in integers of the form $2 = a^2 - 4k$. Since a square integer is never congruent modulo four to two, the theorem is proved.

4. Pic $(S \otimes S)$. Let R be an integral domain whose quotient field, K, has characteristic not 2. Let S be an integral quadratic extension of R, that is, $S = R[\rho]$ where the minimal polynomial of ρ over R is $p(x) = x^2 + ax + b$. Let $\bar{\rho}$ be the second (and distinct) root of $p(x)$. Note that S is an integral domain with quotient field $K(\rho)$, and that $\bar{\rho}$ is in S as a consequence of the familiar formula $\rho + \bar{\rho} = -a$. The main theorem of this section characterizes the Picard group [5, Ch. II, Sec. 4] Pic$(S \otimes_R S)$ of rank one projective $S \otimes_R S$ modules in terms of the units of S and of $S/(\rho - \bar{\rho})/S$. Henceforth $\otimes$ means $\otimes_R$ and S' denotes $S/(\rho - \bar{\rho})S$.

LEMMA 4.0. *$S \otimes S \cong S \times_{S'} S$. That is, in the notation of* [3, IX Sec. 5, p. 478], *there are maps h_1, h_2 making*

(1)
$$\begin{array}{ccc} S \otimes S & \xrightarrow{h_1} & S \\ {\scriptstyle h_2}\downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

a cartesian square (here the unlabelled maps are the natural projections).

Proof. By assumption, S is free over R on 1 and ρ, so $S \otimes S$ is free on 1 and $1 \otimes \rho$ when regarded as an S module on the first factor. For s and t in S, define $h_1(s \otimes 1 + t \otimes \rho) = s + t\rho$ and $h_2(s \otimes 1 + t \otimes \rho) = s + t\bar{\rho}$. Then $h_1(a) - h_2(a) = t(\rho - \bar{\rho})$ for any $a = s \otimes 1 + t \otimes \rho$ in $S \otimes S$. Conversely, suppose $s_1 \equiv s_2 \pmod{(\rho - \bar{\rho})S}$, i.e., $s_1 - s_2 = s_3(\rho - \bar{\rho})$ for some s_3 in S. Then taking $y = s_3$ and $x = s_1 - s_3\rho$ gives $s_1 = x + y\rho = h_1(x \otimes 1 + y \otimes \rho)$ and $s_2 = x + y\bar{\rho} = h_2(x \otimes 1 + y \otimes \rho)$. Thus $\{(s_1, s_2)$ in $S \times S \mid s_1 \equiv s_2 \pmod{(\rho - \bar{\rho})S}\} = \{(h_1(a), h_2(a)) \mid a$ is in $S \otimes S\}$. Since S is an integral domain, it follows that $a \to (h_1(a), h_2(a))$ is a monomorphism of $S \otimes S$ into $S \times S$ so the square (1) satisfies the definition of cartesian.

REMARK. Let R be the ring of integers of an algebraic number field, K, with class number 1, and S the integers of a quadratic extension, L of K. S is finitely generated projective over R (cf. 12, p. 158).

If $\{x_i, \phi_i\}$ is a projective coordinate system, the map $f: S \to R$ given by $f(x) = \Sigma\phi_i(x, x_i)$ is a split R-module epimorphism, so that $S = R1 \oplus \ker F$. Since R is a *PID*, $\ker f$ is free and a simple rank argument (e.g. passing to L) shows $\ker f \cong R \cdot \rho$ for some ρ in $S \subseteq L$. Clearly such a ρ must satisfy a quadratic monic polynomial over R, so that S is quadratic over R in the above sense.

THEOREM 4.1. *Let $S = R[\rho]$ be a commutative integral quadratic extension of an integral domain R, let $\bar{\rho}$ be the conjugate of ρ and let $S' = S/(\rho - \bar{\rho})S$. Then the following sequence is exact:*
$0 \to U(S \otimes S) \to U(S) \times U(S) \to U(S') \to \operatorname{Pic}(S \otimes S) \to \operatorname{Pic} S \times \operatorname{Pic} S \to \operatorname{Pic} S'$.

Proof. In view of Lemma 4.0 the above sequence is given by Theorem 5.3 [3, IX Sec. 5, p. 481].

REMARKS. The maps of the above sequence are those of the Mayer-Vietoris sequence of [3, VII Sec. 4]. In particular, $U(S \otimes S) \to U(S) \times U(S)$ is given by $u \to (h_1(u), h_2(u)^{-1})$ where h_i are the maps in Lemma 4.0, and $U(S) \times U(S) \to U(S')$ is given by $(s, t) \to \pi(s)\pi(t)$ where $\pi: S \to S'$ is the natural projection. Clearly the image of $U(S) \times U(S) \to U(S')$ is the same as the image of π restricted to $U(S)$.

COROLLARY 4.2. *With R and S as in Theorem 4.1, $\operatorname{Pic}(S \otimes S) = 0$ if and only if $\operatorname{Pic} S = 0$ and the natural projection $U(S) \to U(S')$ is surjective.*

Proof. The R-algebra map $\varepsilon_1: S \to S \otimes S$ given by $x \to x \otimes 1$ is split by the map $M: x \otimes y \to xy$. Hence $\operatorname{Pic} S \xrightarrow{\operatorname{Pic} \varepsilon_1} \operatorname{Pic}(S \otimes S) \xrightarrow{\operatorname{Pic} M} \operatorname{Pic} S$ is identity, so that $\operatorname{Pic} \varepsilon_1$ is a monomorphism, i.e., $\operatorname{Pic} S \subseteq \operatorname{Pic}(S \otimes S)$. The corollary is then immediate from Theorem 4.1 and remarks following it.

Now let $K = Q(\sqrt{m})$ be a quadratic field extension of the rationals, and S be its ring of integers. As in § 3, $S = Z[\rho]$ where $\rho = \sqrt{m}$ or $(1 + \sqrt{m})/2$ according to whether $m \equiv 2$ or 3 or $m \equiv 1 \pmod 4$. We can easily compute S':

LEMMA 4.3. *If $m \equiv 1 \pmod 4$, then $S' \cong Z/mZ$.*

Proof. Let $m = 4k + 1$ so that $\rho + 2k = (\sqrt{m})\rho$, and write $x + y\rho = x - 2ky + y(\rho + 2k) = x - 2ky + y\sqrt{m}\rho = x - 2ky + y\rho(\rho - \bar{\rho})$ where x, y lie in Z. Hence $x + y\rho \equiv x - 2ky \pmod{(\rho - \bar{\rho})S}$. Moreover, $m = \sqrt{m}\sqrt{m} = (\rho - \bar{\rho})^2 \equiv 0 \pmod{S(\rho - \bar{\rho})}$. Thus if, for an integer a, $\bar{a}$ denotes the coset of a mod m, we see that $x + y\rho \to \overline{x - 2ky}$ is a ring map of S onto Z/mZ whose kernel, $J = \{x + y\rho \mid x - 2ky = am\}$ is contained in $(\rho - \bar{\rho})S$. Conversely, since $\rho - \bar{\rho} = \sqrt{m} = -1 - 2\rho$ we have $\rho - \bar{\rho} \to -1 - 4k = -m$, so that $(\rho - \bar{\rho})S$ is contained in J. Thus $S' \cong Z/mZ$.

LEMMA 4.4. *If* $m \not\equiv 1 \pmod 4$ *then* $S' \cong T = Z/2mZ + Z/2Z(\sqrt{m})$ (where this ring has the obvious multiplication).

Proof. In this case $\rho = \sqrt{m}$ and $\rho - \bar{\rho} = 2\sqrt{m}$. Let $\sim$ and $\wedge$ denote reduction mod $2m$ and 2 respectively. Then for x, y in Z, $x + y\rho \to \tilde{x} + \hat{y}\sqrt{m}$ is a ring map whose kernel is $\{2ma + 2b\rho \mid a, b \text{ are in } Z\}$. Since $2ma + 2b\rho = 2\sqrt{m}(\sqrt{m}a + b) = (\rho - \bar{\rho})(\sqrt{m}a + b)$, this kernel is just $(\rho - \bar{\rho})S$ and the lemma is proved.

Now S' is finite in either case; It follows from Proposition 5 of [5, Ch. 2, Sec. 5, No. 4] that any semi-local ring has trivial Picard group, hence $\mathrm{Pic}(S') = 0$ under the hypotheses of Lemmas 4.3 or 4.4. Suppose that $\mathrm{Pic}\, S = 0$ and let $\pi\colon U(S) \to U(S')$ denote the map induced by the projection $S \to S'$. Then employing the remarks following Theorem 4.1 the exact sequence of that theorem becomes in this case

$$0 \to \mathrm{Im}\, \pi \to U(S') \to \mathrm{Pic}(S \otimes S) \to 0. \tag{2}$$

THEOREM 4.5. *Let* $K = Q(\sqrt{m})$ *be a quadratic extension of the rational numbers,* Q, *with* m *a square free integer. If* S *denotes the integers of* K, *then* $\mathrm{Pic}(S \otimes S) = 0$ *for* $m = \pm 3, -1, 2,$ *and* 5 *but for no other value of* m.

Proof. For the given values of m, S is a euclidean domain [12, Propn. 6-4-1] hence a PID, or equivalently [cf. 5, Sec. 5, No. 7] $\mathrm{Pic}\, S = 0$. Referring to Lemmas 4.3 and 4.4 we may easily verify the following table by direct calculation

m	S'	$U(S')$
2	$Z/4Z + Z/2Z\sqrt{2}$	$\{\pm\bar{1}, \pm\bar{1} + \sqrt{2}\}$
3	$Z/2Z + Z/2z\sqrt{3}$	$\{\pm\bar{1}, \pm\bar{2} + \sqrt{3}\}$
5	$Z/5Z$	$\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

where $\bar{\ }$ denotes the coset mod 4,6, or 5 respectively.

Now by the Dirichlet units theorem [12, Sec. 6-3], $U(S) = \{\pm \varepsilon^i \mid i \text{ in } Z\}$ where the fundamental unit, ε, is $1 + \sqrt{2}, 2 + \sqrt{3}$, or $(1 + \sqrt{5})/2$ respectively [11, "Tables"]. Referring to Lemmas 4.3 and 4.4 for the definition of π we find in case $m = 2$ that $\pi(\varepsilon) = \bar{1} + \sqrt{2}$, $\pi(-\varepsilon) = -\bar{1} - \sqrt{2} = -\bar{1} + \sqrt{2}$ in S'. In all cases $\pi(-1) = -\bar{1}$ and $\pi(1) = \bar{1}$. Since π is (the restriction of) a ring map, we see that π is onto when $m = 2$. Similarly when $m = 3$, $\pi(\varepsilon) = \bar{2} + \sqrt{3}$ and $\pi(-\varepsilon) = -\bar{2} - \sqrt{3} = -\bar{2} + \sqrt{3}$ and when $m = 5$ $\pi(\varepsilon) = -\bar{2} = \bar{3}$

which generates the cyclic group of units of $S' = Z/5Z$. Thus also in these cases π is onto.

If $m = -3$ then $U(S') = U(Z/3Z) = \{\pm 1\}$. π is again onto because it is the restriction of a ring map. If $m = -1$ then $U(S') = \{1, \sqrt{-1}\}$. By definition $\pi(\sqrt{-1}) = \sqrt{-1}$ so that the fact that π is the restriction of a ring map again implies π is onto. That $\mathrm{Pic}(S \otimes S) = 0$ for the given m now follows from Corollary 4.2.

Now suppose m is not one of the listed integers. By Corollary 4.2 we need only consider integers m for which S is a *PID*. If $m \leqq -5$, the Units Theorem shows $U(S) = \pm 1$. Now S' contains Z/mZ or $Z/2mZ$ according to whether $m \equiv 1 \pmod 4$ or not. Let $m = -p_1p_2 \cdots p_r$ with p_i distinct primes, and consider first $m \equiv 1 \pmod 4$. Then $Z/mZ \cong Z/p_1Z \times \cdots \times Z/p_rZ$ with p_i odd primes. There being only two units in S, if π is to be onto we must clearly have $r = 1$ and $p_r = 3$, so π is not onto. Similarly, if $-5 < m \equiv 3 \pmod 4$, $Z/2mZ \cong Z/2Z \times Z/p_1Z \times \cdots \times Z/p_rZ$ which has the same units as Z/mZ and, as above π is not onto. If $m \equiv 2$ we take $p_1 = 2$, so that $Z/2mZ \cong Z/4Z \times Z/p_2Z \times \cdots \times Z/p_rZ$. Again, if π is to be onto there can be no factors other than p_1, since $Z/4Z$ has 2 units, so that for no $m \leqq -5$ can π be onto.

Consider now $m > 5$. For any unit $a + b\rho$ in S we have that the norm

$$N(a + b\rho) = (a + b\rho)(a + b\bar{\rho})$$

is a unit in Z, so

$$\pm 1 = (a + b\rho)(a + b\bar{\rho}) \equiv (a + b\rho)^2 (\mathrm{mod}(\rho - \bar{\rho})S).$$

Squaring shows that for any unit v in $S' = S/(\rho - \bar{\rho})S$ we have $v^4 = 1$. Now the Units Theorem shows that $U(S)$ is the direct product of the cyclic group $< -1 >$ of order two, generated by -1 with an infinite cyclic group $< \varepsilon >$ for some unit ε, called the *fundamental unit*. It then follows that $\mathrm{Im}\,\pi \in U(S')$ is a group of exponent dividing four, generated by two elements, one, namely $\pi(-1)$, of order at most two. In particular $\mathrm{Im}\,\pi$ has at most eight elements.

Suppose first that $m = 2p_1 \cdots p_r$ with p_i distinct odd primes. Then $S' \supseteqq Z/2mZ = Z/4Z \times Z/p_1Z \times \cdots \times Z/p_rZ$. If this ring is to have at most eight units we must clearly have $p_i \leqq 5$. Indeed $m = 6$ or $m = 10$ are the only possibilities, since $m = 30$ produces more than eight units. However, if $m = 6$ or 10, S is not a *PID* [11, "Tables"] so by Corollary 4.2 we can not have $\mathrm{Pic}(S \otimes S) = 0$. Thus in all possible remaining cases, $n = 2k$ implies π is not onto and again Corollary 4.2 shows $\mathrm{Pic}(S \otimes S) \neq 0$.

Consider next $m \equiv 3 \pmod 4$ and write $m = p_1 \cdots p_r$ as the product of distinct odd primes. Then

$$S' \supsetneqq Z/2mZ = Z/2Z \times Z/p_1Z \times \cdots \times Z/p_rZ.$$

In order to have at most eight units we must have each $p_i \leqq 7$. But some $p_i = 7$ would entail a unit of order three which can not happen. Since $m > 5$, we see that π is onto possibly only if $m = 15$. But in this case S is not a *PID* [11, "Tables"] so again we can not have $\text{Pic}(S \otimes S) = 0$.

Finally there remains the case $m \equiv 1 \pmod 4$. If $m = p_1p_2 \cdots p_r$, then the units of $S = Z/mZ = Z/p_1Z \times \cdots \times Z/p_rZ$ are the same as those of $Z/2mZ = Z/2Z \times Z/p_1Z \times \cdots \times Z/p_rZ$ so the same argument as above for $m \equiv 3$ shows that $\text{Pic}(S \otimes S) = 0$ only for the listed values of $m \equiv 1 \pmod 4$, completing the proof.

5. $B(S/Z)$. All notation is as in [7].

THEOREM 5.0. *Let $K = Q(\sqrt{m})$ with m a square free integer and Q the rationals. Let* S *be the ring of integers of K. Then the split Brauer group $B(S/Z)$ is zero when $m = -3, -1, 2, 3$ or 5.*

Proof. In each case S is euclidean [12, Propn. 6-4-1] hence a *PID*. Thus as remarked in § 4, $\text{Pic } S = 0$, so that $H^0(S/R, \text{Pic})$, being a subgroup of $\text{Pic } S$, is zero. By Theorem 4.3, $\text{Pic}(S \otimes S) = 0$, hence $H^1(S/Z, \text{Pic})$, which is a homomorphic image of a subgroup of $\text{Pic}(S \otimes S)$, is zero. It then follows from Theorem 7.6 of [7] that $B(S/Z) \cong H^2(S/Z, U)$ and the result follows from Theorems 3.0 and 3.2.

Using the global class field theory, one can prove that in fact $B(S/Z) \subseteqq B(Z) = 0$ [9]. Dobbs [8] has exploited this fact to obtain an improvement of our Theorems 3.0 and 3.2. Of course the conclusion of Theorem 4.5 is more than is needed to show $B(S/Z) = 0$. It would suffice to prove directly that $H^1(S/Z, \text{Pic}) = 0$ or that the map $H^1(S/Z, \text{Pic}) \to H^3(S/Z, U)$ given in Theorem 7.6 of [7] is a monomorphism. However, $H^1(S/Z, \text{Pic})$ does not seem amenable to computation at the present time.

REFERENCES

1. S. A. Amitsur, *Homology groups and double complexes for arbitrary fields*, J. Math., Soc., Japan, **14** (1962), 1-25.

2. M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc., **97** (1960), 367-409.

3. H. Bass, *Algebraic K-Theory*, Benjamin, 1968.

4. N. Bourbaki, *Algebre*, Chapitre 7, Hermann, Paris, 1964 (Act. Scient. et Ind. 1261, 2nd ed.); Chapitre 8, 1958 (Act. Scient. et Ind. 1179).

5. ———, *Algebre Commutative*, Chapitre 2, Hermann, Paris, 1961 (Act. Scient. et Ind. 1290).

6. S. Chase, D. K. Harrison and A. Rosenberg, *Galois theory and galois cohomology of commutative rings*, Amer. Math. Soc. Memoir No. 52, 1965.

7. S. Chase and A. Rosenberg. *Amitsur cohomology and the Brauer group*, Amer. Math. Soc. Memoir No. 52, 1965.

8. D. Dobbs, *On inflation and torsion of Amitsur cohomology*, to appear.

9. R. M. Fossum, *The Noetherian Different of Projective Orders*, Thesis, University of Michigan, 1963.

10. J. P Serre, *Corps Locaux*. Hermann, Paris, 1962 (Act. Scient. et Ind. 1926).

11. J. Sommer, *Introduction a la theorie des nombres algebriques*, Hermann, Paris.

12. E. Weiss, *Algebraic Number Theory*. McGraw Hill, New York, 1963.

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STATE UNIVERSITY OF NEW YORK AT ALBANY