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PRIMES IN PRODUCTS OF RINGS

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This paper is an elementary note which indicates how Harrison's primes sit in certain kinds of rings. It is proved that primes behave nicely under finite direct products. Also it is shown that any nil ideal is a subset of every prime. This gives information about the primes of artinian rings.

By ring we mean associative ring with identity.

Harrison introduced the notion of a prime of a ring in [2]. For a ring R, a *preprime* of R is a subset of R which is closed under addition and multiplication and does not contain -1. A *prime* of R is a preprime of R which is not properly contained in any other preprime of R. A prime is called finite if it does not contain 1.

The collection $Y(R)$ of all primes of R is topologized by taking as a basis all subsets $V(E)$ where E is a finite subset of R and $V(E) = \{P \in Y(R) \mid P \cap E = \phi\}$.

Some important special cases of the following result were known to Harrison and his students as early as 1966. However the proofs known then were long and depended heavily on the special conditions. One of the cases in which the result was known was where the rings are both commutative and all the primes are taken to be finite. This case is also handled by Connell in [1] where he deals with a notion of prime which generalizes the notion of a finite prime of a commutative ring.

THEOREM 1. *Let R and S be rings. The primes of the direct product $R \oplus S$ are exactly the subsets $P \times S = \{(p, s) \in R \oplus S \mid p \in P\}$ where P is a prime of R and $R \times Q = \{(r, q) \in R \oplus S \mid q \in Q\}$ where Q is a prime of S.*

Proof. Let $\Pi_R: R \oplus S \rightarrow R$ and $\Pi_S: R \oplus S \rightarrow S$ be the canonical projections.

Let H be any prime of $R \oplus S$. Clearly $\Pi_R(H)$ and $\Pi_S(H)$ are both closed under addition and multiplication. Just suppose $-1 \in \Pi_R(H)$ and $-1 \in \Pi_S(H)$. Then there exists $s \in S$ and $r \in R$ with $(-1, s)$ and $(r, -1)$ in H. But then $(-1, -1) = (-1, s)(r, -1) + (-1, s) + (r, -1)$ is in H, a contradiction. So say $-1 \notin \Pi_R(H)$. Then $\Pi_R(H)$ is a preprime of R. Choose a prime P of R such that $\Pi_R(H) \subset P$. Then clearly $P \times S$ is a preprime of $R \oplus S$ and $H \subset P \times S$. So $H = P \times S$. Similarly if $-1 \notin \Pi_S(H)$ then there exists a prime Q of S such that $H = R \times Q$.

Now let P be a prime of R. Then $P \times S$ is a preprime of $R \oplus S$. Let H be a prime of $R \oplus S$ containing $P \times S$. Using the above part of the proof one checks that $H = P' \times S$ for some prime P' of R. But then since P is a prime, $P = P'$. So $P \times S$ is a prime. Similarly, for $Q \in Y(S)$, $R \times Q \in Y(R \oplus S)$.

In the above, no mention is made of the topology. However, we do have

COROLLARY 2. *For any nonzero rings R and S, $Y(R \oplus S)$ is naturally homeomorphic to $Y(R) \cup Y(S)$ with the disjoint union topology.*

Proof. Let $A = \{P \times S \mid P \in Y(R)\}$ and $B = \{R \times Q \mid Q \in Y(S)\}$. Since R and S are both nonzero, A and B are disjoint. Using Theorem 1 we have $Y(R \oplus S) = A \cup B$ and that $A = V(\{(-1, 0)\})$ and $B = V(\{(0, -1)\})$, so A and B are both open. Noting that A is naturally homeomorphic with $Y(R)$ and B is naturally homeomorphic with $Y(S)$ we have the corollary.

Clearly Theorem 1 and its corollary can be extended to any finite product of rings. However the corresponding statement for an infinite product of rings is not true. In the infinite case there can be primes other than the ones given by one coordinate.

It follows from Corollary 2 that for an arbitrary ring R, $Y(R)$ is far from connected. However, for R an integral domain $Y(R)$ is connected; in fact we have

PROPOSITION 3. *If R is a commutative integral domain, then $Y(R)$ is irreducible, i.e., any two nonvoid open subsets of $Y(R)$ have nonvoid intersection.*

Proof. This follows easily by using the result of Harrison and Manis [2, 2.6]. For if $V(G)$ and $V(H)$ are two nonvoid basic open subsets of $Y(R)$ then $0 \notin G \cup H$ and so no power of the product of the elements of $G \cup H$ is zero. Hence by [2, 2.6] there is a prime P with $P \in V(G \cup H) = V(G) \cap V(H)$.

In order to use Theorem 1 to study $Y(R)$ for artinian rings we need the following two observations.

Note 4. For any ring R and any two-sided ideal $\mathfrak{a}$ of R the space $Y(R/\mathfrak{a})$ is naturally homeomorphic to the subspace of $Y(R)$ composed of all primes containing $\mathfrak{a}$. The homeomorphism is given by $P \to f(P)$ where $f\colon R \to R/\mathfrak{a}$ is the natural map.

This note is made more interesting by the following

PROPOSITION 5. *Any nil ideal of a ring R is contained in every prime of R.*

Proof. Let N be a nil ideal of R and P a prime of R. Just suppose $N \not\supset P$, then $P \subsetneqq N + P$ and $N + P$ is closed under addition and multiplication. So $-1 \in N + P$. Say $-1 = n + p$ for $n \in N$ and $p \in P$. Then $p + 1$ is in N so there exists a positive integer m such that

$$0 = (p + 1)^m = \sum_{k=0}^{m} \binom{m}{k} p^k .$$

Hence

$$-1 = \sum_{k=1}^{m} \binom{m}{k} p^k \in P ,$$

a contradiction.

Now if R is an artinian ring with Jacobson radical J, then J is nilpotent so by the above proposition J is contained in every prime of R. R/J has an Artin-Wedderburn decomposition

$$R/J \simeq M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$$

as a direct product of finitely many full matrix rings over division rings. Thus using Note 4 we have

COROLLARY 6. *If R is artinian with notation as above, $Y(R)$ is naturally homeomorphic to the disjoint union*

$$\bigcup_{i=1}^{k} Y(M_{n_i}(D_i)) .$$

In the special case where R is a finite ring the D_i are all finite fields and in this case $Y(M_{n_i}(D_i))$ is completely analyzed by Rutherford in [3]. There it is shown that for any finite dimensional vector space V over a locally finite field K, the primes of the full ring E of K-endomorphisms of V are exactly the subsets $T(W, L) \subset E$ where L is a subspace of V and W is a subspace of L with $\dim_K(L/W) = 1$ and $T(W, L) = \{f \in E \mid f(L) \subset W\}$. Using this and Corollary 6 we have a complete analysis of $Y(R)$ for finite rings.

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