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A GENERALIZATION OF SEPARABLE GROUPS

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This paper introduces a new class of torsion free abelian groups, the class of quasi-separable groups, which is the quasi-isomorphism analog of the class of separable groups and which properly contains the latter. Our purpose is two-fold: first, to further explore the phenomena of quasi-isomorphism, which has proved fruitful in the study of torsion free groups, and second, to shed further light on separable groups.

The term "group" herein refers to a torsion free abelian group. As is customary when dealing with quasi-isomorphism, we assume that all groups are subgroups of a fixed vector space V over the rational number field Q. $L(V)$ denotes the algebra of linear transformations of V. $L(V)$ is equipped with the finite topology [7] throughout and topological terms refer to this topology unless otherwise stated. G always denotes a full subgroup of V, i.e., a subgroup with torsion quotient V/G; G is full in V if and only if V is its unique minimal divisible extension. $QE(G)$ is the quasi-endomorphism algebra of G and $QF(G)$ is the ideal of $QE(G)$ consisting of elements of finite rank.

Our approach is to recall that there is a one-to-one correspondence between quasi-decompositions of a group G and idempotents in $QE(G)$ [8]. Thus a group with "many" quasi-decompositions has "many" quasi-endomorphisms of a particular type. In §1, quasi-separable groups are defined and basic properties are explored. A principal result is that every pure subgroup of finite rank in G is a quasi-summand of G if and only if G is quasi-separable with linearly ordered type set, $T(G)$. In §2, a characterization of homogeneous quasi-separable groups is obtained, namely, G is homogeneous and quasi-separable if and only if $QF(G)$ is dense in the finite topology of $L(V)$. In §3, attention focuses on separable groups. It is shown that a countable group G is homogeneous and completely decomposable if and only if $QE(G)$ is dense. Finally, a description of homogeneous separable groups is obtained in terms of their endomorphisms. For example, a countable group G is homogeneous and completely decomposable if and only if for any pair of independent elements a_1, a_2 in G and any arbitrary pair of elements b_1, b_2 in G, there exists an endomorphism f of G such that $fa_i = nb_i, i = 1, 2,$ n some positive integer.

General abelian group theory [5] is assumed. By this date, quasi-isomorphism is a familiar concept of this theory so basic facts are used here without comment; a complete background may be obtained from [1, 2, 8, 9]. $\stackrel{\bullet}{\subseteq}$ and $\doteq$ denote quasi-contained and quasi-equal, respectively. Recall that $QE(G) = \{f \in L(V)\colon fG \stackrel{\bullet}{\subseteq} G\}$. Each endomorphism of G has a unique extension to a linear transformation of V and we use the same symbol to denote both. $h(a)$ denotes the height of the element a; if it is not clear from context in which group height is computed, a subscript is appended, e.g., $h_G(a)$. Similarly, $t(a)$ denotes the type of the element a; $t(H)$ may also denote the type of a homogeneous group H. Notation is abused for the sake of conciseness; e.g., the same symbol Z is used to denote both the ring of integers and its additive group. S^* denotes the subspace spanned by the subset S of V; it is also used to denote the subalgebra generated by a subset of $L(V)$. All sums are direct; e.g., notation such as $G \doteq A + B$ implies that A and B are disjoint subgroups of V and we call A a quasi-summand of G. Additional notation is introduced as needed.

1.0. Quasi-separable groups.

DEFINITION 1.1. Call a group G quasi-separable if and only if every finite subset of G is contained in a completely decomposable quasi-summand.

REMARK 1.2. Suppose G is quasi-separable and suppose F is a finite subset of G; by definition $G \doteq A + B$ for some groups A and B contained in V, with A completely decomposable and containing F. Clearly A may be assumed to have finite rank without any loss of generality. Now $G \doteq A \cap G + B \cap G$ and $F \subseteq A \cap G$, but $A \cap G$ need not be completely decomposable even if A has finite rank; see for example Lemma 9.3 [2]. However, if A has finite rank and $T(A)$ is linearly ordered (especially f A is homogeneous), then $A \cap G$ is also completely decomposable by Corollary 9.6 [1]. Thus if $T(G)$ is linearly ordered, A may be assumed to be a completely decomposable, pure subgroup [1, p. 95] of finite rank in G.

The following modular law will prove indispensable.

PROPOSITION 1.3. *Suppose* $H \stackrel{\bullet}{\subseteq} A + B$ *and* $A \stackrel{\bullet}{\subseteq} H$ *for groups* H, A, *and* B. *Then* $H \doteq A + H \cap B$.

Proof. For some positive integer n, $nA \subseteq H$ so

$$n(A + H \cap B) \subseteq H .$$

If $mH \subseteq A + B$ for m a positive integer, then $nmH \subseteq nA + nB$;

i.e., for $c \in H$, mnc may be written $mnc = na + nb$ with $a \in A$ and $b \in B$. Now $nb = mnc - na \in H \cap B$ so $mnH \subseteqq A + H \cap B$.

REMARK 1.4. Let n be a positive integer. Consider a group having the property: (1) every pure subgroup of rank n is a quasi-summand. It is easy to see that every pure subgroup of rank n is a quasi-summand of G if and only if $QE(G)$ contains a projection onto any n-dimensional subspace of V. Consequently if G has property (1), so does any quasi-summand of G. Also, by Proposition 1.3, if G satisfies (1), so does any pure subgroup of G. Corresponding results hold for the property: (2) every pure subgroup is a quasi-summand.

LEMMA 1.5. *If every pure subgroup of rank one is a quasi-summand of G, then every pure subgroup of finite rank is a quasi-summand which is quasi-equal to a completely decomposable group.*

Proof. Assume the result for pure subgroups of rank $\leqq n$ and let H be a pure subgroup of rank $n + 1 \geqq 2$. Let $A \subset H$ be pure of rank n; by hypothesis $G \doteq A + B$ with A quasi-equal to a completely decomposable group; take B pure in G [1, p. 95]. By Proposition 1.3, $H \doteq A + H \cap B$; clearly $H \cap B$ is a pure subgroup of rank one in B. By Remark 1.4, $B \doteq H \cap B + C$ and so $G \doteq A + H \cap B + C \doteq H + C$, which completes the proof.

We shall shortly be able to strengthen the conclusion of Lemma 1.5 (see Corollary 1.7). A complete description of groups with the property that every pure subgroup of finite rank is a quasi-summand can be obtained from the following theorem, which is the quasi-isomorphism analog of Theorem 46.8 [5].

THEOREM 1.6. *Every pure subgroup of G is a quasi-summand if and only if $G = D + G_1 + \cdots + G_n$ with D divisible and the G_i reduced rank-one groups satisfying $t(G_1) \leqq \cdots \leqq t(G_n)$.*

Proof. Suppose G has the property that every pure subgroup is a quasi-summand and write $G = D + H$ with D divisible and H reduced; by Remark 1.4, H inherits this property. To see that H has finite rank, suppose $\{a_i\}_{i=1}^{\infty}$ is an independent set in H. Let A be the pure subgroup of H generated by $\{a_i - (i + 1)a_{i+1}\}_{i=1}^{\infty}$; $a_1 \notin A$. Now H/A contains a divisible subgroup generated by $\{a_i + A\}_{i=1}^{\infty}$, so A could not be a quasi-summand of the reduced group H [2, p. 26]. Thus H has finite rank and by Lemma 1.5, $H \doteq H_1 + \cdots + H_n$ with H_i of rank one, $i = 1, \cdots, n$. It will be sufficient to show that the types of any two of the H_i are comparable, for then a suitable relabeling of the H_i and Corollary 9.6 [1] will complete the proof. Let B and C be distinct among the H_i; by Remark 1.4, every pure subgroup of $B + C$ is a quasi-summand since $B + C$ is a quasi-summand of H. Suppose the types of B and C are incomparable; then $B + C$ contains elements of three different types, $t(B), t(C)$, and $t(B) \cap t(C)$. Pick nonzero elements b and c of B and C, respectively, and let M be the pure subgroup of $B + C$ generated by $b + c$. But $B + C \doteq M + N$ is impossible because $M + N$ cannot contain both elements of type $t(B)$ and of type $t(C)$, since $t(M) = t(B) \cap t(C)$ [2, p. 26]. This contradiction shows that $t(B)$ and $t(C)$ are in fact comparable and so completes the first half of the proof. Conversely suppose $G = D + H$ with D divisible, $H = G_1 + \cdots + G_n$, and the G_i reduced rank-one groups satisfying $t(G_1) \leqq \cdots \leqq t(G_n)$. First, to see that it will be sufficient to treat the case $D = 0$, recall that any pure subgroup A of G decomposes into $A = B + C$ with B divisible and C reduced and that $D \cap C = 0$ because C is pure in G. Thus the complement H of D may be chosen to contain C [5, p. 63]. Since B is a direct summand of D, it will be enough to show that C is a quasi-summand of H, so we assume $D = 0$. By Remark 1.4 and Lemma 1.5, it will be sufficient to show that $QE(G)$ contains a projection onto any one-dimensional subspace of V. Let $x \in V$ be nonzero; $kx \in G$ for some positive integer k and so $kx = a_1 + \cdots + a_n$ with $a_i \in G_i$, $i = 1, \cdots, n$. Let a_j be the first nonzero a_i;

$$t_G(kx) = t_G(a_j) = t(G_j) .$$

If S denotes the pure subgroup of G generated by kx, then G_j is isomorphic to S via some map f. Since G_j has rank one, for some non-zero integers r and s, $rf^{-1}(kx) = sa_j$. If g denotes the map from G onto $S \subseteqq G$ induced by f, then $(s/r)g \in QE(G)$ projects V onto the subspace spanned by x.

COROLLARY 1.7. *These properties of a group G are equivalent*: (1) *Every pure subgroup of rank one in G is a quasi-summand*; (2) *every pure subgroup of finite rank in G is a completely decomposable quasi-summand*; (3) *G is quasi-separable with linearly ordered type set.*

Proof. Assume (1) is true and let S be a pure subgroup of finite rank in G. By Lemma 1.5, S is a quasi-summand of G and thus by Remark 1.4, every pure subgroup of S is a quasi-summand of S. Theorem 1.6 shows that S is completely decomposable with linearly ordered type set. Thus we have (1) implies (2) and (2) implies (3). Finally, suppose (3) holds and let H be a pure subgroup of rank one in G. By Remark 1.2, H is contained in a pure subgroup S of G which is a completely decomposable group of finite rank with linearly ordered type set. By Theorem 1.6, H is a quasi-summand of S and thus of G.

From the foregoing results it is perhaps clear that a quasi-separable group need not be separable; a specific example is the following. It is well known that the subgroup S of $\pi = \prod_{i=1}^{\infty} Z$ generated by 2π and $\Sigma = \sum_{i=1}^{\infty} Z \subseteqq \pi$ is not separable. Since $S \doteq \pi$, both groups have the same quasi-endomorphism algebra [8]. It is also known that π is homogeneous and separable, so by Theorem 2.5, S is quasi-separable. In fact, there exist rank-two groups which are quasi-separable but not separable, i.e., not the direct sum of two rank-one groups; see for example Lemma 9.3 [2].

Just as for separable groups, the direct sum of a collection of quasi-separable groups is quasi-separable and the tensor product of two quasi-separable groups is quasi-separable.

Having proved basic results about quasi-separable groups, we turn our attention to the homogeneous case.

2.0. **Homogeneous quasi-separable groups.** We proceed to obtain a characterization of homogeneous quasi-separable groups in terms of quasi-endomorphisms. Intuitively, a group is homogeneous and quasi-separable precisely when it has "enough" quasi-endomorphisms; this is formulated in terms of density in the finite topology [7] of $L(V)$.

Recall [9] that a group is irreducible if and only if it has no nontrivial, pure, fully invariant subgroups, that an irreducible group is homogeneous, and that G is an irreducible group if and only if V is an irreducible $QE(G)$-module. After Jacobson [7], call a subset S of $L(V)$ k-fold transitive if and only if given any $j \leqq k$ linearly independent vectors $x_1, \cdots, x_j$ in V and any j vectors $y_1, \cdots, y_j$ in V, there exists $f \in S$ such that $fx_i = y_i$, $i = 1, \cdots, j$. Note well that G is irreducible, and thus homogeneous, if and only if $QE(G)$ is one-fold transitive.

REMARK 2.1. For a subring R of $L(V)$ the following conditions are equivalent: (1) R is two-fold transitive; (2) R is k-fold transitive for every k; (3) R is dense in $L(V)$. This follows immediately from Jacobson [7, p. 32].

LEMMA 2.2. *Let H be a pure subgroup of G and let f be any quasi-endomorphism of G such that $f(H^*) \subseteqq H^*$. Then the restriction of f to H^* is a quasi-endomorphism of H.*

Proof. Let n be a positive integer such that $n(fG) \subseteqq G$; then

$$n(fH) \subseteqq G \cap (fH)^* \subseteqq G \cap (H^*) = H .$$

PROPOSITION 2.3. (1) *$QE(G)$ is dense if and only if G is irreducible and Q is the centralizer of $QE(G)$ in $L(V)$.*

(2) *If $QE(G)$ is dense, then G is homogeneous and every pure subgroup of finite rank in G is completely decomposable.*

(3) *$QF(G)$ is an ideal of $QE(G)$; if $QE(G)$ is dense and $QF(G) \neq 0$, then $QF(G)$ is also dense.*

Proof. (1) follows from a remark of Jacobson [7, p.32] and the fact that G is irreducible if and only if V is an irreducible $QE(G)$-module. Let H be a pure subgroup of finite rank in G. In order to prove (2), it will suffice to show that $QE(H) = L(H^*)$ by Corollary 1.5 [4]. Let $x_1, \cdots, x_n$ be a basis of H^* and let $f \in L(H^*)$. By density and Remark 2.1, some $g \in QE(G)$ maps x_i to fx_i, $i = 1, \cdots, n$, and so $g(H^*) \subseteqq H^*$. By Lemma 2.2, g restricted to H^* is a quasi-endomorphism of H and so $QE(H) = L(H^*)$. In (3), it is clear that $QF(G)$ is an ideal of $QE(G)$; Theorem 4 [7, p.33] completes the proof.

LEMMA 2.4. *If $QF(G)$ is dense, then it contains a projection onto any finite dimensional subspace of V and thus every pure subgroup of finite rank in G is a completely decomposable quasi-summand.*

Proof. Let $x_1, \cdots, x_n$ be independent in V. By density and Remark 2.1, some $f \in QF(G)$ leaves the x_i invariant. Extend $x_1, \cdots, x_n$ to a basis $x_1, \cdots, x_n, y_1, \cdots, y_m$ of fV. Again, some $g \in QF(G)$ leaves the x_i invariant and annihilates $y_1, \cdots, y_m$. Now gf projects V onto the subspace spanned by $x_1, \cdots, x_n$. Suppose H is a pure subgroup of finite rank in G; by (2) of Proposition 2.3, H is completely decomposable. We have just proved that $QF(G)$ contains a projection e of V onto H^*. Now $G \doteq eV \cap G + (1 - e)V \cap G$ and $eV \cap G = H$ because H is pure.

We are now prepared to prove

THEOREM 2.5. *These are equivalent:*

(1) *G is homogeneous and quasi-separable.*

(2) *$QF(G)$ is dense in the finite topology of $L(V)$.*

(3) *$QF(G)$ is one-fold transitive and every pure subgroup of finite rank in G is completely decomposable.*

Proof. (1) implies (2). By Remark 2.1, it will be sufficient to show that $QF(G)$ is two-fold transitive. Let x_1 and x_2 be independent in V and let y_1 and y_2 be arbitrary elements of V. Since G is a full subgroup of V, there is some positive integer n such that nx_1, nx_2, ny_1, and ny_2 are all in G; suppose these elements are contained in a completely decomposable quasi-summand, H, of finite rank. H is homogeneous because G is, so by Corollary 1.5 [4], $QE(H) = L(H^*)$. If e is an idempotent associated with H, $QE(H) = eQE(G)e$ [8], so if $f \in L(H^*)$ sends x_i to y_i, $i = 1, 2$, f is induced by ege for some $g \in QE(G)$. Now $ege \in QF(G)$, so $QF(G)$ is two-fold transitive and thus dense.

That (2) implies (3) follows from Remark 2.1 and Proposition 2.3.

(3) implies (1). G is certainly homogeneous because $QE(G)$ is one-fold transitive. By Corollary 1.7 and Remark 1.4, it will suffice to prove that $QE(G)$ contains a projection onto any one-dimensional subspace of V, so let x be any nonzero element in V. By hypothesis some $f \in QF(G)$ leaves x invariant; $A = fV \cap G$ is pure of finite rank in G and so is completely decomposable. $B = \{x\}^* \cap G$ is a direct summand of A [5, p. 178]. If g projects A onto B, then $gf \in QE(G)$ projects V onto $\{x\}^*$.

Under the hypothesis of Theorem 2.5, $QE(G)$ is primitive with socle $QF(G)$ by the Structure Theorem [7, p. 75].

3.0. Applications to separable groups. Here we prove that countable groups G with $QE(G)$ dense in $L(V)$ are homogeneous and completely decomposable; this is accomplished with the aid of a generalization of Pontryagin's criterion for countable free groups. k-fold transitivity of quasi-endomorphisms is interpreted in terms of endomorphisms to provide further insight into homogeneous quasi-separable groups. This suggests properties of endomorphisms both necessary and sufficient for a group to be homogeneous and separable.

LEMMA 3.1. *A countable homogeneous group is completely decomposable if and only if each pure subgroup of finite rank is completely decomposable.*

Proof. The necessity obtains by Theorem 46.6 [5]. For the sufficiency, let $\{a_i\}_{i=1}^{\infty}$ be an enumeration of a countable homogeneous group G, each of whose pure subgroups of finite rank is completely decomposable. Let H_n denote the pure subgroup generated by $a_1, \cdots, a_n$ and set $G_1 = H_1$. Then in general, $H_{n+1} = H_n + G_{n+1}$ [5, p.178] with G_{n+1} either 0 or of rank one. Now $G = \sum_{n=1}^{\infty} G_n$.

THEOREM 3.2. *A countable group G is homogeneous and completely decomposable if and only if $QE(G)$ is dense.*

Proof. The necessity follows from Theorem 2.5 and the sufficiency from Proposition 2.3 (2) and Lemma 3.1.

COROLLARY 3.3. *A countable, homogeneous, quasi-separable group is completely decomposable.*

COROLLARY 3.4. *If $QE(G)$ is dense then G is $\aleph_1$-completely decomposable in the sense that every countable pure subgroup is completely decomposable.*

The discussion now turns to an interpretation in terms of endomorphisms of some properties of quasi-endomorphisms encountered in §2. $E(G)$ denotes the endomorphism ring of G and $F(G)$ denotes those endomorphisms of G which have finite rank.

DEFINITION 3.5. A subset S of $E(G)$ is called k-fold transitive if and only if given $j \leqq k$ independent elements $a_1, \cdots, a_j$ of G and any j elements $b_1, \cdots, b_j$ of G, there exists an endomorphism $f \in S$ and a positive integer n such that $fa_i = nb_i$, $i = 1, \cdots, j$.

PROPOSITION 3.6. *The pure subring R of $E(G)$ is k-fold transitive if and only if $R^*(\subseteqq L(V))$ is k-fold transitive.*

Proof. A straightforward computation using the fact that $E(G)$ is full in $QE(G)$ and using the one-to-one correspondence between pure subrings of $E(G)$ and subalgebras of $QE(G)$ [5, p.271].

REMARK 3.7. The above implies the following; (5) is of particular interest.

(1) G is irreducible if and only if $E(G)$ is one-fold transitive.

(2) $F(G)$ is a pure ideal of $E(G)$ and $F(G)^* = QF(G)$.

(3) The pure subring R of $E(G)$ is two-fold transitive if and only if R^* is dense.

(4) G is homogeneous and quasi-separable if and only if $F(G)$ is two-fold transitive.

(5) A countable group G is homogeneous and completely decomposable if and only if $E(G)$ is two-fold transitive.

A property somewhat stronger than two-fold transitivity may be required of $F(G)$ to conclude that G is homogeneous and separable.

DEFINITION 3.8. Call a subset S of $E(G)$ fully k-fold transitive if and only if S is k-fold transitive and in addition for any nonzero elements a and b of G such that $h(a) \leqq h(b)$, some $f \in S$ maps a to b.

LEMMA 3.9. *Let a and b be nonzero elements of rank-one groups A and B, respectively. Then some $f \in \mathrm{Hom}_Z(A, B)$ maps a to b if and only if $h_A(a) \leqq h_B(b)$.*

Proof. Only the sufficiency need be checked and this can be done computationally by using the characterization of subgroups of Q found in [3].

THEOREM 3.10. *The following statements about the group G are equivalent.*

(1) *G is homogeneous and separable.*

(2) *$F(G)$ is fully two-fold transitive.*

(3) *$F(G)$ is fully one-fold transitive and every pure subgroup of finite rank in G is completely decomposable.*

(4) *G is homogeneous, every pure subgroup of finite rank is completely decomposable, and $F(G)$ is dense in the finite topology of $E(G)$.*

Proof. We prove that (1) and (2) are equivalent, then (2) and (3), and finally (1) and (4).

(1) implies (2). By Remark 3.7 (4), $F(G)$ is two-fold transitive. Let a and b be any two nonzero elements of G satisfying $h(a) \leqq h(b)$ and let A and B denote the pure subgroups of G generated by a and b respectively. By Lemma 3.9, some $f \in \mathrm{Hom}_Z(A, B)$ maps a to b. By [5, p.178], there is a projection g of G onto A. Now $fg \in F(G)$ sends a to b, so $F(G)$ is fully two-fold transitive.

(2) implies (1). By Remark 3.7 (4) and Corollary 1.7, G is homogeneous and every pure subgroup A of rank one is a quasi-summand; it will be sufficient to show that A is in fact a direct summand [5, p.178]. Write $G \doteq A + C$ with C pure in G. By [1, p.96],

$$G = B + C$$

with B isomorphic to A via some map f; let g be the projection of G onto B. Pick a nonzero element $a \in A$; $h(a) = h(f^{-1}a)$ and height is unambiguous since all relevant groups are pure subgroups of G. By hypothesis, some $r \in F(G)$ maps a to $f^{-1}a$. Let $s = fgr$; $sa = a$. $\{c \in G\colon sc = c\}$ is a nontrivial pure subgroup of G contained in A and so equals A, i.e., s is an idempotent.

(2) and (3) are equivalent by Remark 3.7 (4) and Theorem 2.5.

(1) implies (4). Since (1) implies (3), it will be enough to prove that $F(G)$ is dense in the finite topology of $E(G)$. Let f be any endomorphism of G and let $a_1, \cdots, a_n$ be arbitrary elements of G. Now a_i, fa_i, $i = 1, \cdots, n$, are all contained in some direct summand

of finite rank. If g is a projection associated with this summand, $gf \in F(G)$ is in the open neighborhood of f,

$$\{h \in E(G)\colon\ ha_i = fa_i,\ i = 1, \cdots, n\}\ .$$

(4) implies (1). It will be sufficient to see that any pure subgroup A of finite rank in G is a direct summand [5, p.178]. By density, some $f \in F(G)$ leaves A invariant because the identity map does. Let B denote the pure subgroup of G generated by fG; by hypothesis B is completely decomposable and A is a direct summand of B by [5, p.178]. If g projects B onto A then gf projects G onto A.

REMARK 3.11. Full two-fold transitivity cannot be strengthened in the following sense. Given a_1, a_2 independent in G and b_1, b_2 arbitrary in G such that $h(a_i) \leqq h(b_i)$, in general there is no endomorphism mapping a_i to b_i, $i = 1, 2$. Furthermore in Theorem 3.10 (3), the complete decomposability of pure subgroups of finite rank is essential, as the following discussion indicates. Let K be any subfield of the p-adic number field F_p and let $R = K \cap J_p$, J_p the subring of p-adic integers; R is a pure subring of J_p and so is indecomposable [5, p.150]. By standard arguments [5, p.212], $E(R) = R$, i.e., every endomorphism of the additive group of R is induced by ring multiplication. Now it is easy to see that $E(R)$ is fully one-fold transitive, for if a and b are nonzero elements of R, $a = p^m u$, $b = p^n v$ with u and v units in J_p [6, p.225] and hence in R by purity; also

$$u^{-1} \in R = K \cap J_p\ .$$

Now $h(a) \leqq h(b)$ if and only if $m \leqq n$ [6, p.225], so if $m \leqq n$,

$$p^{n-m} v u^{-1} \in R = E(R)$$

maps a to b; otherwise vu^{-1} maps a to $p^{m-n} b$. Thus $E(R)$ is fully one-fold transitive. In particular for K an algebraic number field [6, p.229], $E(R) = F(R)$, $F(R)$ is fully one-fold transitive but not (fully) two-fold transitive, and R is homogeneous but not (quasi-) separable.

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