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ISOMETRIC IMMERSIONS OF SPACE FORMS IN SPACE FORMS

JOHN DOUGLAS MOORE

Let M be a connected n-dimensional space form isometrically immersed in a simply connected $(2n\text{-}1)$-dimensional space form of strictly larger curvature. If M is minimal, it is proven that it must be a piece of the flat Clifford torus in the $(2n\text{-}1)$-sphere. If M is complete and simply connected, it is proven that M possesses a global coordinate system whose coordinate vectors are unit-length asymptotic vectors.

Introduction. A well-known theorem of David Hilbert states that a complete two-dimensional riemannian manifold of constant negative curvature cannot be isometrically immersed in three-dimensional euclidean space [5], [7, p. 265]. There is reason to believe that the natural generalization of Hilbert's theorem to higher dimensions would be the following conjecture: A complete n-dimensional riemannian manifold of constant negative curvature cannot be isometrically immersed in E^{2n-1}. If completeness is strengthened to compactness the conjecture is known to be true by work of Chern, Kuiper, and Otsuki [6, vol. 2, p. 29].

The local problem of isometrically immersing a space form in a space form was studied by Élie Cartan [3]. He used his theory of exterior differential systems to show, among other things, that real analytic n-dimensional submanifolds of constant negative curvature in $(2n\text{-}1)$-dimensional euclidean space E^{2n-1} depend upon $n(n\text{-}1)$ functions of a single variable. Cartan also showed that no n-dimensional hyperbolic space form can be isometrically immersed in E^{2n-2}. To construct an explicit example, we choose nonzero real numbers a_i, $1 \leqq i \leqq n-1$, so that $\sum_i a_i^2 = 1$, and we define an immersion from

$$D = \{(y_1, y_2, \cdots, y_n) \in \boldsymbol{R}^n \mid y_n < 0\}$$

into E^{2n-1} with rectangular cartesian coordinates $x_1, x_2, \cdots, x_{2n-1}$ by the equations

$$\begin{aligned} x_{2i-1} &= a_i e^{y_n} \cos(y_i/a_i) \,, \\ x_{2i} &= a_i e^{y_n} \sin(y_i/a_i) \,, \qquad\qquad 1 \leqq i \leqq n-1, \\ x_{2n-1} &= \int_0^{y_n} (1 - e^{2u})^{1/2} du \,. \end{aligned}$$

We find that the submanifold metric on D is of constant negative curvature; however D is not complete in this metric.

In §3 of this paper we prove that one of the main steps in the proof of Hilbert's theorem, the construction of a global coordinate system whose coordinate vectors are unit-length asymptotic vectors, can be generalized to the n-dimensional context. Our treatment is based upon a theorem of Cartan, a proof of which is given in § 1. Section 2 is devoted to the local properties of space forms isometrically immersed in space forms, and includes a rigidity theorem for minimal submanifolds of constant curvature.

Unless otherwise stated all manifolds are connected and C^∞.

1. Exteriorly orthogonal symmetric bilinear forms. Let V be an n-dimensional real vector space and let $\Phi^1, \Phi^2, \cdots, \Phi^n$ be n symmetric bilinear forms on V. We say that $\Phi^1, \Phi^2, \cdots, \Phi^n$ are *exteriorly orthogonal* if

$$\sum_{\lambda=1}^{n} [\Phi^\lambda(X, Y)\Phi^\lambda(Z, W) - \Phi^\lambda(X, W)\Phi^\lambda(Z, Y)] = 0$$

for $X, Y, Z, W \in V$.

THEOREM 1. (Élie Cartan [3]). *Suppose that $\Phi^1, \Phi^2, \cdots, \Phi^n$ are n exteriorly orthogonal symmetric bilinear forms on an n-dimensional real vector space V with the following property: if X is a vector in V such that $\Phi^\lambda(X, Y) = 0$ for $1 \leqq \lambda \leqq n$ and for all $Y \in V$, then $X = 0$. Then there exists a real orthogonal matrix (a^λ_μ) and n linear functionals $\varphi^1, \varphi^2, \cdots, \varphi^n$ such that*

$$\Phi^\lambda = \sum_\mu a^\lambda_\mu \varphi^\mu \otimes \varphi^\mu , \qquad 1 \leqq \lambda \leqq n .$$

It follows that $\Phi^1, \Phi^2, \cdots, \Phi^n$ are simultaneously diagonalized with respect to the basis dual to $\{\varphi^1, \varphi^2, \cdots, \varphi^n\}$. Theorem 1 is trivial when $n = 1$ and when $n = 2$ it is a consequence of the following well-known fact: two symmetric bilinear forms, one of which is positive definite, can be simultaneously diagonalized.

We will find it convenient to regard Φ^λ as a linear transformation from V to the dual space V^* so that it induces a linear map

$$\Phi^\lambda \wedge \Phi^\lambda \colon V \wedge V \to V^* \wedge V^* .$$

Then $\Phi^\lambda \wedge \Phi^\lambda = 0$ if and only if $\Phi^\lambda = \pm \varphi^\lambda \otimes \varphi^\lambda$ for some linear functional φ^λ. We can now restate Theorem 1 as follows: Suppose that $\Phi^1, \Phi^2, \cdots, \Phi^n$ are linear transformations from an n-dimensional real vector space to its dual such that $[\Phi^\lambda(X)](Y) = [\Phi^\lambda(Y)](X)$. If

$$\bigcap_\lambda \ker(\Phi^\lambda) = (0) \quad \text{and} \quad \sum_\lambda \Phi^\lambda \wedge \Phi^\lambda = 0 ,$$

then there exists a real orthogonal matrix (a^λ_μ) such that if

$$\Psi^\lambda = \sum_\mu a^\lambda_\mu \Phi^\mu\ , \quad \text{then} \quad \Psi^\lambda \wedge \Psi^\lambda = 0 \quad \text{for} \quad 1 \leqq \lambda \leqq n\ .$$

The first step in the proof of Theorem 1 consists of showing that there exists a vector X in V such that $\Phi^1(X), \Phi^2(X), \cdots, \Phi^n(X)$ are linearly independent. We prove this by contradiction. If $X \in V$, let $U^*(X)$ be the subspace of V^* generated by $\{\Phi^\lambda(X): 1 \leqq \lambda \leqq n\}$ and let p be the maximum dimension of $U^*(X)$ for $X \in V$. We assume that $p < n$. If M is a vector for which the maximum dimension p is attained, we can assume without loss of generality that

$$\Phi^1(M), \Phi^2(M), \cdots, \Phi^p(M)$$

are linearly independent, and $\Phi^{p+1}(M) = \cdots = \Phi^n(M) = 0$. If Y is any other vector in V, then

$$\sum_{\alpha=1}^p \Phi^\alpha(M) \wedge \Phi^\alpha(Y) = 0\ ,$$

so that by Cartan's lemma there exists a $p \times p$ symmetric matrix (c^α_β) such that

$$\Phi^\alpha(Y) = \sum_{\beta=1}^p c^\alpha_\beta \Phi^\beta(M)\ , \qquad 1 \leqq \alpha \leqq p\ . \tag{1}$$

If we let W^* be the subspace of V^* generated by

$$\{\Phi^\alpha(X):\ X \in V,\ 1 \leqq \alpha \leqq p\}\ ,$$

then (1) shows that W^* is exactly p-dimensional. Since $p < n$ there exists a nonzero vector Z in V which is annihilated by W^*. But by hypothesis there exists λ, $1 \leqq \lambda \leqq n$, and a vector $N \in V$ such that $\Phi^\lambda(Z, N) \neq 0$. Since Z is annihilated by W^*, $\lambda \geqq p + 1$. If $\varepsilon > 0$ is sufficiently small, $\{\Phi^\alpha[(\cos \varepsilon)M + (\sin \varepsilon)N]\ 1 \leqq \alpha \leqq p\}$ will generate W^* and $\Phi^\lambda[(\cos \varepsilon)M + (\sin \varepsilon)N]$ will be outside of W^*. Hence

$$U^*[(\cos \varepsilon)M + (\sin \varepsilon)N]$$

is at least $(p + 1)$-dimensional; this contradicts the definition of p, and the first step is established.

Let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V such that

$$\Phi^1(v_1), \Phi^2(v_1), \cdots, \Phi^n(v_1)$$

are linearly independent. Then we can apply Cartan's lemma to the equation

$$\sum_\lambda \Phi^\lambda(v_1) \wedge \Phi^\lambda(v_i) = 0$$

and conclude that there exists a symmetric matrix $C(i) = (c(i)^\lambda_\mu)$ such that

$$\Phi^\lambda(v_i) = \sum_\mu c(i)^\lambda_\mu \Phi^\mu(v_1) , \qquad 1 \leqq \lambda \leqq n .$$

(Notice that $C(1)$ is the identity matrix.) We next observe that it follows from the equation

$$\sum_\lambda \Phi^\lambda(v_i) \wedge \Phi^\lambda(v_j) = 0$$

that the matrices $C(i)$ and $C(j)$ commute with each other. By a well-known theorem from linear algebra there exists an orthogonal matrix $A = (a^\lambda_\mu)$ such that $A[C(i)][{}^tA]$ is diagonal for $1 \leqq i \leqq n$. If we let $\Psi^\lambda = \sum_\mu a^\lambda_\mu \Phi^\mu$ then $\Psi^\lambda(v_i)$ is a constant multiple of $\Psi^\lambda(v_1)$ for $1 \leqq i \leqq n$, so that

$$\Psi^\lambda(v_i) \wedge \Psi^\lambda(v_j) = 0 , \qquad 1 \leqq i, j, \lambda \leqq n .$$

It follows that $\Psi^\lambda \wedge \Psi^\lambda = 0$, $1 \leqq \lambda \leqq n$, and Theorem 1 is proven.

An examination of the above proof shows that $\Psi^1, \Psi^2, \cdots, \Psi^n$ are uniquely determined up to a permutation. Hence the linear functionals $\varphi^1, \varphi^2, \cdots, \varphi^n$ are uniquely determined up to changes of sign and a possible permutation.

2. Submanifolds of constant curvature: local theory. In the rest of this paper, our setup will be as follows: we will let M be an n-dimensional riemannian manifold of constant curvature k isometrically immersed in a $(2n-1)$-dimensional riemannian manifold N of constant curvature K. We will use the following conventions on ranges of indices:

$$1 \leqq i, j, k, l \leqq n , \quad n+1 \leqq \lambda, u \leqq 2n-1 , \quad 1 \leqq A, B, C \leqq 2n-1 .$$

Let $e_1, e_2, \cdots, e_{2n-1}$ be a moving oriented orthonormal frame on an open set U in N, chosen so that at points of a suitable open subset V of the submanifold M the first n frame vectors are tangent to M. Let $\theta^1, \theta^2, \cdots, \theta^{2n-1}$ be the dual orthonormal coframe. A fundamental theorem of riemannian geometry states that there exists a unique collection of 1-forms θ^A_B on U which satisfy the structure equation

$$d\theta^A = -\sum_B \theta^A_B \wedge \theta^B, \; \theta^A_B = -\theta^B_A . \tag{2}$$

The fact that N has constant curvature K is expressed by the equation

$$d\theta^A_B = -\sum_C \theta^A_C \wedge \theta^C_B + K\theta^A \wedge \theta^B . \tag{3}$$

If we restrict these equations to the open subset V of M and make use of the fact that $\theta^\lambda = 0$ on V, we obtain from (2) the equations

$$(4) \qquad d\theta^i = -\sum_k \theta^i_k \wedge \theta^k,\ 0 = -\sum_k \theta^\lambda_k \wedge \theta^k .$$

The second of these implies via Cartan's lemma that

$$(5) \qquad \theta^\lambda_i = \sum_j b^\lambda_{ij}\theta^j,\ b^\lambda_{ij} = b^\lambda_{ji} ,$$

where the b^λ_{ij}'s are differentiable functions on V called the components of the second fundamental forms. From equation (3) we obtain the equation

$$d\theta^i_j = -\sum_k \theta^i_k \wedge \theta^k_j - \sum_\lambda \theta^i_\lambda \wedge \theta^\lambda_j + K\theta^i \wedge \theta^j .$$

Since M is of constant curvature k

$$-\sum_\lambda \theta^i_\lambda \wedge \theta^\lambda_j = (k-K)\theta^i \wedge \theta^j ,$$

or equivalently

$$(6) \qquad \sum_\lambda (b^\lambda_{ij}b^\lambda_{kl} - b^\lambda_{il}b^\lambda_{kj}) = (k-K)(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}) ,$$

where δ_{ij} is the usual Kronecker delta.

Assume now that $k < K$. Equation (6) then states that the second fundamental forms $\Phi^\lambda = \sum_i \theta^\lambda_i \otimes \theta^i$ and the symmetric bilinear form

$$\Psi = \sqrt{K-k}\,(\sum_i \theta^i \otimes \theta^i)$$

are exteriorly orthogonal, and Theorem 1 implies that they can be simultaneously diagonalized by a basis for the tangent space to M. Since the basis diagonalizes Ψ it can be chosen to be orthonormal, and hence we can assume that the moving frame $e_1, e_2, \cdots, e_{2n-1}$ chosen in the preceding paragraphs satisfies the equations $b^\lambda_{ij} = 0$ for $i \neq j$. In view of the remark at the end of § 1, any two diagonalizing orthonormal bases differ at most by changes of sign and a possible permutation. Hence if M is simply connected we can choose a global moving frame $e_1, e_2, \cdots, e_n$ on M which diagonalizes the second fundamental forms. In particular, the universal covering space of M is parallelizable.

In terms of the diagonalizing moving frame, equation (6) takes the simpler form

$$(7) \qquad \sum_\lambda b^\lambda_{ii}b^\lambda_{jj} = (k-K) , \qquad i \neq j .$$

We claim that it follows from this equation that there exist unique positive functions $x_1, x_2, \cdots, x_n$ such that

$$(8) \qquad \sum_i b^\lambda_{ii}x_i^2 = 0 \text{ and } \sum_i x_i^2 = 1 .$$

Indeed, such functions need to satisfy the equation

$$0 = \sum_{i,\lambda} b^\lambda_{jj} b^\lambda_{ii} x_i^2 = \sum_\lambda b^\lambda_{jj} b^\lambda_{jj} x_j^2 - (K - k)(1 - x_j^2) ,$$

from which it follows that

$$\sum_\lambda b^\lambda_{ii} b^\lambda_{ii} = (K - k)(1 - x_i^2)/x_i^2 . \tag{9}$$

We can solve for x_i to obtain the expression

$$x_i = \left[\left(\sum_\lambda b^\lambda_{ii} b^\lambda_{ii}\right)\Big/(K - k) + 1\right]^{-1/2} , \tag{10}$$

and check that the functions defined by this equation satisfy equations (8). A slight modification of this argument shows that any $n - 1$ of the "principal normal curvature vectors" $\sum_\lambda b^\lambda_{ii} e_\lambda$ are linearly independent.

A restatement of what we proved in the preceding paragraph is that there exist exactly 2^n unit-length vectors on which all the second fundamental forms vanish simultaneously. They are all of the form

$$\pm x_1 e_1 \pm x_2 e_2 \pm \cdots \pm x_n e_n , \tag{11}$$

where the signs can be chosen in 2^n ways, and they are called *asymptotic vectors*.

We remark that the normal bundle of M in N has zero curvature because the curvature forms of the normal bundle are $-\sum_i \theta^\lambda_i \wedge \theta^i_\mu$ and both θ^λ_i and θ^i_μ are multiples of θ^i. Hence without loss of generality we will assume that $e_{n+1}, \cdots, e_{2n-1}$ have been chosen so that $\theta^\lambda_\mu = 0$.

Our next objective is to find an expression for the differential 1-forms θ^i_j in terms of the functions x_i. For this purpose we will use the tensor b^λ_{ijk} defined by the following equation

$$db^\lambda_{ij} + \sum_\mu b^\mu_{ij} \theta^\lambda_\mu - \sum_k b^\lambda_{kj} \theta^k_i - \sum_k b^\lambda_{ik} \theta^k_j = \sum_k b^\lambda_{ijk} \theta^k . \tag{12}$$

The exterior derivative of equation (5) shows that the tensor b^λ_{ijk} is symmetric in its lower indices. If we make use of the facts that $b^\lambda_{ij} = 0$ for $i \neq j$ and $\theta^\lambda_\mu = 0$, we can simplify (12) and obtain the equations

$$db^\lambda_{ii} = \sum_k b^\lambda_{iik} \theta^k , \tag{13}$$

$$(b^\lambda_{jj} - b^\lambda_{ii}) \theta^i_j = \sum_k b^\lambda_{ijk} \theta^k , \qquad i \neq j . \tag{14}$$

If we choose e_{n+1} at a point $x \in M$ so that $b^{n+2}_{11}(x) = \cdots = b^{2n-1}_{11}(x) = 0$, then it follows from equation (7) that

$$b_{ii}^{n+1}(x) = (k - K)/b_{11}^{n+1}(x) .$$

Equation (14) therefore implies that $b_{ijk}^{n+1}(x) = 0$ for $i, j, 1$ distinct. It follows that $b_{ijk}^{n+1}(x) = 0$ for i, j, k distinct, and since the principal normal curvature vectors span the normal space, $b_{ijk}^{\lambda}(x) = 0$ for i, j, k distinct. Since x is arbitrary, equation (14) now becomes

$$(b_{jj}^{\lambda} - b_{ii}^{\lambda})\theta_j^i = b_{iji}^{\lambda}\theta^i + b_{ijj}^{\lambda}\theta^j , \qquad i \neq j .$$

We multiply this last equation by b_{ii}^{λ} and sum with respect to λ to conclude that

$$(k - K - \sum_{\lambda} b_{ii}^{\lambda}b_{ii}^{\lambda})\theta_j^i = \sum_{\lambda} b_{ii}^{\lambda}b_{iij}^{\lambda}\theta^i + \sum_{\lambda} b_{ii}^{\lambda}b_{jji}^{\lambda}\theta^j , \qquad i \neq j .$$

We now need to use the following fact which is a consequence of (9):

$$2\sum_{\lambda} b_{ii}^{\lambda}b_{iij}^{\lambda} = (K - k)e_j[(1 - x_i^2)/x_i^2] . \tag{15}$$

We can use this to derive the following equation for the 1-forms θ_j^i:

$$\theta_j^i = (1/x_i)e_j(x_i)\theta^i + (\text{something})\theta^j .$$

Using skew-symmetry we conclude that

$$\theta_j^i = (1/x_i)e_j(x_i)\theta^i - (1/x_j)e_i(x_j)\theta^j . \tag{16}$$

As an application of these ideas we prove the following theorem closely related to recent work of do Carmo and Wallach [2]:

THEOREM 2. *Let M be a connected n-dimensional riemannian manifold of constant curvature k isometrically and minimally immersed in a simply connected $(2n - 1)$-dimensional riemannian manifold N of constant curvature K. Then either M is totally geodesic or it is flat. In the flat case it is immersed as a piece of the n-dimensional Clifford torus in the $(2n - 1)$-sphere.*

The proof is local. The fact that the immersion is minimal is expressed by the equation

$$\sum_i b_{ii}^{\lambda} = 0 \tag{17}$$

which together with equation (6) implies that

$$\sum_{i,\lambda} b_{ij}^{\lambda}b_{ik}^{\lambda} = (n - 1)(K - k)\delta_{jk} .$$

Hence $k \leqq K$ and if $k = K$ then the submanifold M is totally geodesic. Therefore we assume without loss of generality that $k < K$.

In the case where $k < K$ we will actually prove a little more than the theorem states: if the hypothesis that M be minimal is replaced by the weaker condition that its mean curvature vector be parallel, it still follows that M is flat.

Since the normal moving frame vectors are parallel, the mean curvature vector is parallel if and only if there exist constants c^λ such that

$$\sum_i b^\lambda_{ii} = c^\lambda .$$

On the other hand, equations (13) and (7) imply that

$$\sum_i b^\lambda_{iij} = 0, \text{ and } \sum_\lambda b^\lambda_{iij} b^\lambda_{kk} = -\sum_\lambda b^\lambda_{ii} b^\lambda_{kkj} \text{ if } i \neq k .$$

Hence we conclude that

$$\begin{aligned} \sum_{i,\lambda} b^\lambda_{iij} b^\lambda_{ii} &= -\sum_{\substack{\lambda \\ k\neq i}} b^\lambda_{kkj} b^\lambda_{ii} = \sum_{\substack{\lambda \\ k\neq i}} b^\lambda_{kk} b^\lambda_{iij} \\ &= \sum_{i,\lambda} c^\lambda b^\lambda_{iij} - \sum_{i,\lambda} b^\lambda_{ii} b^\lambda_{iij} = -\sum_{i,\lambda} b^\lambda_{ii} b^\lambda_{iij} . \end{aligned}$$

It follows that $\sum_{i,\lambda} b^\lambda_{iij} b^\lambda_{ii} = 0$, and hence equation (15) implies that $e_j(x_i) = 0$. Now by equation (16) the differential forms θ^i_j vanish, proving that M is flat.

To finish the proof of the theorem, we notice that if M is minimal the principal normal curvature vectors (i.e., the b^λ_{ii}'s) are determined up to a rotation of $e_{n+1}, \cdots, e_{2n-1}$ by equations (7) and (17). Since the b^λ_{ii}'s determine the θ^λ_i's and $\theta^i_j = 0 = \theta^\lambda_\mu$, it follows from the classical rigidity theorem [1, p. 202] that locally there is at most one minimal flat n-dimensional submanifold of N, up to a rigid motion. Therefore M must be a piece of the Clifford torus, and the theorem is proven.

3. The global existence of asymptotic coordinates. If M is complete and simply connected, then any choice of signs in expression (11) determines a globally defined unit-length asymptotic vector field on M. If n unit-length asymptotic vector fields are linearly independent at one point, they are linearly independent everywhere.

THEOREM 3. *If M is a complete simply connected riemannian manifold of constant curvature k isometrically immersed in a $(2n-1)$-dimensional riemannian manifold N of constant curvature $K > k$, then any n linearly independent unit-length asymptotic vector fields $Z_1, Z_2, \cdots, Z_n$ determine a global coordinate system whose coordinate vectors are the Z_i's.*

First we establish local existence. Because of the theorem of Frobenius, it suffices to show that the Lie bracket of any two asymptotic vector fields is zero. But

$$\begin{aligned}\theta^i([x_je_j, x_ke_k]) &= x_je_j(\theta^i(x_ke_k)) - x_ke_k(\theta^i(x_je_j)) - 2d\theta^i(x_je_j, x_ke_k)\\ &= x_je_j(\theta^i(x_ke_k)) - x_ke_k(\theta^i(x_je_j)) + 2\sum_l \theta^i_l \wedge \theta^l(x_je_j, x_ke_k)\\ &= \delta_{ik}x_je_j(x_k) - \delta_{ij}x_ke_k(x_j) + \delta_{ij}x_ke_k(x_j) - \delta_{ik}x_je_j(x_k)\\ &= 0 .\end{aligned}$$

In this derivation we have used equations (4) and (16). Since the asymptotic vectors are sums of $\pm x_ie_i$, local existence is proven.

To prove global existence, we let $\varphi_i(x, t)$, $x \in M$, $t \in \boldsymbol{R}$ be the one-parameter group of transformations corresponding to Z_i. Since Z_i is a vector field of unit length, it follows from the theory of ordinary differential equations [4, p. 15] that $\varphi_i(x, t)$ is defined for all values of x and t. Let x_0 be a fixed point in M and define a function F: $\boldsymbol{R}^n \to M$ by

$$F(t_1, t_2, \cdots, t_n) = \varphi_n(\varphi_{n-1}(\cdots\varphi_2(\varphi_1(x_0, t_1), t_2), \cdots), t_n) .$$

Since the Lie bracket $[Z_i, Z_j]$ vanishes, the one-parameter groups φ_i and φ_j commute. Using this fact we can verify the following equation:

$$(18)\quad F(s_1 + t_1, \cdots, s_n + t_n) = \varphi_n(\varphi_{n-1}(\cdots\varphi_1(F(s_1, \cdots, s_n), t_1), \cdots), t_n) .$$

We claim that F is a covering map. Let x be a point in the manifold M and let U_x be an open neighborhood of x on which local asymptotic coordinates $z_1, z_2, \cdots, z_n$ exist, and we can assume that $z_1(x) = z_2(x) = \cdots = z_n(x) = 0$. For $\delta > 0$, let

$$B_\delta(x) = \{y \in U_x: |z_i(y)| < \delta\}$$

and choose ε so small that $(z_1, z_2, \cdots, z_n)$ give a diffeomorphism from $B_{2\varepsilon}(x)$ onto an open ball of radius 2ε in $\boldsymbol{R}^n$. Let $\tilde{x}_\alpha$, $\alpha \in A$, be the points in $F^{-1}(x)$, and let $B_\delta(\tilde{x}_\alpha)$ denote the open ball of radius δ around x_α. To show that F is a covering map, it suffices to check the following facts:

1. $F \mid B_{2\varepsilon}(\tilde{x}_\alpha)$ is a diffeomorphism from $B_{2\varepsilon}(\tilde{x}_\alpha)$ onto $B_{2\varepsilon}(x)$ for $\alpha \in A$.
2. $B_\varepsilon(\tilde{x}_\alpha) \cap B_\varepsilon(\tilde{x}_\beta) = \phi$ if $\tilde{x}_\alpha \neq \tilde{x}_\beta$.
3. $\tilde{y} \in F^{-1}(B_\varepsilon(x)) \Rightarrow \tilde{y} \in B_\varepsilon(\tilde{x}_\alpha)$ for some $\alpha \in A$.

To prove 1, we need only check that the local asymptotic coordinates define an inverse to $F|B_{2\varepsilon}(\tilde{x}_\alpha)$ using equation (18). 2 follows from 1, and 3 follows from the fact that $\tilde{y} - (z_1(F(\tilde{y})), \cdots, z_n(F(\tilde{y})))$ goes to x under F.

Thus F is a covering map, and since M is simply connected it is a diffeomorphism. Therefore F defines a global coordinate system whose coordinate vectors are the Z_i's and Theorem 3 is proven.

A straightforward modification of the above proof establishes the existence of "principal coordinates" whose coordinate vectors are $x_1e_1, x_2e_2, \cdots, x_ne_n$.

Since $\boldsymbol{R}^n$ is not a covering space for the n-sphere when $n > 1$, we obtain the positive curvature analogue of our conjecture:

COROLLARY. *A complete n-dimensional riemannian manifold of constant positive curvature k cannot be isometrically immersed in a* $(2n - 1)$*-sphere of constant curvature* $K > k$.

The corresponding local assertion is false, as Cartan proved in [3]. An n-sphere of constant curvature can be isometrically immersed in a $(2n + 1)$-sphere of constant curvature by first embedding it in E^{n+1} in the usual fashion, and then immersing E^{n+1} in the $(2n + 1)$-sphere as a flat torus.

If M is a complete simply connected space form as in Theorem 3, we will use the term "asymptotic surface" to denote a complete two-dimensional submanifold generated by two unit-length asymptotic vector fields. Every asymptotic surface possesses a global Tchebychef net ([7], p. 198) and it follows from the formula of Hazzidakkis that the integral of the Gaussian curvature over any parallelogram of the Tchebychef net is bounded in absolute value by 2π.

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UNIVERSITY OF CALIFORNIA, SANTA BARBARA