PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 1, 1972

# DIAGONAL SIMILARITY OF IRREDUCIBLE MATRICES TO ROW STOCHASTIC MATRICES

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**By using the Perron-Frobenius Theorem it is easily shown that if $A$ is an irreducible matrix then there is a diagonal matrix $D$ with positive main diagonal so that $DAD^{-1} = rS$ where $r$ is a positive scalar and $S$ a stochastic matrix. This paper gives a short proof of this result without direct appeal to the Perron-Frobenius Theorem.**

Definitions and Notations. Let $n \geqq 2$ be an integer. Let $N = \{1, 2, \cdots, n\}$. An $n \times n$ nonnegative matrix $A$ is said to be reducible if there is a permutation matrix $P$ so that

$PAP^T = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$ where $A_1$ and $A_2$ are square. If $A$ is not reducible we say that $A$ is irreducible. By agreement each $1 \times 1$ matrix is irreducible.

Denote by

$$u(A) = \min_M \left[ \max_{\substack{i \in M \\ j \notin M}} a_{ij} \right]$$

where the minimum is over all proper subsets of $N$.

$$r(A) = \max_{i \in N} \sum_{k \in N} a_{ik}\ , \qquad p(A) = \min_{i \in N} \sum_{k \in N} a_{ik}$$

$$D = \{d = (d_1, d_2, \cdots, d_n) \mid \text{each } d_k > 0 \text{ and } \min_k d_k = 1\}\ .$$

$f(d) = \max_{i\, j \in N} \mid \sum_{k \in N} d_i a_{ik} d_k^{-1} - \sum_{k \in N} d_j a_{jk} d_k^{-1} \mid$ where each $d_k > 0$ and $A$ is irreducible. Finally let $S(A)$ denote the positive number so that $S(A) \cdot u(A) - r(A) = f(e)$ where $e = (1, 1, \cdots, 1)$.

RESULTS.

LEMMA 1: *$f(d) = f(\lambda \cdot d)$ for each $\lambda > 0$.*

LEMMA 2. *If $(d_1, d_2, \cdots, d_n) \in D$, and $\max_{k \in N} d_k > [S(A)]^{n-1}$, then $f(d) > f(e)$.*

*Proof*. Reorder $(d_1, d_2, \cdots, d_n)$ to $(d_{i_1}, d_{i_2}, \cdots, d_{i_n})$ so that $d_{i_1} \geqq d_{i_2} \geqq \cdots \geqq d_{i_n}$. Let $s$ denote the smallest integer so that $(d_{i_s}/d_{i_{s+1}}) > S(A)$. That there is such an $s$ follows since $(d_{i_k}/d_{i_{k+1}}) \leqq S(A)$ for each $k \in \{1, 2, \cdots, n-1\}$ would imply that

$$d_{i_1} = \frac{d_{i_1}}{d_{i_n}} = \prod_{k=1}^{n-1} \left(\frac{d_{i_k}}{d_{i_{k+1}}}\right) \leqq [S(A)]^{n-1} .$$

Let $M = \{d_{i_1}, d_{i_2}, \cdots, d_{i_s}\}$. Note that $M \neq N$. Since $A$ is irreducible there is an $a_{pq} = \max_{i \in M, j \notin M} a_{ij} > 0$. Then since $p \in M$ and $q \notin M$

$$\frac{d_p}{d_q} > S(A),\ \frac{d_{i_n}}{d_k} \leqq 1 \text{ for each } k \in N ,$$
$$\sum_{k \in N} d_p a_{pk} d_k^{-1} > S(A) \cdot u(A) , \quad \text{and} \quad \sum_{k \in N} d_{i_n} a_{i_n k} d_k^{-1} \leqq r(A) .$$

From this it follows that

$$f(d) \geqq \left| \sum_{k \in N} d_p a_{pk} d_k^{-1} - \sum_{k \in N} d_{i_n} a_{i_n k} d_k^{-1} \right| > S(A) \cdot u(A) - r(A) = f(e) .$$

LEMMA 3. *$f$ achieves a minimum in $D$.*

*Proof.* The proof follows from Lemma 2, the fact that $f$ is continuous on the compact set $\{d \mid d \in D \text{ and } \max_k d_k \leqq [S(A)]^{n-1}\}$, and $e \in D$.

THEOREM. *The minimum of $f$ in $D$ is 0, i.e.,* $\mathrm{Min}_{d_k > 0\ k \in N} f(d) = 0$.

*Proof.* We first prove the theorem for positive matrices. Suppose $A > 0$ and $f$ achieves its minimum at $d^0 = (d_1^0, d_2^0, \cdots, d_n^0) \in D$. Further suppose $f(d^0) > 0$. Let $D_0 = \text{diagonal } (d_1^0, d_2^0, \cdots, d_n^0)$. Let $D_0 A D_0^{-1} = B$. If $P$ is a permutation matrix then $(PD_0P^T)PAP^T(PD_0^{-1}P^T) = PBP^T$. Hence we may assume that

$$\sum_{k \in N} b_{1k} \geqq \sum_{k \in N} b_{2k} \geqq \cdots \geqq \sum_{k \in N} b_{nk} .$$

Let

$$M_1 = \left\{i \,\middle|\, \sum_{k \in N} b_{ik} = \sum_{k \in N} b_{1k}\right\} \qquad M_2 = \left\{i \,\middle|\, \sum_{k \in N} b_{ik} = \sum_{k \in N} b_{nk}\right\} .$$

Let

$$d_k = \begin{cases} 1 - \varepsilon & k \in M_1 \\ (1 - \varepsilon)^{-1} & k \in M_2 \\ 1 & \text{otherwise} . \end{cases}$$

Consider $DBD^{-1}$ and let $g(\varepsilon)$
$= \sum_{k \in N} d_i b_{ik} d_k^{-1} - \sum_{k \in N} d_j b_{jk} d_k^{-1}$ $\quad i \in M_1,\ j \in M_2$.

Then

$$g'(0) = -\sum_{\substack{k \notin M_1 \\ k \notin M_2}} b_{ik} - 2 \sum_{k \in M_2} b_{ik} - 2 \sum_{k \in M_1} b_{jk} - \sum_{\substack{k \notin M_1 \\ k \notin M_2}} b_{jk} < 0 .$$

Hence for sufficiently small $\varepsilon$,

$$f_A[d_1d_1^0, d_2d_2^0, \cdots, d_nd_n^0] < f(d^0) .$$

However, this contradicts $f$ having its minimum at $d^0$. Therefore, if $A > 0$, $\min_{d_k>0\ k\in N} f(d) = 0$.

Now suppose $A$ is irreducible. For each positive integer $k$, let $A_k = A + (1/k)J$ where $J$ is the $n \times n$ matrix of ones so that $\lim_{m\to\infty} A_m = A$. For each $A_m$ there is a diagonal matrix $D_m = \text{diag.}\ (d_1^m, d_2^m, \cdots, d_n^m)$, $(d_1^m, d_2^m, \cdots, d_n^m) \in D$, so that $D_mA_mD_m^{-1}$ has equal row sums. Further

$$1 \leqq d_k^m \leqq [S(A_m)]^{n-1} \text{ for each } k \in N .$$

The $S(A_m)$'s are easily seen to be bounded, and hence the $D_m$'s are bounded having a limit point $D$. Let $\{D_{m'}\}$ denote a subsequence of $\{D_m\}$ so that $\lim_{m\to\infty} D_{m'} = D$. Then $\lim_{m\to\infty} D_{m'}A_{m'}D_{m'}^{-1} = DAD^{-1}$ which has all its row sums equal. Hence $\min_{d_k>0\ k\in N} f(d) = 0$.

COROLLARY. *If $A$ is an irreducible matrix then there is a diagonal matrix $D$ with positive main diagonal so that $DAD^{-1} = rS$ where $S$ is a row stochastic matrix and $r$ a positive number.*

We also include the following corollary to Lemma 2.

COROLLARY. *If $A$ is irreducible with Perron eigenvector $x = (x_1, x_2, \cdots, x_n)$ then* $\max_{i\ j} x_i/x_j \leqq [S(A)]^{n-1} = ((2r(A) - p(A)/u(A))^{n-1}$.

We include this bound as the bound involves the quantity $u(A)$ which to our knowledge is new.

## REFERENCE

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Received December 4, 1970 and in revised form May 6, 1971.

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