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CYCLES IN k-STRONG TOURNAMENTS

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A tournament T_n with n nodes is k-strong if k is the largest integer such that for every partition of the nodes of T_n into two nonempty subsets A and B there are at least k arcs that go from nodes of A to nodes of B and conversely. The main result is that every k-strong tournament has at least k different spanning cycles.

1. Introduction. A *tournament* T_n consists of a finite set of *nodes* $1, 2, \cdots, n$ such that each pair of distinct nodes i and j is joined by exactly one of the arcs $\overrightarrow{ij}$ or $\overrightarrow{ji}$. If the arc $\overrightarrow{ij}$ is in T_n we say that i *beats* j or j *loses to* i and write $i \rightarrow j$. If each node of a subtournament A beats each node of a subtournament B we write $A \rightarrow B$ and let $A + B$ denote the tournament determined by the nodes of A and B. A tournament T_n is *k-strong* if k is the largest integer such that for every partition of the nodes of T_n into two nonempty subsets A and B there are at least k arcs that go from nodes of A to nodes of B and conversely; a tournament T_n is *strong* if $n = 1$ or if it is k-strong for some positive integer k. If a tournament T_n is not strong, or *weak*, it has a unique expression of the type $T_n = A + B + \cdots + J$ where the nonempty *components* $A, B, \cdots, J$ all are strong; we call A and J the *top* and *bottom* components of T_n. (The top or bottom component of a strong tournament is the tournament itself.)

An *l-path is a sequence* $\mathscr{P} = \{p_1, p_2, \cdots, p_{l+1}\}$ of nodes such that $p_i \rightarrow p_{i+1}$ for $1 \leqq i \leqq l$; we assume the nodes of $\mathscr{P}$ are distinct except that p_{l+1} and p_1 may be the same in which case we call the sequence an *l-cycle*; it is sometimes convenient to regard a single node as a 0-path or a 1-cycle. A *spanning* path or cycle of T_n is one that involves every node of T_n.

Camion [1] proved that every strong tournament contains a spanning cycle. Our main object is to prove the following result.

THEOREM 1. *Any k-strong tournament contains at least k spanning cycles.*

More generally, we shall prove the following result.

THEOREM 2. *Let p denote any node of any k-strong tournament T_n; if $3 \leqq l \leqq n$, then p is contained in at least k l-cycles.*

In what follows we assume that the node p and the k-strong tournament T_n are fixed. The case $k = 1$ is treated, in effect, in [2; p. 6] so we may suppose that $k \geqq 2$; since each node of T_n must beat and lose to at least k other nodes, it follows that $2k + 1 \leqq n$ or $k \leqq 1/2(n - 1)$. Before proving the theorem we make some observations about paths and the structure of the k-strong tournament T_n.

2. Three lemmas.

The following result is obvious.

LEMMA 1. *Let $\mathscr{P}$ denote an l-path from node u to node v. If node w is not contained in $\mathscr{P}$ and $u \rightarrow w$ and $w \rightarrow v$, then w can be inserted in the path to form an $(l + 1)$-path from u to v; in particular w can be inserted immediately before the first node of $\mathscr{P}$ it beats.*

LEMMA 2. *If u and v are any nodes of the top and bottom components of a weak tournament W_t and $1 \leqq l \leqq t - 1$, then there exists an l-path in W_t that starts with u and ends with v; furthermore, if $2 \leqq l \leqq t - 1$ this path may be assumed to contain any given node belonging to any intermediate component of W_t.*

This may be proved by applying the following observations to the components of W_t: If a tournament Z_k is strong and $0 \leqq l \leqq k - 1$, then it contains a spanning cycle and, hence, each node is the first node, and the last node, of at least one l-path in Z_k; and, if $R \rightarrow S$, then any c-path of R may be followed by any d-path of S to form a $(c + d + 1)$-path of $R + S$.

LEMMA 3. *Let G denote any minimal subtournament of the k-strong tournament T_n whose removal leaves a weak subtournament W of the form $W = Q + R + S$ where Q and S are strong and R may be empty; then each node of G loses to at least one node of S and beats at least one node of Q, and there are at least k arcs from nodes of G to nodes of Q and at least k arcs from nodes of S to nodes of G.*

The conclusion in this lemma follows from the fact that G is minimal and T_n is k-strong. The existence of such a subtournament G will be shown before each application of this lemma.

We now proceed to the proof of Theorem 2; we have to use different arguments when l lies in different intervals.

3. Proof when $l = 3$.

Let B and L denote the set of nodes that beat and lose to p, respectively. Since T_n is k-strong B and L are nonempty and there are at least k arcs $\overrightarrow{uv}$ that go from a node

u of L to a node v of B. The theorem now follows when $l = 3$ since each such $\overrightarrow{uv}$ determines a different 3-cycle $\{p, u, v, p\}$ containing p.

4. **Proof when $l = 4$.** If w is any node that beats p, let B, L, M, and N denote the set of nodes that beat both w and p, lose to both w and p, beat w and lose to p, and lose to w and beat p, respectively. If $L = \phi$, then M must contain at least k nodes and N must contain at least $k - 1$ nodes, since p and w must each beat at least k nodes. In this case there are at least $k(k - 1) \geqq k$ different 4-cycles of the type $\{p, u, w, v, p\}$ containing p, where $u \in M$ and $v \in N$. We may suppose, therefore, that $L \neq \phi$.

There are at least k arcs of the type $\overrightarrow{uv}$ where $u \in L$ and $v \in B \cup M \cup N$. If $v \in B \cup M$, then the 4-cycle $\{p, u, v, w, p\}$ contains p. If $v \in N$ and v beats some other node y of N, then the 4-cycle $\{p, u, v, y, p\}$ contains p; if there is no such node y but u loses to some other node z of L, then the 4-cycle $\{p, z, u, v, p\}$ contains p. Thus, there are at least k different 4-cycles containing p except, possibly, when there exists an arc $\overrightarrow{uv}$ from L to N such that u beats the remaining nodes of L and v loses to the remaining nodes of N; there is at most one such arc $\overrightarrow{uv}$ so in this case the preceding construction provides at least $k - 1$ 4-cycles containing p.

If $z \in M$, then $\{p, z, w, v, p\}$ is a new 4-cycle containing p. Thus we may suppose that $M = \phi$; this implies L has at least k nodes since p beats at least k nodes. If there exists an arc $\overrightarrow{zy}$ where $z \neq u$, $z \in L$, and $y \in B$ then $\{p, u, z, y, p\}$ is a new 4-cycle containing p. Thus we may suppose that u is the only node of L that beats any nodes of B. This implies, since T_n is k-strong, that there must be at least k arcs of the type $\overrightarrow{zy}$ where $z \neq u$, $z \in L$, and $y \in N$. In this case, however, there are at least k 4-cycles of the type $\{p, u, z, y, p\}$ containing p. This completes the proof of the theorem when $l = 4$.

5. **Proof when $5 \leqq l \leqq n - k + l$.** Let $\mathscr{C}$ denote any $(l - 2)$-cycle containing p; such a cycle exists, either by virtue of an induction hypothesis or as a consequence of the result cited in the introduction. Let B and L denote the set of nodes that beat and lose to every node of $\mathscr{C}$, respectively, and let M denote the set of the remaining nodes of T_n that aren't in $\mathscr{C}$.

If $L \neq \phi$, there exist at least k arcs of the type $\overrightarrow{uv}$ where $u \in L$ and $v \in B \cup M$. For each such node v there exists at least one node q of $\mathscr{C}$ such that $v \rightarrow q$. If we insert the nodes u and v immediately before q in $\mathscr{C}$ we obtain an l-cycle containing p; different arcs $\overrightarrow{uv}$

clearly yield different l-cycles. A similar argument may be applied to B if $B \neq \phi$ so we may now assume that $L = B = \phi$ and $M \neq \phi$.

If $u \in M$, then there exists a pair of consecutive nodes r and s of $\mathscr{C}$ such that $r \rightarrow u$ and $u \rightarrow s$. Thus u can be inserted between r and s in $\mathscr{C}$ to form an $(l-1)$-cycle $\mathscr{C}_1$ containing p. Any other node v of M can now be inserted between some pair of consecutive nodes of $\mathscr{C}_1$ to form an l-cycle $\mathscr{C}_2$ containing p. Different cycles $\mathscr{C}_2$ are formed when different pairs of nodes of M are inserted in $\mathscr{C}$. Thus, there are at least $\binom{n-(l-2)}{2} \geqq \binom{k+1}{2} \geqq k$ different l-cycles containing p when $5 \leqq l \leqq n-k+1$. (This argument can be applied for somewhat larger values of l as well.)

6. **Proof when** $n - k + 2 \leqq l \leqq n - 1$. Let T_l denote any subtournament of T_n with l nodes that contains the node p. If T_l is strong, then it contains an l-cycle containing p, by Camion's theorem. Thus, if each such subtournament T_l is strong, then p is contained in at least $\binom{n-1}{l-1} \geqq n-1 > k$ l-cycles in T_n.

We may suppose, therefore, that there exists a minimal subtournament G of T_n, with $g \leqq n - l$ nodes, whose removal leaves a weak subtournament W containing node p. Then W can be expressed in the form $W = Q + R + S$ where Q and S are strong and R may be empty.

There are at least k arcs $\overrightarrow{xq}$ in T_n that go from a node x of G to a node q of Q, and for each such node x there exists at least one node s of S such that $s \rightarrow x$; this follows from Lemma 3. We shall show that for each such pair of nodes q and s, there exists an $(l-2)$-path $\mathscr{P}$ in W that starts with q, contains the node p, and ends with s.

If $p \in R$, then the existence of $\mathscr{P}$ follows immediately from Lemma 2 since W has $n-g$ nodes and $2 \leqq l-2 \leqq n-g-1$. If $p \in Q$, let $\mathscr{P}_1$ denote any spanning path of Q that starts with q. We observe that if Q has m nodes then $m \leqq l-3$ since otherwise node s would lose to at least $l-2 \geqq (n-k+2)-2 = n-k$ nodes and this is impossible since T_n is k-strong. Let $\mathscr{P}_2$ denote any $(l-m-2)$-path of $R+S$ that ends with s; the existence of $\mathscr{P}_2$ follows from Lemma 2 since $R+S$ has $n-g-m$ nodes and $1 \leqq l-m-2 \leqq n-g-m-1$. If $\mathscr{P} = \mathscr{P}_1 + \mathscr{P}_2$ then $\mathscr{P}$ is an $(l-2)$-path in W with the required properties and we can also find such a path when $p \in S$ by a similar argument.

This suffices to complete the proof when $n-k+2 \leqq l \leqq n-1$ since $\{x\} + \mathscr{P} + \{x\}$ is an l-cycle containing p and it is clear that different arcs $\overrightarrow{qx}$ yield different l-cycles.

7. **Proof when $l = n$; a special case.** Since T_n is k-strong, there exists a partition of the nodes of T_n into two subsets A and B such that precisely k arcs go from nodes of A to nodes of B. At least one of these subsets has more than k nodes; if the nodes in this subset that are incident with the k arcs that go from A to B are removed, then the subtournament determined by the remaining nodes is weak. It follows, therefore, that there exists a smallest subtournament G, with at most k nodes, whose removal leaves a weak subtournament W of the form $W = Q + R + S$ where Q and S are strong and R may be empty. We may now apply Lemma 3 to T_n. There are at least k arcs that go from a node of G to a node of Q and we shall prove the case $l = n$ of the theorem, in general, by constructing a different n-cycle of T_n for each such arc; the node p plays no special role in this case since it automatically belongs to every n-cycle. First, however, we dispose of a special case.

Suppose R is empty and $Q = \{q\}$ so that $W = \{q\} + S$. Then G must have precisely k nodes all of which beat q for otherwise there wouldn't be k arcs going from G to Q. Consequently, S has $n - 1 - k \geqq k$ nodes. There must be at least k nodes S that don't lose to all nodes of G for otherwise these nodes would determine a subtournament smaller than G whose removal from T_n would leave a weak subtournament.

Let s denote any node of S that beats some node x of G. It follows from Lemma 2, that there exists a spanning path $\mathscr{P}_1$ of W that starts with q and ends with s and a path $\mathscr{P}_2$ in G that starts with x and contains all nodes of G except those belonging to components of G that are above the component X that contains x. Hence, the cycle $\mathscr{C} = \mathscr{P}_2 + \mathscr{P}_1 + \{x\}$ contains all nodes of T_n except those nodes, if any, belonging to components of G above X. These nodes, however, can all be inserted in $\mathscr{C}$ by Lemma 1, since they all beat x and lose to at least one node of S. The node s in the resulting n-cycle is the last node of S that occurs before the node q. Thus, in this way we can construct a different n-cycle for each of the at least k nodes of S that beat some nodes of G. Similarly, the theorem holds when $W = Q + S$ and S consists of a single node.

8. **Proof when $l = n$; the general case.** Let $\overrightarrow{xq}$ denote any arc that goes from a node x of G to a node q of Q in the tournament T_n. Next, let $\overrightarrow{sy}$ denote any arc that goes from a node s of S to a node y of the top component of G; if the component X of G containing x is the top component of G let y be the immediate successor of x in some fixed spanning cycle of X unless $X = \{x\}$ in which case let $y = x$. Finally, let $\mathscr{P}(q, s)$ denote some spanning path of W that starts with q and ends with s and let $\mathscr{P}(y, x)$ denote a path from

y to x in G that contains all the nodes in components of G that are not below x; it is not difficult to see that these paths exist and that we may suppose q loses to the last node of Q other than itself that occurs in $\mathscr{P}(q, s)$.

Insert as many as possible of the nodes in the components of G below X between q and s in the path $\mathscr{P}(q, s)$ to form an augmented path $\mathscr{P}'(q, s)$ and let $\mathscr{P}(f, g)$ denote any spanning path, starting and ending with some nodes f and g, of the subtournament F determined by those nodes that can't be so inserted; it may be that $\mathscr{P}(f, g)$ is empty or consists of a single node. If t is any node of f, then (i) $t \to q$, (ii) $s \to t$, and (iii) $t \to u$, where u is the immediate successor of q in $\mathscr{P}'(q, s)$. The node t beats at least one node of Q and loses to at least one node of S; hence, by Lemma 1, it could be inserted in $\mathscr{P}'(q, s)$ unless (i) and (ii) hold. Since t doesn't beat itself or node x, and since there are at most $k - 2$ other nodes of G, it must be that t beats at least one other node of W besides q if it is to beat at least k nodes altogether; this implies (iii) in view of Lemma 1.

If at least one node of the component of G immediately below X is in $\mathscr{P}'(q, s)$ or if X is the bottom component of G let

$$\mathscr{C} = \mathscr{C}(x, q) = \{x\} + \mathscr{P}(f, g) + \mathscr{P}'(q, s) + \mathscr{P}(y, x).$$

This is an n-cycle in view of the preceding remarks; we shall call it a type I cycle. The nodes s and q can be identified as the last node of S and the first node of Q encountered in traversing the cycle from any node of S to any node of Q. The node x can be identified as the last node between s and q in $\mathscr{C}$ that belongs to a component X of G with the property that no node of X or any component of G above X is between q and s in $\mathscr{C}$. Thus different arcs $\overrightarrow{xq}$ determine different type I cycles, if they determine any at all.

Let us now suppose that X is not the bottom component of G and that no node of the component immediately below X belongs to $\mathscr{P}'(q, s)$. In this case we are unable to identify the node x used in defining the cycle $\mathscr{C}(x, q)$ so we must use a different construction.

Let $\mathscr{P}(u, v)$ denote the nonempty path such that $\mathscr{P}'(q, s) = \{q\} + \mathscr{P}(u, v) + \{s\}$. Node x does not lose to itself or to the node f (which definitely exists in the present case), so it must lose to at least two nodes of $\mathscr{P}'(q, s)$ if it is to lose to at least k nodes altogether; but $x \to q$, so x must lose to at least one node of $\mathscr{P}(u, v)$. If t is any other node of $\mathscr{P}(u, x)$ then t does not lose to itself, its immediate successor in $\mathscr{P}(y, x)$, or to f; hence, t must lose to at least three nodes of $\mathscr{P}'(q, s)$ if it is to lose to k nodes altogether. It follows that every node of $\mathscr{P}(y, x)$ loses to at least one node of $\mathscr{P}(u, v)$.

If every node of $\mathscr{P}(y, x)$ beats v then these nodes can all be inserted in the path $\mathscr{P}(u, v)$ to form an augmented path $\mathscr{P}'(u, v)$ by Lemma 1; this can be done in such a way that the nodes of $\mathscr{P}(y, x)$ occur in the same order in $\mathscr{P}'(u, v)$ as they do in $\mathscr{P}(y, x)$. In this case let

$$\mathscr{C} = \mathscr{C}(x, q) = \mathscr{P}(f, g) + \{q\} + \mathscr{P}'(u, v) + \{s, f\}.$$

That this is an n-cycle follows from properties (i) and (ii) of the nodes F, among other things; we shall call this a type II cycle. The nodes s and q can be identified in the same way as before. The node x can be identified as the last node between q and s that comes from G and beats f, the immediate successor of s in $\mathscr{C}$ (we use the assumption about the nodes in the component of G containing f here). Thus, different arcs $\overrightarrow{xq}$ determine different type II cycles, if they determine any at all. We can distinguish between cycles of types I and II because the node following s belongs to the top component of G in a type I cycle but not in a type II cycle.

If not all nodes of $\mathscr{P}(y, x)$ beat v, let w denote the first node of this path that loses to v. The nodes, if any, of $\mathscr{P}(y, x)$ that precede W can be inserted, as before, in $\mathscr{P}(u, v)$ to form an augmented path $\mathscr{P}'(u, v)$. If $\mathscr{P}(w, x)$ denotes the subpath determined by the remaining nodes of $\mathscr{P}(y, x)$, let

$$\mathscr{C} = \mathscr{C}(x, q) = \{x, q, s, f\} + \mathscr{P}(f, g) + \mathscr{P}'(u, v) + \mathscr{P}(w, x) .$$

That this is an n-cycle follows from properties (i) and (iii) of the nodes of F; we shall call this a type III cycle. There are at most two nodes of Q that are immediately followed by a node of S in $\mathscr{C}$. If there is only one such node then this node must be q, and if there are two then q is the node that loses to the other one. Thus we can identify the node q in $\mathscr{C}$ and x is the immediate predecessor of q. Hence, different arcs $\overleftarrow{xq}$ determine different type III cycles, if they determine any at all.

It remains to show that we can distinguish a type III cycle from a type I or II cycle. Some node of Q is followed immediately by a node of S in a type III cycle but not in a type I or II cycle when R, the subtournament determined by the intermediate components of W, is nonempty. Thus we may suppose that $W = Q + S$ where the strong components Q and S have at least three nodes each, in view of the case treated in § 7. In this case, however, the first node of Q that occurs after a node of S is the same for all nodes of S in a type I or II cycle but not in a type III cycle.

Thus, in the general case, we can construct a different n-cycle $\mathscr{C}(x, q)$ corresponding to each arc $\overrightarrow{xq}$ from a node of G to a node of Q. As there are at least k such arcs, this completes the proof of the theorem.

REFERENCES

1. P. Camion, *Chemins et circuits hamiltoniens des graphes complets*, C. R. Acad. Sci. Paris, **249** (1959), 2151-2152.
2. J. W. Moon, *Topics on Tournaments*, New York: Holt, Rinehert, and Winston, 1968.

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