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THE DIOPHANTINE EQUATION $u(u+1)(u+2)(u+3) = v(v+1)(v+2)$

DAVID W. BOYD AND H. H. KISILEVSKY

In this paper we demonstrate that the equation of the title has exactly three solutions in positive integers, namely: $1 \cdot 2 \cdot 3 \cdot 4 = 2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 4 \cdot 5 = 4 \cdot 5 \cdot 6$ **and** $19 \cdot 20 \cdot 21 \cdot 22 = 55 \cdot 56 \cdot 57$. **The method of proof is to reduce the equation to the form** $y^2 = x^3 - x + 1$.

According to the studies of Mordell, this equation has only a finite number of solutions and each corresponds to a representation of unity by a binary quartic form. We thus can reduce the problem to a study of the solutions of two binary quartic forms. These forms can be regarded as the norms of certain units in an order $\mathcal{O}$ of the field $Q(\alpha)$ where $\alpha^4 - 8\alpha + 4 = 0$. We next determine the two fundamental units of this order. Representing the units we seek in terms of these fundamental units, we obtain exponential equations which are then solved by the p-adic method of Th. Skolem.

This technique was used by Ljunggren [5] in treating the equations $y^2 = x^3 - 7$, and $y^2 = x^3 - 15$.

1. Define $y = u^2 + 3u + 1$ and $x = v + 1$, and then the equation of the title becomes

$$y^2 = x^2 - x + 1 \,. \tag{1}$$

Mordell [6] shows that this has only a finite number of integer solutions. A recent result of Baker [2] shows that for all integer solutions of (1), one has

$$\max(|x|, |y|) < \exp(10^{6 \cdot 10^6}) \,.$$

We discuss in Section 5 a feasible computational approach to solving (1) which uses the results of §§ 2 and 3 and the method of Baker and Davenport [3].

2. We now reduce our problem to the study of two binary quartic forms. This reduction follows the paper [7] of Mordell in which he treats the problem $u(u + 1)(u + 2) = v(v + 1)$. Consider (1) as an equation in the field $Q(\theta)$, where $\theta^3 - \theta + 1 = 0$. The discriminant of $x^3 - x + 1$ is -23, and hence

(a) $1, \theta, \theta^2$ is an integral basis (-23 is square free),

(b) there is a single fundamental unit, which we may take to be

$-\theta$, according to Delone and Fadeev [4], p. 303,

(c) the class number of $\boldsymbol{Q}(\theta)$ is 1([4], p. 141).

We may now write (1) in the form

$$y^2 = (x-\theta)(x^2+\theta x+\theta^2-1)\ . \tag{2}$$

If π is a prime in $\boldsymbol{Q}(\theta)$ which divides both factors of (2), then $x \equiv \theta \pmod{\pi}$ hence $3\theta^2 - 1 \equiv 0 \pmod{\pi}$. However $N(3\theta^2-1) = +23$ so $3\theta^2 - 1$ is a prime, hence $3\theta^2-1$ is the only possible prime factor of $x-\theta$ and $x^2+\theta x+\theta^2-1$. Thus, (2) together with (a), (b), (c) above implies that there are integers k, l, a, b, c such that

$$x-\theta = \pm(-\theta)^k(3\theta^2-1)^l(a+b\theta+c\theta^2)^2\ . \tag{3}$$

We may assume k and l are either 0 or 1 by absorbing extra factors of $-\theta$ and $(3\theta^2-1)$ into the squared term. Taking norms in (3) shows that, for some integer r,

$$y^2 = N(x-\theta) = \pm(23)^l r^2\ .$$

This implies that $l = 0$ and that the $+$ sign should be used. Thus we have two cases

$$k=0:\quad x-\theta = (a+b\theta+c\theta^2)^2 \tag{4}$$

$$k=1:\quad x-\theta = -\theta(a+b\theta+c\theta^2)^2\ . \tag{5}$$

In case $k=0$, equation (4) becomes

$$x-\theta = (a^2-2bc)+(2ab+2bc-c^2)\theta+(b^2+2ac+c^2)\theta^2$$

which gives the following three equations:

$$x = a^2-2bc \tag{6}$$

$$0 = b^2+2ac+c^2 \tag{7}$$

$$1 = c^2-2ab-2bc\ . \tag{8}$$

If $a+c\neq 0$, then (8) implies that $b=(c^2-1)/2(a+c)$, and substitution in (7) yields

$$0 = (c^2-1)^2+4c(2a+c)(a+c)^2\ . \tag{9}$$

Equation (9) implies that $c\,|\,(c^2-1)^2$ so $c^2=1$, but then $2a+c=0$ and this means $a=\pm 1/2$ which is inadmissable.

Hence, $a+c=0$, and so $c^2=1$ from (8). Now using (7), we have $b^2=c^2$, and hence $x=a^2-2bc=1\pm 2$, so

$$x = 3 \text{ or } -1$$

corresponding to

$$y = 5 \text{ or } 1\ .$$

The solution $x = -1$ is trivial in that it corresponds to $v(v+1)(v+2) = (-2)(-1)(0)$, while $x = 3$ leads to $(u, v) = (1, 2)$.

The case $k = 1$ leads to a more difficult situation. Here, we obtain from (5) the equations:

$$x = b^2 + 2ac + c^2 \tag{10}$$

$$0 = c^2 - 2ab - 2bc \tag{11}$$

$$1 = a^2 - 2bc + b^2 + 2ac + c^2 . \tag{12}$$

Let us define $a + c = d$ so that (10), (11) and (12) become

$$x = b^2 + 2cd - c^2 \tag{13}$$

$$0 = c^2 - 2bd \tag{14}$$

$$1 = d^2 - 2bc + b^2 . \tag{15}$$

Our first observation is that if (b, c, d) is a solution of (14) and (15) then so is $(-b, -c, -d)$ and these give the same value for x. Hence we may assume $b \geqq 0$, say. Then (14) implies $d \geqq 0$, and then (15) implies $c \geqq 0$ since $d^2 + b^2 \geqq 1$ for any solution of (15).

Now, (15) shows that g.c.d$(d, b) = 1$ and from (14), 2bd is a square so that one of d, b is a square and the other twice a square. Thus, there are relatively prime p and q with $(b, d) = (q^2, 2p^2)$ or else $(2q^2, p^2)$ giving again two cases to consider:

$$\begin{aligned} &(b, c, d) = (q^2, 2pq, 2p^2) \\ &\qquad 4p^4 - 4pq^3 + q^4 = 1,\ x = q^4 + 8p^3q - 4p^2q^2 \end{aligned} \tag{16}$$

$$\begin{aligned} &(b, c, d) = (2q^2, 2pq, p^2) \\ &\qquad p^4 - 8pq^3 + 4q^4 = 1,\ x = 4q^4 + 4p^3q - 4p^2q^2 . \end{aligned} \tag{17}$$

By inspection, (16) has the solutions $(p, q) = (0, 1), (1, 1)$ giving

$$x = 1 \text{ or } 5 .$$

And, (17) has the solutions $(p, q) = (1, 0), (1, 2)$ giving

$$x = 0 \text{ or } 56 .$$

For our problem $x = 0$ and 1 are trivial cases, while $x = 5, 56$ give the two solutions $(u, v) = (2, 4), (19, 55)$ which were mentioned in the introduction.

The remainder of the paper is devoted to proving that we have found all solutions of (16) and (17).

To investigate (16), (17) completely we notice their connection with the field $\boldsymbol{Q}(\alpha)$ where $\alpha^4 - 8\alpha + 4 = 0$. Observe that in this field $\alpha^2/2 = \eta$ is an integer since $\eta^2 = 2\alpha - 1$, and also $2/\alpha = 4 - \alpha^3/2$ is an integer. Multiplying equation (16) by 4, and letting N denote the norm in $\boldsymbol{Q}(\alpha)$,

$$N(q\alpha - 2p) = 16p^4 - 16pq^3 + 4q^4 = 4 \tag{18}$$

while equation (17) may be interpreted as

$$N(q\alpha - p) = p^4 - 8pq^3 + 4q^4 = 1\,. \tag{19}$$

We can simplify (18) by noticing that $(q\alpha - 2p)/\alpha = (q - 4p) + p(\alpha^3/2)$ is an integer, and since $N(\alpha) = 4$, we have from (18) that

$$N((q - 4p) + p\alpha^3/2) = 1\,. \tag{20}$$

Thus our problem reduces to finding all units in $\boldsymbol{Q}(\alpha)$ which are of one of the forms $r + s\alpha$ and $r + s(\alpha\eta)$. The units lie in the order $\mathcal{O} = \boldsymbol{Z}(1, \alpha, \eta, \alpha\eta)$, so our first problem is to determine the fundamental units for $\mathcal{O}$. There are two such units since α has two real and two complex conjugates. We mention in passing that $\mathcal{O}$ is not the maximal order in $\boldsymbol{Q}(\alpha)$. This can be shown to be $\boldsymbol{Z}(1, \beta, \beta^2, \beta^3)$, where $\beta = -1 + \alpha/2 + \alpha^3/4$. (Note that $\beta^2 = \alpha$). The field disciminant is thus $-64\cdot 23$, and one can verify that the class number of $\boldsymbol{Q}(\alpha)$ is 1.

3. We shall show in this section that $\eta = \alpha^2/2$ and $\zeta = \alpha - \eta$ are fundamental units in the order $\mathcal{O} = \boldsymbol{Z}(1, \alpha, \eta, \alpha\eta)$. For this purpose we need the following numerical data for the conjugates of α, η and ζ:

$$\begin{aligned} \alpha_1 &= .508347 \\ \alpha_2 &= 1.793580 \\ \alpha_{3\,4} &= -1.150964 \pm i(1.749969) \\ |\alpha_{3,4}| &= 2.094543\,. \end{aligned}$$

Thus

$$\begin{aligned} \eta_1 &= .129209 & \zeta_1 &= .379139 \\ \eta_2 &= 1.608465 & \zeta_2 &= .185115 \\ |\eta_{3,4}| &= 2.193555 & |\zeta_{3\,4}| &= 3.774679\,. \end{aligned}$$

It is clear from this data that ζ and η are independent units since if $\eta^m = \zeta^n$ for nonzero m and n, we would have $\eta_1^m = \zeta_1^n$ and $\eta_2^m = \zeta_2^n$ which are inconsistent since $0 < \eta_1, \zeta_1, \zeta_2 < 1$ while $\eta_2 > 1$. To proceed further, we use the following lemma:

LEMMA. *Let $\boldsymbol{Q}(\alpha)$ be an algebraic number field, where α has s real and $2t$ complex conjugates. Let $\omega_1, \cdots, \omega_n$ be linearly independent integers in $\boldsymbol{Q}(\alpha)$. Let $\omega_j^{(i)}$ denote the ith conjugate of ω_j, and let $A = (\omega_j^{(i)}: 1 \leqq i, j \leqq n)$. Write $A^{-1} = (\nu_i^{(j)}: 1 \leqq i, j \leqq n)$. Sup-*

pose that $r = s + t - 1$ *and that* $\eta_1, \cdots, \eta_r$ *are independent units in the order* $\mathcal{O} = \boldsymbol{Z}(\omega_1, \cdots, \omega_n)$. *Write*

$$e_{ik} = \sum_j (|v_i^{(j)}|\,|\eta_k^{(j)}|^{r/2} : j = 1, \cdots, n) \text{ for } k = 1, 2, \cdots, r;\ i = 1, \cdots, n$$

and

$$e_{i,r+1} = \sum(|\nu_i^{(j)}| : j = 1, \cdots, n), \text{ for } 1, \cdots, n\,.$$

If $\eta_1 \cdots \eta_r$ *are not fundamental units in* $\mathcal{O}$, *then there is a unit* μ *in* $\mathcal{O}$ *different from* $\eta_1, \cdots, \eta_r$, *with* $\mu = \Sigma_i a_i \omega_i$, $a_i \in \boldsymbol{Z}$, *and*

$$|a_i| \leqq \max(e_{i1}, \cdots, e_{i,r+1}) \textit{ for } i = 1, \cdots, n\,. \tag{21}$$

Proof. If $\eta_1, \cdots, \eta_r$ are not fundamental then there is a unit μ and rational numbers $c_1, \cdots, c_r$ with $0 \leqq c_i$ for all i and $\sum_{i=1}^r c_i \leqq r/2$, such that

$$\mu = \eta_1^{c_1} \cdots \eta_r^{c_r}\,. \tag{22}$$

Define $b_i = 2c_i/r$ so that $b_1 + \cdots + b_r \leqq 1$.
Let $b_{r+n} = 1 - b_1 - \cdots - b_r$ and let $\eta_{r+1} = 1$.

From (22), if $\mu = a_1\omega_1 + \cdots + a_n\omega_n$, we obtain

$$\begin{aligned} |a_i| &= |\textstyle\sum_j \nu_i^{(j)} \mu^{(j)}| \\ &\leqq \textstyle\sum_j |\nu_i^{(j)} (\eta_1^{(j)})^{c_1} \cdots (\eta_{r+1}^{(j)})^{c_{r+1}}| \\ &\leqq e_{i1}^{b_1} \cdots e_{i,r+1}^{b_{r+1}} \end{aligned}$$

using Hölder's inequality.

To obtain (21), we estimate the products by

$$|a_i| \leqq e_{i1}^{b_1} \cdots e_{i,r+1}^{b_{r+1}} \leqq \max(e_{i1}, \cdots, e_{i,r+1})\,.$$

The matrix A in our case has ith row $(1, \alpha_i, \eta_i, \alpha_i\eta_i)$. To obtain A^{-1}, we regard $(1, \alpha_i, \alpha_i^2, \alpha_i^3)$ as a left eigenvector of the companion matrix of $x^4 - 8x + 4$ for the eigenvalue α_i. A right eigenvector for α_i is easily seen to be $(-4, \alpha_j^3, \alpha_j^2, \alpha_j)^T$. Hence A^{-1} has jth column $8^{-1}(3\alpha_j - 2)^{-1}(-4, \alpha_j^3, 2\alpha_j^2, 2\alpha_i)^T$. Write W as the 4×3 matrix whose first and second columns are the absolute values of the corresponding columns of A^{-1} and whose third column is the sum of the absolute values of the third and fourth columns of A^{-1}. Then we have approximately:

$$W = \begin{pmatrix} 1.0527 & .1479 & .1321 \\ .0346 & .2133 & .3035 \\ .1360 & .2379 & .2898 \\ .2676 & .1326 & .1384 \end{pmatrix} = (w(i, j))\,.$$

If $e(i) = \Sigma(w(i, j)|\eta_j|: j = 1, 2, 3)$, $z(i) = \Sigma(w(i, j)|\zeta_j|: j = 1, 2, 3)$, and $w(i) = \Sigma(w(i, j): j = 1, 2, 3)$ we have

$$\begin{aligned} e &= (\ .6637, 1.0133, 1.0359, .5514) \\ z &= (\ .9252, 1.1982, 1.1895, .6483) \\ w &= (1.3327,\ \ .5514,\ \ .6637, .5386)\ . \end{aligned}$$

From equation (22) of our lemma, there is a unit $\mu = a + b\alpha + c\eta + d\alpha\eta$ with

$$|a| \leqq 1, |b| \leqq 1,\ |c| \leqq 1, d = 0\ .$$

It is now easy to check that these 27 possibilities give no new units. (The computation is lessened by noticing that we must have $0 < \mu_1 < 1$ and $0 < \mu_2$).

4. In this section we solve equations (18) and (20) by an application of 2-adic analysis, as in [8] and [5].

We have shown in 3 that $\eta = \alpha^2/2$ and $\zeta = \alpha - \eta$ are fundamental units in $\mathcal{O} = \boldsymbol{Z}(1, \alpha, \eta, \alpha\eta)$. According to equations (18) and (20), we wish to determine all units in $\mathcal{O}$ of either of the two forms $r + s\alpha$ or $r + s(\alpha\eta)$, $r, s \in \boldsymbol{Z}$. We treat first $r + s\alpha$. For any such unit there are integers m, n such that

$$\pm (r + s\alpha) = \eta^m\zeta^n\ . \tag{23}$$

We will examine (23) modulo 2^k for all k. First observe that

$$\eta^2 = -1 + 2\alpha$$

and

$$\zeta^2 = -1 + 2\alpha + \alpha^2 - \alpha^3 = -1 + 2\omega$$

where

$$\omega = \alpha + \eta - \alpha\eta \in \mathcal{O}\ .$$

Treating (23) modulo 2, if $m = 2u + b$, $n = 2v + c$, $b, c \in \{0, 1\}$, then

$$\pm (r + s\alpha) = (1 - 2\alpha)^u(1 - 2\omega)^v\eta^b\zeta^c \equiv \eta^b\zeta^c\ .$$

Clearly $(b, c) = (1, 0), (0, 1)$ are not possible, and we rule out $(1, 1)$ by noticing that

$$\eta\zeta = \eta\alpha - \eta^2 = -1 + 2\alpha - \eta\alpha \equiv 1 + \eta\alpha\ .$$

Thus $(b, c) = (0, 0)$ is the only possibility. Now, proceeding modulo 4 we write $u = 2w + e$, $v = 2z + f$ with $e, f \in \{0, 1\}$. Then

$$(1 - 2\alpha)^{2w+e}(1 - 2\omega)^{2z+f} \equiv (1 - 2\alpha)^e(1 - 2\omega)^f (\mathrm{mod}\ 4)\ .$$

Treating the four cases for (e, f) we conclude that only $(e, f) = (0, 0)$ and $(1, 0)$ are possible.

First consider $(e, f) = (0, 0)$.

For any $k \geqq 1$

$$(1 - 2\alpha)^{2^k} = 1 - 2^{k+1}\alpha - 2^{k+2}\eta + O(2^{k+3})$$
$$(1 - 2\omega)^{2^k} = 1 - 2^{k+1}(1 + \alpha + \eta + \alpha\eta) + O(2^{k+3}).$$

If $(u, v) \neq (0, 0)$, then $u = 2^k x$, $v = 2^k y$ with one of x and y odd, and $k \geqq 1$. Then

$$(1 - 2\alpha)^{2^k x}(1 - 2\omega)^{2^k y} = 1 - 2^{k+1}(x\alpha + y + y\alpha + x\eta + y\alpha\eta) - 2^{k+2}(x\eta) + O(2^{k+3}) \tag{24}$$

Equating the coefficient of η and $\alpha\eta$ in (24) to zero we have

$$0 = y + 2x + O(2^2) \tag{25}$$
$$0 = y + O(2^2) . \tag{26}$$

Subtracting gives

$$0 = x + O(2) . \tag{27}$$

Hence (26) and (27) imply x and y are even, a contradiction, and thus $(u, v) = (0, 0)$.

Now consider $(e, f) = (1, 0)$. Unless $(u, v) = (1, 0)$ we have $(u, v) = (2^k x + 1, 2^k y)$ with one of x and y odd. Then

$$(1 - 2\alpha)^u(1 - 2\omega)^v = 1 - 2\alpha - 2^{k+1}(x\alpha + y + y\alpha + y\eta + y\alpha\eta) - 2^{k+2}(x\eta - y\alpha - y\alpha\eta) + O(2^{k+3}) . \tag{28}$$

Equating coefficients of η and $\alpha\eta$ to zero, we have

$$0 = y + 2x + O(2^2) \tag{29}$$
$$0 = y + O(2^2) . \tag{30}$$

Equation (30) implies $y = O(2^2)$ and then (29) implies $x = O(2)$, a contradiction. Hence $(u, v) = (1, 0)$.

Now we turn from equation (19) to equation (20) and seek units of the form $r + s\alpha\eta$.

We first examine $\eta^m\zeta^n = \pm(r + s\alpha\eta)$ modulo 2, and find that $(m, n) \equiv (0, 0)$ or $(1, 1) \pmod 2$. Treating $(m, n) \equiv (0, 0)$ first, we may write $m = 2u$, $n = 2v$ and then, working modulo 4,

$$\begin{aligned}\eta^{2u}\zeta^{2v} &\equiv (1 - 2\alpha)^u(1 - 2\omega)^v \\ &\equiv (1 - 2u\alpha)(1 - 2r\omega) \equiv 1 - 2u\alpha - 2v(\alpha + \eta - 2\eta) \pmod 4 .\end{aligned}$$

Equating coefficients of η and α to zero shows first that v is even then that u is even. Thus, if $(u, v) \neq (0, 0)$ we may write $u = 2^k x$, $v = 2^k y$ where $k \geqq 1$ and are of x, y is odd. Then we wish to solve

$$\pm (r + s\alpha\eta) = (1 - 2\alpha)^{2^k x}(1 - 2\omega)^{2^k y} . \tag{31}$$

Using (24), we find that

$$0 = x + y + O(2^2) \tag{32}$$

$$0 = y + 2x + O(2^2) . \tag{33}$$

Equation (33) shows y is even and (32) shows x is even, a contradiction; hence $(u, v) = (0, 0)$.

For the final case we have $(m, n) \equiv (1, 1) (\bmod 2)$, hence we may write $m = 2u - 1$, $n = 2v + 1$. We observe that $\eta^{-1}\zeta = 2/\alpha - 1 = 3 - \alpha\eta$. We wish to solve

$$\pm (r + s\alpha\eta) = \eta^{2u}\zeta^{2v}\eta^{-1}\zeta$$

which becomes, modulo 4,

$$\begin{aligned} \pm (r + s\alpha\eta) &\equiv (1 - 2\alpha)^u (1 - 2\omega)^v (3 - \alpha\eta) \pmod 4 \\ &\equiv - 1 - \alpha\eta + 2u\alpha + 2v(\eta - \alpha\eta) \pmod 4 . \end{aligned}$$

This implies that u and v are even. Thus, if $(u, v) \neq (0, 0)$ then $u = 2^k x$, $v = 2^k y$ with $k \geqq 1$ and one of x, y odd. Then

$$\begin{aligned} \pm (r + s\alpha\eta) &= (1 - 2\alpha)^{2^k x}(1 - 2\omega)^{2^k y}(3 - \alpha\eta) \\ &= 3 - \alpha\eta - 2^{k+1}(x\alpha + y + y\eta) + O(2^{k+2}) \end{aligned}$$

which gives

$$0 = x + O(2) \tag{29}$$

$$0 = y + O(2) . \tag{30}$$

This implies x and y are even, a contradiction. Hence $(u, v) = (0, 0)$ and $(m, n) = (-1, 1)$.

This completes the verification that the only solutions of (16) and (17) are the ones obtained by inspection. Thus the only solutions of the equation of the title are those listed in the introduction.

5. Rather than using the 2-adic method of §4, one may reduce the equation $\eta^m\zeta^n = \pm (r + s\alpha)$ or $\pm (r + s\alpha\eta)$ to an inequality in linear forms of logarithms of algebraic numbers. If A^{-1} is as in §3, if η_i, ζ_i denote the conjugates of η and ζ, and if $\gamma = \eta^m\zeta^n = a + b\alpha + c\eta + d\alpha\eta$ is a unit, then

$$(a, b, c, d)^T = A^{-1}(\eta_1^m\zeta_1^n, \cdots, \eta_4^m\zeta_4^n)^T .$$

Equating c and d to zero gives two pairs of equations from which the term $\eta_4^m \zeta_4^n$ can be eliminated to give

$$\sum_{j=1}^{3} \alpha_j(\alpha_j - \alpha_4)(3\alpha_j - 2)^{-1}\eta_j^m \zeta_j^n = 0 .$$

Equating b and c to zero gives a similar equation. One can show, by consideration of the size of the various quantities η_j, ζ_j that m and n must be of opposite signs. Again considering the relative sizes of the η_j and ζ_j we obtain four inequalities:

$$|m \log(\eta_1\eta_2^2) - n \log(\zeta_1\zeta_2^2) - \log \beta^{(k)}| < (2\cdot 9)e^{-m(2\cdot 63)}$$

and

$$|m \log(\eta_1^2\eta_2) - n \log(\zeta_1^2\zeta_2) - \log \gamma^{(k)}| < 5e^{-m(2.63)}$$

where $\beta^{(k)}$, $\gamma^{(k)}$, $k = 1, 2$ are in $\boldsymbol{Q}(\alpha)$.

From a result of Baker [1], one can then show that $\max(|m|, |n|) < 10^{843}$. The method used in Baker and Davenport [3] can then be applied and one would expect the problem to be computationally feasible.

REMARK. We should perhaps note that we have in fact found all solutions of the equation $k(k+2) = l(l+1)(l+2)$ in positive integers, and these are $4\cdot 6 = 2\cdot 3\cdot 4$, $10\cdot 12 = 4\cdot 5\cdot 6$ and $418\cdot 420 = 55\cdot 56\cdot 57$. This is clear since from the beginning we dealt with the equation $(y-1)(y+1) = (x-1)x(x+1)$.

Another equation which can be reduced to this is the following: $m(m+1) = n(n+1)(2n+1) (= 6(1^2 + 2^2 + \cdots + n^2))$, for if n is a solution of this equation, let $l = 2n$, $k = 2m$ and then $k(k+2) = l(l+1)(l+2)$. The only even solutions for l are $l = 2$ and $l = 4$ so the only solutions of the equation for m and n are $2\cdot 3 = 1\cdot 2\cdot 3$, and $5\cdot 6 = 2\cdot 3\cdot 5$.

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CALIFORNIA INSTITUTE OF TECHNOLOGY