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STRONG CONCENTRATION OF THE SPECTRA OF SELF-ADJOINT OPERATORS

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Let H be a self-adjoint operator with spectral measure $E(S)$ over the Borel sets S of the real line. The spectrum of H is said to be strongly concentrated on S if whenever H_n converges strongly to H in the generalized sense it is true that $E_n(S)$ converges strongly to the identity. Sufficient conditions on H are given for this to occur for a given arbitrary Borel set S and necessary and sufficient conditions when S is the spectrum of H. In addition several more workable sufficient conditions are cited and a few examples illustrating the results are given.

Many authors have studied the changes in the spectra of a sequence of self-adjoint operators H_n as it converges strongly in some sense to a self-adjoint operator—e.g., [2], [3], [5], [6], [7, pp. 471-477], [8], [11]. It is known that while as point sets the spectra of H_n do not necessarily converge to the spectrum of H, nevertheless in some sense the spectra of H_n are concentrated on that of H. This spectral concentration phenomenon is described through the spectral measures E_n, E of the operators involved. In particular since $E(\Sigma)$ is the identity when Σ is the spectrum of H it is reasonable to say that the spectrum of the sequence H_n is concentrated on Σ if $E_n(\Sigma)$ converges to the identity as $n \rightarrow \infty$. Our main results concern necessary and sufficient conditions for this to occur for an arbitrary sequence H_n converging strongly to a fixed operator H. We make extensive use of the properties of the spectral measure $E(S)$ over the Borel sets S of the real line for which a general reference is [4] §§ X. 2 and XII. 2.

1. Preliminaries. Throughout this paper the following notation will be adhered to. H will denote a self-adjoint operator over a Hilbert space $\boldsymbol{H}$. Its domain will be denoted by $D(H)$ and its spectrum by Σ (which is always a closed subset of the real line R). The resolution of the identity of H will be denoted by $E(\lambda)$, $-\infty < \lambda < \infty$, and the associated projection-valued spectral measure by $E(S)$ over all Borel subsets S of R. By convention we take $E(\lambda)$ to be right continuous, i.e., $E(\lambda + 0) = E(\lambda)$. For a sequence of self-adjoint operators H_n, $n = 1, 2, \cdots$, over $\boldsymbol{H}$ the quantities $D(H_n)$, Σ_n, $E_n(\lambda)$, and $E_n(S)$ are defined accordingly.

According to a definition of Rellich (cf. [9] or [7, p. 429]) we say that the sequence of self-adjoint operators H_n coverges strongly to H in the generalized sense (denoted $H_n \underset{s}{\rightarrow} H$ in the generalized sense) if there exists a dense linear manifold D in $\mathbf{H}$ such that the following conditions are satisfied:

(i) $D \subseteq D(H_n)$ for all n sufficiently large

(ii) the closure of H restricted to D is again H

(iii) $\lim_{n\to\infty} H_n u = Hu$ for all $u \in D$.

If the operators H_n and H are all bounded the above definition reduces to ordinary strong convergence which we denote simply $H_n \underset{s}{\rightarrow} H$.

The following theorem of Rellich (cf. [9] or [7, p. 432]) will be basic to our analysis:

THEOREM 1.1. *Let the sequence of self adjoint operators H_n converge strongly in the generalized sense to the self-adjoint operator H. Then if λ is not an eigenvalue of H we have*

$$E_n(\lambda) \underset{s}{\longrightarrow} E(\lambda)$$

and

$$E_n(\lambda - 0) \underset{s}{\longrightarrow} E(\lambda)\ .$$

Next we give the definition of a spectral concentration phenomenon suggested by Titchmarsh (cf. [11], [12, p. 261]) and later refined by Conley and Rejto (cf. [2], [3]).

DEFINITION 1.2[1]. The spectrum of H_n is asymptotically concentrated on the Borel set S if $E_n(S) \underset{s}{\rightarrow} I$.

Asymptotic concentration is thus a property of a *sequence* H_n (and a subset S). We now introduce the definition of a concentration phenomenon associated with a *single* self-adjoint operator H.

DEFINITION 1.3. The spectrum of H is strongly concentrated on the Borel set S if whenever $H_n \underset{s}{\rightarrow} H$ in the generalized sense it is true that $E_n(S) \rightarrow I$.

Hence if the spectrum of H is strongly concentrated on S then the spectrum of any sequence H_n which converges strongly to H in the generalized sense is asympotically concentrated on S.

The following lemma states, as we would expect, that if the

[1] Actually asymptotic concentration is defined more generally to allow the subset S to depend on n. We then say the spectrum of H_n is asymptotically concentrated on the sets S_n if $E_n(S_n) \underset{s}{\rightarrow} I$. We shall not need this generalization however.

spectrum of an operator is strongly concentrated on a set it is also strongly concentrated on any larger set. The proof follows easily from the fact that if $S \subseteqq S'$ then $E(S) \leqq E(S')$.

LEMMA 1.4. *If the spectrum of H is strongly concentrated on S and if $S \subseteqq S'$ then the spectrum of H is strongly concentrated on S'.*

2. **Main results.** Our main interest will be to see how small we may make the set S. To this end the following theorem (cf. [7, p. 472]), reworded using our terminology, is of interest:

THEOREM 2.1. *Let S be an open set which contains the spectrum Σ of H. Then the spectrum of H is strongly concentrated on S.*

We shall next strengthen this theorem to the case where S is not necessarily open. Let $\operatorname{int} S$ denote the interior of the set S and let ∂S denote its boundary. Then

THEOREM 2.2. *Let S be a Borel set which contains the spectrum Σ of H. Then if $E(\partial S) = 0$ the spectrum of H is strongly concentrated on* $\operatorname{int} S$.

Proof. We have

$$\Sigma \subseteqq S \subseteqq (\operatorname{int} S) \cup \partial S \text{ and } (\operatorname{int} S) \cap \partial S = \varnothing$$

hence

$$E(\Sigma) \leqq E(S) \leqq E(\operatorname{int} S) + E(\partial S) .$$

But $E(\Sigma) = I$ and $E(\partial S) = 0$. Hence $E(\operatorname{int} S) = I$.

Since $\operatorname{int} S$ is an open subset of R it may be expressed as a countable union of disjoint open intervals, say

$$\operatorname{int} S = \bigcup_{k=1}^{\infty} I_k \text{ where } I_k = (\alpha_k, \beta_k) .$$

Since the spectral measure is strongly countably additive we therefore have

$$\sum_{k=1}^{\infty} E(I_k) = I . \tag{1}$$

Furthermore none of the endpoints of the intervals (α_k, β_k) can be eigenvalues of H, for the endpoints belong to ∂S and $E(\partial S) = 0$ while for eigenvalues λ_0 we have $E(\{\lambda_0\}) \neq 0$. Since (1) is in the sense of strong convergence of the sum, given any $u \in \boldsymbol{H}$ and any $\varepsilon > 0$

there exists a K such that

$$\left\| (I - \sum_{k=1}^{K} E(I_k))u \right\| < \varepsilon/2 \ .$$

Now let H_n be any sequence which converges strongly to H in the generalized sense. Then by Theorem 1.1 we have

$$E_n(\beta_k - 0) \xrightarrow[s]{} E(\beta_k) \qquad \text{as } n \longrightarrow \infty$$

and

$$E_n(\alpha_k) \xrightarrow[s]{} E(\alpha_k) \qquad \text{as } n \longrightarrow \infty \ .$$

For the above value of K we may therefore find a value N such that for $n \geqq N$

$$\| (E_n(\beta_k - 0) - E(\beta_k))u \| < \varepsilon/4K$$
$$\| (E_n(\beta_k - 0) - E(\alpha_k))u \| < \varepsilon/4K$$

for $k = 1, 2, \cdots, K$.

By definition of the spectral measure we have

$$E(I_k) = E(\beta_k) - E(\alpha_k) \qquad (\text{since } E(\beta_k - 0) = E(\beta_k))$$

and

$$E_n(I_k) = E_n(\beta_k - 0) - E_n(\alpha_k) \ .$$

Hence by successive use of the triangle inequality we have for $n \geqq N$

$$\begin{aligned} \left\| \left(I - \sum_{k=1}^{K} E_n(I_k)\right)u \right\| &\leqq \left\| \left(I - \sum_{k=1}^{K} E(I_k)\right)u \right\| \\ &+ \left\| \sum_{k=1}^{K} E(I_k)u - \sum_{k=1}^{K} E_n(I_k)u \right\| < \varepsilon/2 + \sum_{k=1}^{K} \| (E(\beta_k) - E_n(\beta_k - 0))u \| \\ &+ \sum_{k=1}^{K} \| (E_n(\alpha_k) - E(\alpha_k))u \| < \varepsilon/2 + \sum_{k=1}^{K} \varepsilon/4K + \sum_{k=1}^{K} \varepsilon/4K = \varepsilon \ . \end{aligned} \tag{2}$$

Now for all K we have

$$\bigcup_{k=1}^{K} I_k \subseteqq \bigcup_{k=1}^{\infty} I_k = \operatorname{int} S$$

hence

$$\sum_{k=1}^{K} E_n(I_k) \leqq E_n(\operatorname{int} S)$$

and so

$$I - E_n(\text{int } S) \leqq I - \sum_{k=1}^{K} E_n(I_k) \ .$$

Therefore from (2) above we have for $n \geqq N$

$$\| (I - E_n(\text{int } S))u \| < \varepsilon$$

which implies that $E_n(\text{int } S) \underset{s}{\longrightarrow} I$.

Notice that Theorem 2.2 is indeed a generalization of Theorem 2.1 for if S is open then $\text{int } S = S$ and if $\Sigma \subseteqq S$ then ∂S must lie in the resolvent set of H, hence $E(\partial S) = 0$.

This theorem suggests that under certain conditions we may expect the spectrum of H to be strongly concentrated on itself, by which we mean

DEFINITION 2.3. The spectrum Σ of H is strongly concentrated on itself if whenever $H_n \underset{s}{\longrightarrow} H$ in the generalized sense it is true that $E_n(\Sigma) \rightarrow I$.

The condition suggested by Theorem 2.2 will be shown also to be necessary in the following

THEOREM 2.4. *The spectrum Σ of H is strongly concentrated on itself if and only if $E(\partial \Sigma) = 0$.*

Proof. First assume that $E(\partial \Sigma) = 0$. Then from Theorem 2.2 with $S = \Sigma$ we have that the spectrum of H is strongly concentrated on $\text{int } \Sigma$ hence on itself.

To prove the converse we must show that if $E(\partial \Sigma) \neq 0$ then there exists a sequence H_n which converges strongly to H in the generalized sense for which $E_n(\Sigma)$ does not converge strongly to the identity. To construct this sequence we will need the following subspaces

$$\boldsymbol{H}_B = E(\partial \Sigma)\boldsymbol{H} \text{ and } \boldsymbol{H}_I = E(\text{int } \Sigma)\boldsymbol{H} \ .$$

Since $\Sigma = \text{int } \Sigma \cup \partial \Sigma$ and $\text{int } \Sigma \cap \partial \Sigma = \varnothing$ the closed subspaces $\boldsymbol{H}_B$ and $\boldsymbol{H}_I$ are orthogonal and span the whole space $\boldsymbol{H}$. Furthermore they each reduce the operator H. Let us denote the part of H restricted to $\boldsymbol{H}_B$ by H_B and the part of H restricted to $\boldsymbol{H}_I$ by H_I. Let Σ_B and Σ_I be their corresponding spectra. Then we have $\Sigma = \Sigma_B \cup \Sigma_I$, $\Sigma_B \subseteqq \partial \Sigma$, and $\Sigma_I = \overline{\text{int } \Sigma}$ where $\overline{\text{int } \Sigma}$ is the closure of the set $\text{int } \Sigma$.

We may now define the sequence H_n in each of the subspaces $\boldsymbol{H}_B$ and $\boldsymbol{H}_I$. We set

$$H_{I,n}u = Hu \text{ for } u \in \boldsymbol{H}_I \cap D(H)$$

and

$$H_{B,n}u = \sum_{k=-\infty}^{\infty} \lambda_{kn} E(I_{kn})u \qquad \text{for } u \in \boldsymbol{H}_B \cap D(H) .$$

$H_{B,n}$ is an approximation to the representation $H = \int \lambda dE(\lambda)$ restricted to $\boldsymbol{H}_B$, which converges strongly to H_B in the generalized sense as $n \to \infty$. For each fixed n the intervals I_{kn}, $k = 0, \pm 1, \cdots$, are to be a subdivision of the real line chosen so that the end points do not fall on $\partial \Sigma$. The length of the largest interval is to approach zero as $n \to \infty$. Each interval I_{kn} which contains a point of $\partial \Sigma$ also contains points not belonging to Σ, from which we choose λ_{kn}. In the intervals which do not contain points of $\partial \Sigma$ we have that $E(I_{kn})u = 0$ for $u \in \boldsymbol{H}_B \cap D(H)$, hence the choice of λ_{kn} is immaterial. The spectrum of $H_{B,n}$ for each fixed n consists of those λ_{kn}, $k = 0, \pm 1, \cdots$, for which $\partial \Sigma \cap I_{kn} \neq \emptyset$, and so is disjoint from Σ.

Finally let

$$H_n = H_{I,n} \oplus H_{B,n} .$$

Then $H_{ns} \to H$ in the generalized sense since $H_{I,n}u \to H_I u$ and $H_{B,n}v \to H_B v$ for all $u \in \boldsymbol{H}_I \cap D(H)$ and all $v \in \boldsymbol{H}_B \cap D(H)$. To show that $E_n(\Sigma)$ does not converge strongly to the identity, let v be any nonzero element of $\boldsymbol{H}_B$. Then

$$E_n(\Sigma)v = E_{B,n}(\Sigma)v .$$

But the spectrum of $H_{B,n}$ is disjoint from Σ, hence $E_{B,n}(\Sigma) = 0$. Therefore $E_n(\Sigma)v = 0$ and so $E_n(\Sigma)v$ does not converge to v.

Actually this theorem also shows the spectrum of H to be strongly concentrated on a slightly smaller set, as follows.

COROLLARY 2.5. *The spectrum Σ of H is concentrated on itself if and only if it is concentrated on* $\operatorname{int} \Sigma$.

Proof. The "if" part follows from Lemma 1.4. Conversely if the spectrum of H is concentrated on itself then $E(\partial \Sigma) = 0$ and hence from Theorem 2.2 with $S = \Sigma$ it is concentrated on $\operatorname{int} \Sigma$.

The condition $E(\partial \Sigma) = 0$ requires that no eigenvalues of H lie on $\partial \Sigma$ (in particular H may have no isolated eigenvalues). However more is required. To give sufficient conditions for $E(\partial \Sigma) = 0$ let us denote by $\boldsymbol{H}_{AC}$ the set of all $u \in \boldsymbol{H}$ for which $(E(\lambda)u, u)$ is absolutely continuous in λ. It is known (cf. [7, p. 516]) that $\boldsymbol{H}_{AC}$ is a closed subspace of $\boldsymbol{H}$ which reduces H. Let us further denote by $m(S)$ the Lebesgue measure of the set S. Then we have

THEOREM 2.6. *If* $m(\partial\Sigma) = 0$ *and* $\boldsymbol{H}_B \subseteqq \boldsymbol{H}_{AC}$ *then* $E(\partial\Sigma) = 0$.

Proof. If $u \in \boldsymbol{H}_I$ then $E(\operatorname{int}\Sigma)u = u$. Hence

$$E(\partial\Sigma)u = E(\partial\Sigma)E(\operatorname{int}\Sigma)u = 0 \text{ since } \partial\Sigma \cap \operatorname{int}\Sigma = \varnothing .$$

And if $u \in \boldsymbol{H}_B$ then $u \in \boldsymbol{H}_{AC}$ and since then $(E(\lambda)u, u)$ is absolutely continuous we have

$$\| E(\partial\Sigma)u \|^2 = (E(\partial\Sigma)u, u) = \int_{\partial\Sigma} d(E(\lambda)u, u) = 0$$

as $m(\partial\Sigma) = 0$.

As $\boldsymbol{H}_I$ and $\boldsymbol{H}_B$ span $\boldsymbol{H}$ we have $E(\partial\Sigma)u = 0$ for all $u \in \boldsymbol{H}$.

The orthogonal complement of $\boldsymbol{H}_{AC}$ is denoted by $\boldsymbol{H}_S$ and is identical to the set of all $u \in \boldsymbol{H}$ for which $(E(\lambda)u, u)$ is a singular function of λ. The spectrum of H restricted to $\boldsymbol{H}_{AC}$ or $\boldsymbol{H}_S$ is called, respectively, the absolutely continuous spectrum and the singular spectrum of H (denoted Σ_{AC} and Σ_S). The condition $\boldsymbol{H}_B \subseteqq \boldsymbol{H}_{AC}$ in Theorem 2.6 implies, but is not implied by, the condition $\partial\Sigma \subseteqq \Sigma_{AC}$. The following corollary gives sufficient conditions for $E(\partial\Sigma) = 0$ in terms of the singular spectrum.

COROLLARY 2.7. *If* $m(\partial\Sigma) = 0$ *and* $\partial\Sigma \cap \Sigma_S = \varnothing$ *then* $E(\partial\Sigma) = 0$.

Proof. If $\partial\Sigma \cap \Sigma_S = \varnothing$ then $E(\partial\Sigma)E(\Sigma_S) = 0$ and hence $E(\partial\Sigma)\boldsymbol{H}$ and $E(\Sigma_S)\boldsymbol{H}$ are orthogonal subspaces. Now $\boldsymbol{H}_B = E(\partial\Sigma)\boldsymbol{H}$ and $\boldsymbol{H}_S \subseteqq E(\Sigma_S)\boldsymbol{H}$ since $\boldsymbol{H}_S$ reduces H, consequently $\boldsymbol{H}_B$ and $\boldsymbol{H}_S$ are orthogonal. But since $\boldsymbol{H}_{AC}$ is the orthogonal complement of $\boldsymbol{H}_S$ it follows that $\boldsymbol{H}_B \subseteqq \boldsymbol{H}_{AC}$ and so the conditions of Theorem 2.6 are satisfied.

A weaker but somewhat more useful sufficient condition is the following:

THEOREM 2.8. *Let* $\partial\Sigma$ *be countable and not contain any eigenvalues of* H. *Then* $E(\partial\Sigma) = 0$.

Proof. Let $\partial\Sigma = \bigcup_{k=1}^{\infty} \{\lambda_k\}$. Then

$$E(\partial\Sigma) = \sum_{k=1}^{\infty} E(\{\lambda_k\}) = \sum_{k=1}^{\infty} (E(\lambda_k) - E(\lambda_k - 0)) = 0$$

since $E(\lambda)$ is continuous at any point λ not an eigenvalue of H.

3. Examples and applications. Our first example will be to show that the condition $m(\partial\Sigma) = 0$ in theorem 2.6 is essential. Let $\{r_n\}$ $n = 1, 2, \cdots$, be an enumeration of the rational numbers in the closed interval $[0, 1]$. Let ε be any positive number less than 1 and let

$$0_n = \{x \in (0, 1) \mid |x - r_n| < \varepsilon/2^n\}$$

and

$$Q = \bigcup_{n=1}^{\infty} 0_n .$$

Then 0_n is an open set with Lebesgue measure less than or equal to $\varepsilon/2^n$, and Q is open with

$$m(Q) \leqq \sum_{n=1}^{\infty} m(0_n) \leqq \sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon .$$

If we set

$$P = ([0, 1] - Q) \cup \{0\} \cup \{1\}$$

then P is a closed nowhere dense subset of the unit interval consisting entirely of irrational numbers (plus the endpoints 0 and 1). Furthermore $m(P) \geqq 1 - \varepsilon$. We define a Borel measure σ on R by

$$\sigma(S) = m(S \cap P)$$

with associated generating function

$$\sigma(x) = \sigma((-\infty, x)) .$$

Our Hilbert space $\boldsymbol{H}$ will be $L^2_\sigma(R)$, consisting of all σ-measurable functions $f(x)$ on R for which

$$\int |f(x)|^2 d\sigma(x) < \infty .$$

The multiplication operator

$$(Hf)(x) = xf(x)$$

is then self-adjoint on $\boldsymbol{H}$ with spectrum $\Sigma = P$ (cf. [1, pp. 103–106]). But P is closed and nowhere dense, hence $\partial P = \partial\Sigma = \Sigma$. Hence $E(\partial\Sigma) = E(\Sigma) = I$ and so the spectrum of H cannot be concentrated on itself. Furthermore since σ is a restriction of Lebesgue measure it is absolutely continuous, hence $\boldsymbol{H}_{AC} = \boldsymbol{H}_B = \boldsymbol{H}$.

Our second example will be to show that the condition $\boldsymbol{H}_B \subseteqq \boldsymbol{H}_{AC}$ is also essential in Theorem 2.6. Let $c(x)$ be the Cantor Ternary function (cf. [10, p. 39]) on the unit interval and let $c(S)$ be the associated Cantor measure on R. Our Hilbert space will be $L^2_c(R)$ and H will again be multiplication by the independent variable. $c(x)$ is a continuous non-absolutely continuous function whose only points of increase are on the Cantor set C. Hence $\Sigma = C$. But C is closed and nowhere dense so that $\partial C = \partial \Sigma = \Sigma$. Therefore $E(\partial \Sigma) = E(\Sigma) = I$ and so the spectrum of H cannot be strongly concentrated on itself. And since the Cantor set has Lebesgue measure zero we have $m(\Sigma) = 0$. The theorem fails, of course, since H has no absolutely continuous spectrum and $\boldsymbol{H}_B = \boldsymbol{H}$.

Our final example will be a positive one of interest in itself. Let $\boldsymbol{H} = L^2(R)$ and let H be the Schroedinger operator

$$(Hf)(x) = -\frac{d^2f}{dx^2} + g(x)f$$

acting on the class of functions $f(x)$ with absolutely continuous first derivatives for which $Hf \in \boldsymbol{H}$. Here $g(x)$ is a continuous real-valued periodic function. H is the Hamiltonian operator of a one-dimensional quantum mechanical particle moving in a periodic potential (a crystal for example). It is known (cf. [12, Chapter XXI]) that H is self-adjoint with a purely continuous spectrum consisting of a sequence of closed intervals bounded below, extending to $+\infty$, and separated by a finite or infinite number of gaps (these are the so-called energy bands of solid state physics). Since the conditions of Theorem 2.8 are satisfied the spectrum of H is strongly concentrated on itself.

Let us consider the following sequence of operators:

$$(H_nf)(x) = -\frac{d^2f}{dx^2} + g_n(x)f$$

where

$$g_n(x) = \begin{cases} g(x) & \text{for } |x| \leqq n \\ 0 & \text{for } |x| > n \,. \end{cases}$$

The operators H_n are self-adjoint over the same domain as H and they converge strongly to H in the generalized sense since for $f \in D(H) = D(H_n)$ we have

$$\| Hf - H_nf \|^2 = \int_{|x|>n} |g(x)|^2 \, |f(x)|^2 \, dx \longrightarrow 0 \text{ as } n \longrightarrow \infty \,.$$

The operator H_n is the Hamiltonian operator of a particle moving in a crystal of finite extent. Its spectrum, since $g_n(x)$ is continuous with compact support, consists of a continuous portion $[0, \infty)$ and at most a finite number of negative eigenvalues with finite multiplicity.

The quantity $\| E(S)f \|^2$ in quantum mechanics represents the probability of measuring the value of the energy of the particle in the state f within the subset S. While for a finite crystal the energy may assume any value from 0 to $+\infty$, for an infinite crystal the energy must lie within the energy bands of the operator H. The fact that the spectrum of H is strongly concentrated on itself then assures us that for a finite crystal and a fixed state f we may make the probability of finding the energy outside the energy bands of the infinite crystal as small as we desire by taking the crystal sufficiently large (i.e., by choosing n sufficiently large).

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