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HOMOLOGY OF A GROUP EXTENSION

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A topological method has been used by Ganea to derive the homology exact sequence of a central extension. In the same spirit a homology exact sequence is constructed for a group extension with certain homological restrictions. An immediate consequence is an exact sequence of Kervaire which is of some significance in algebraic K-theory.

Let

$$1 \longrightarrow N \rightarrow G \longrightarrow Q \longrightarrow 1 \tag{1}$$

be an extension of groups. Each element g of G induces an automorphism $\theta(g): N \rightarrow N$ via $\theta(g)n = gng^{-1}$ for $n \in N$. In what follows we denote by $H_k(G)$ the kth homology group of G with coefficients in the additive group of integers Z, on which G operates trivially. Let Γ_k denote the subgroup of $H_k(N)$ generated by $\theta(g)_*c - c$, $c \in H_k(N)$, $g \in G$. We say that G operates trivially on $H_k(N)$ if $\Gamma_k = \{0\}$. Let $N \tilde{\times} G$ be the semi-direct product of N and G with respect to the operation $\theta(g)$ and let P_k denote the kernel of $\pi_*: H_k(N \tilde{\times} G) \rightarrow H_k(G)$, where $\pi: N \tilde{\times} G \rightarrow G$ is given by $\pi(n, g) = g$. We shall prove

THEOREM 1. *Suppose $n = 1$ or $H_k(N) = 0$ for $1 \leqq k \leqq n-1$ $(n \geqq 2)$. Then there exists an exact sequence*

$$P_{2n} \longrightarrow H_{2n}(G) \longrightarrow H_{2n}(Q) \longrightarrow P_{2n-1} \longrightarrow \cdots \longrightarrow P_{n+1} \longrightarrow H_{n+1}(G)$$
$$\longrightarrow H_{n+1}(Q) \longrightarrow H_n(N)/\Gamma_n \longrightarrow H_n(G) \longrightarrow H_n(Q) \longrightarrow 0\ .$$

Further assume G operates trivially on $H_n(N)$ and that $H_1(Q) = 0$. Then there exists an exact sequence

$$H_{n+1}(N) \longrightarrow H_{n+1}(G) \longrightarrow H_{n+1}(Q) \longrightarrow H_n(N) \longrightarrow H_n(G) \longrightarrow H_n(Q) \longrightarrow 0\ .$$

We note that the first part of Theorem 1 for $n = 1$ is just Theorem 3.1 of [7].

Now we call an epimorphism $f: H \rightarrow H'$ *central* if Ker f is contained in the center of H. Let

$$\begin{array}{ccccccccc} & & \tilde{N} & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{Q} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array} \tag{2}$$

be a commutative diagram of groups and homomorphisms such that the rows and columns are exact.

THEOREM 2. *In the situation* (2) *suppose* $\tilde{N} \to N$ *and* $\tilde{Q} \to Q$ *are central,* $H_1(\tilde{N}) = H_1(\tilde{Q}) = H_1(N) = 0$ *and that* G *operates trivially on* $H_2(N)$. *Then there exists an exact sequence*

$$H_3(\tilde{N}) \longrightarrow H_3(\tilde{G}) \longrightarrow H_3(\tilde{Q}) \longrightarrow H_2(N) \longrightarrow H_2(G) \longrightarrow H_2(Q) \longrightarrow 0 .$$

As a special case of Theorem 2 we obtain Prop. 2 of Kervaire [6] (cf. [1]).

THEOREM 3. *In* (1) *let* Q *be the additive group of integers* Z *and let* e *be an element of* G *which maps to* $+1 \in Z$. *Then there exists a long exact sequence*

$$\cdots \longrightarrow H_k(N) \xrightarrow{1-\theta(e)_*} H_k(N) \longrightarrow H_k(G) \longrightarrow H_{k-1}(N)$$
$$\cdots \longrightarrow H_1(N) \longrightarrow H_1(G) \longrightarrow Z \longrightarrow 0 .$$

1. Topological preliminaries. In this section two lemmas are established which play a vital role in the proofs of Theorems. We work in the category of based spaces which have the homotopy type of a CW complex. We use the notation $\vee$ for path-composition. The multiplication of elements of fundamental groups are indicated by juxtaposition. Given a map $f: X \to Y$ we denote by $f_\#$ the homomorphism induced on fundamental groups.

Given a map $p: E \to B$, let $\rho: E_p \to E$ denote the fibre of p, that is, $E_p = \{(x, \beta) \in E \times B^I; \beta(0) = *, \beta(1) = p(x)\}$ with $\rho(x, \beta) = x$, where $*$ stands for the base point. ΩB, the space of loops on B, acts on E_p through $\mu: \Omega B \times E_p \to E_p$, $\mu(\omega, (x, \beta)) = (x, \omega \vee \beta)$. We define

$$\tilde{\mu}: \tilde{H}_0(\Omega B) \otimes H_k(E_p) \longrightarrow H_k(E_p)$$

to be the composite

$$\tilde{H}_0(\Omega B) \otimes H_k(E_p) \subset H_0(\Omega B) \otimes H_k(E_p) \longrightarrow H_k(\Omega B \times E_p) \xrightarrow{\mu_*} H_k(E_p) ,$$

where the middle arrow comes from Künneth theorem and $\tilde{H}_0(\Omega B)$ may be identified with the subring of the integral group ring of $\pi_1(B)$ generated by $\omega - 1$, $\omega \in \pi_1(B)$.

Now let p be a Hurewicz fibration with fibre inclusion $i: F \to E$. As shown by Eckmann-Hilton [2; Prop. 3.10 and Theorem 3.11], the above μ determines an action of ΩB on F, which is denoted by the same letter μ. We say that $\pi_1(B)$ operates trivially on $H_k(F)$ if the above $\tilde{\mu}$ is trivial.

Let S denote the suspension functor and let C_p denote the cofibre of p, that is, $C_p = B \bigcup_p CE$ (with $(x, 1)$ and $p(x)$ identified). Let

$$\sigma: SF \longrightarrow C_p$$

denote the canonical embedding defined by $\sigma(x, t) = (x, t) \in CE$, $x \in F$, $0 \leqq t \leqq 1$.

LEMMA 1.1. *Suppose that B is path-connected and that F is homology $(n-1)$-connected, $n \geqq 1$. Then σ is homology $(n+1)$-connected and the sequence*

$$\tilde{H}_0(\Omega B) \otimes H_n(F) \xrightarrow{\tilde{\mu}} H_n(F) \xrightarrow{\sigma *} H_{n+1}(C_p) \longrightarrow 0$$

is exact.

Proof. According to Ganea [3; Theorem 1.1], the extension $r: C_i \rightarrow B$ of p to $E \cup CF$ has the fibre equivalent to $\Omega B * F$, which is n-connected. Thus the argument in [3; Theorem 2.2] is valid in our case, hence there is an exact sequence

$$H_k(\Omega B * F) \xrightarrow{H *} H_k(SF) \xrightarrow{\sigma *} H_k(C_p) \longrightarrow H_{k-1}(\Omega B * F)$$

for $k \leqq n+1$, where $H: \Omega B * F \rightarrow SF$ is the map obtained from μ by the Hopf construction. It is immediate that H_* coincides with $\tilde{\mu}$ on $H_{n+1}(\Omega B * F) \cong \tilde{H}_0(\Omega B) \otimes H_n(F)$, which proves the assertion.

COROLLARY 1.2. *In addition to the assumption of Lemma 1.1, suppose further $\pi_1(B)$ operates trivially on $H_n(F)$ and that $H_1(B) = 0$. Then σ is homology $(n+2)$-connected.*

Proof. Since C_i is the double mapping cylinder of $* \leftarrow F \xrightarrow{i} E$, $r: C_i \leftarrow B$ is homotopically equivalent to the Whitney join

$$p_B \oplus p: PB \oplus E \longrightarrow B$$

of the path-fibration $p_B: PB \rightarrow B$ and p ([5]. For the notation see [7]). It follows from the construction of a lifting function of Whitney join (See Hall [5; §3]) that, in $p_B \oplus p$, ΩB operates on $\Omega B * F$ through $\nu: \Omega B \times (\Omega B * F) \rightarrow \Omega B * F$ as the join of the actions in each fibration; thus, $\nu(\alpha, (1-t)B \oplus tx) = (1-t)(\beta \vee \alpha^{-1}) \oplus t\mu(\alpha, x)$ for $\alpha, \beta \in \Omega B$, $x \in F$, $0 \leqq t \leqq 1$. Consequently, $\tilde{\nu}$ is given by

$$\tilde{\nu}((\alpha - 1) \otimes ((\beta - 1) \otimes c)) = (\beta - 1)(\alpha^{-1} - 1) \otimes c$$

under the assumption $\tilde{\mu}((\alpha - 1) \otimes c) = 0$.

Applying Lemma 1.1 to $p_B \oplus p$, we get an exact sequence

$$\tilde{H}_0(\Omega B) \otimes H_{n+1}(\Omega B * F) \xrightarrow{\tilde{\nu}} H_{n+1}(\Omega B * F) \longrightarrow H_{n+2}(C_{p_B \oplus p}) \longrightarrow 0 .$$

Since $\pi_1(B) = [\pi_1(B), \pi_1(B)]$ by assumption and since the identity

$$\alpha\beta\alpha^{-1}\beta^{-1} - 1 = (\alpha\beta\alpha^{-1} - 1)(\beta^{-1} - 1) + (\alpha - 1)(\beta\alpha^{-1} - 1) - (\beta\alpha^{-1} - 1)(\alpha - 1) - (\beta - 1)(\beta^{-1} - 1)$$

holds in the integral group ring of $\pi_1(B)$ we may infer that $\tilde{\nu}$ is epic. This implies that $H_{n+2}(C_{p_B \oplus p}) \cong H_{n+2}(C_r) = 0$. Since C_σ is of the same homotopy type as C_r by [3; Prop. 1.6], we see that σ is homology $(n + 2)$-connected.

Next consider an extension of groups (1). We may construct a Hurewicz fibration $p: E \to B$ of aspherical spaces with fibre inclusion $i: F \to E$ so that the sequence

$$1 \longrightarrow \pi_1(F) \xrightarrow{i_\sharp} \pi_1(E) \xrightarrow{p_\sharp} \pi_1(B) \longrightarrow 1$$

coincides with the given extension (1). We shall relate $\theta(g)_*$ to the action $\tilde{\mu}$ of $\pi_1(B)$ on $H_*(F)$.

As in the beginning of this section, we may replace $i: F \to E$ by $\rho: E_p \to E$. Let $g \in G = \pi_1(E)$ and let $\overline{\theta(g)}$ denote a map $(E_p, *) \to (E_p, *)$ induced by $\theta(g)$. Take $\alpha: (I, \dot{I}) \to (E, *)$ which represents g. Define a path $\Delta(\alpha)$ in E_p joining $(*, *)$ with $(*, p\alpha \vee *)$ by setting

$$\Delta(\alpha)(t) = (\alpha(t), \bar{\alpha}_t)\,,$$

$$\bar{\alpha}_t(s) = \begin{cases} p\alpha(2s) & 0 \leqq 2s \leqq t \\ p\alpha(t) & t \leqq 2s \leqq 2\,. \end{cases}$$

μ defines a map $\mu(\alpha): (E_p, *) \to (E_p, (*, p\alpha \vee *))$ given by

$$\mu(\alpha)(x, \beta) = \mu(p\alpha; (x, \beta)) = (x, p\alpha \vee \beta)\,.$$

Since E_p has a non-degenerate base point [8], we obtain a map

$$\overline{\mu(\alpha)}: (E_p, *) \longrightarrow (E_p, *)$$

which is $\Delta(\alpha)$-homotopic to $\mu(\alpha)$.

LEMMA 1.5. *There is a based homotopy between* $\overline{\mu(\alpha)}$ *and* $\overline{\theta(g)}$.

Proof. It suffices to prove that, for each loop $\omega: (I, \dot{I}) \to (E_p, *)$, we have $\overline{\mu(\alpha)}_\sharp\omega = \overline{\theta(g)}_\sharp\omega$. We see that $\Delta(\alpha) \vee \mu(\alpha)\omega \vee \Delta(\alpha)^{-1}: (I, \dot{I}) \to (E_p, *)$ is $\Delta(\alpha)$-homotopic to $\mu(\alpha)\omega$ and that $\rho(\Delta(\alpha) \vee \mu(\alpha)\omega \vee \Delta(\alpha)^{-1}) = \alpha \vee \rho\omega \vee \alpha^{-1}$. Thus, by [8; Lemma 7.3.2(b)],

$$\overline{\mu(\alpha)}\omega \simeq \Delta(\alpha) \vee \mu(\alpha)\omega \vee \Delta(\alpha)^{-1}$$

and, since $\rho\overline{\theta(g)}\omega = \alpha \vee \rho\omega \vee \alpha^{-1}$ by definition and since $\rho_\sharp$ is monic, it follows that $\Delta(\alpha) \vee \mu(\alpha)\omega \vee \Delta(\alpha)^{-1} \simeq \overline{\theta(g)}\omega$. These yield $\overline{\mu(\alpha)}\omega \simeq \overline{\theta(g)}\omega$, as desired.

COROLLARY 1.4. $\tilde{\mu}((p_\# \alpha - 1) \otimes c) = \theta(g)_* c - c$ *for* $c \in H_k(F)$, $\alpha \in \pi_1(E)$.

For, we have

$$\begin{aligned}\tilde{\mu}((p_\# \alpha - 1) \otimes c) &= \mu_*(p_\# \alpha, c) - c = \mu(\alpha)_* c - c \\ &= \overline{\mu(\alpha)}_* c - c = \overline{\theta(g)}_* c - c \qquad \text{by Lemma 1.3} \\ &= \theta(g)_* c - c\end{aligned}$$

2. Proof of Theorem 1. Let $p: E \to B$ be a fibration with fibre inclusion $i: F \to E$ which is used in the proof of Lemma 1.3. Introduce the following commutative diagram

$$(3) \qquad \begin{array}{ccccccccc} K & \xrightarrow{p_1} & E & \longrightarrow & C_{p_1} & \longrightarrow & SK & \xrightarrow{Sp_1} & SE \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle Sp_2} & & \downarrow{\scriptstyle Sp} \\ E & \xrightarrow{p} & B & \longrightarrow & C_p & \longrightarrow & SE & \xrightarrow{Sp} & SB \end{array}$$

in which the square in the left corner is the pull-back of p by p, χ is induced by it and the rows are Puppe sequences for p_1 and p. Since $F*F$ is $2n$-connected, it follows that χ is homology $(2n+1)$-connected (cf. [7; 1.1 and 1.2]).

Since p_1 admits a cross-section, $H_k(C_{p_1})$, identified with a subgroup of $H_{k-1}(K)$, coincides with the kernel of $p_{1*}: H_{k-1}(K) \to H_{k-1}(E)$. As shown in [7; 3.1], $\pi_1(K) \cong N \overset{\sim}{\times} G$ and, under this isomorphism, $p_{1\#}(n, g) = g$, which implies that $\operatorname{Ker} p_{1*} = P_{k-1}$.

Observe that the composite $SF \xrightarrow{\sigma} C_p \longrightarrow SE$ coincides with S_I. Lemma 1.1 applied to p yields an exact sequence

$$\tilde{H}_0(\Omega B) \otimes H_n(F) \xrightarrow{\tilde{\mu}} H_{n+1}(SF) \xrightarrow{\sigma_*} C_{n+1}(C_p) \longrightarrow 0$$

and bijections $\sigma_*: H_k(SF) \to H_k(C_p)$ for $k \leqq n$. It follows from Corollary 1.4 that $\operatorname{Im} \tilde{\mu} = \Gamma_k$, hence $H_{n+1}(C_p) \cong H_n(N)/\Gamma_n$. Thus we obtain an exact sequence stated in Theorem 1, which completes the proof of the first part of Theorem 1.

Further assume $H_1(B) = 0$ and that $\Gamma_n = 0$; then, by Corollary 1.2, $\sigma_*: H_{n+2}(SF) \to H_{n+2}(C_p)$ is epic, hence there is an exact sequence

$$H_{n+1}(F) \longrightarrow H_{n+1}(E) \longrightarrow H_{n+1}(B) \longrightarrow H_n(F) \longrightarrow H_n(E) \longrightarrow H_n(B) \longrightarrow 0 .$$

which yields the second part of Theorem 1.

3. Proof of Theorem 2. First we shall prove

LEMMA 3.1. (Kervaire [6; Lemma 3]) *Let* $1 \to N \to G \to Q \to 1$ *be a central extension of groups. If* $H_k(G) = 0$ *for* $1 \leqq k \leqq n$, *then the sequence*

$$H_{n+2}(G) \longrightarrow H_{n+2}(Q) \longrightarrow H_{n+2}(N, 2; Z) \longrightarrow H_{n+1}(G) \longrightarrow H_{n+1}(Q) \longrightarrow H_{n+1}(N, 2; Z)$$

is exact. In particular, if $H_1(G) = 0$, *then* $H_3(G) \to H_3(Q)$ *is epic and* $H_2(G) \to H_2(Q)$ *is monic.*

Proof. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be as in the proof of Lemma 1.3. As shown by Ganea [4], p is homotopically equivalent to the principal fibration $E_\phi \to B$ induced by a map $\phi: B \to C = K(N, 2)$. Let $\tilde{\phi}: C_p \to C$ denote the canonical extensin of ϕ to $B \bigcup_p CE$. By [3; Theorem 1.1] the fibre of $\tilde{\phi}$ is equivalent to $E * \Omega C$, which is $(n+2)$-connected. This implies that $\tilde{\phi}$ is $(n+3)$-connected. Thus, by replacing $H_k(C_p)$ for $k \leqq n+2$ by $H_k(C)$ in the Puppe sequence of p, there is obtained the desired exact sequence. The second part follows from the fact that $H_3(N, 2; Z) = 0$.

We now proceed to the proof of Theorem 2. Let $\bar{N}$ denote the kernel of $\tilde{G} \to \tilde{Q}$ in (2). Then the diagram (2) may be enlarged to the following

$$\begin{array}{ccccc} \tilde{N} & & & & \\ \downarrow{\scriptstyle \xi} & \searrow & & & \\ \bar{N} & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{Q} \\ \downarrow{\scriptstyle \eta} & & \downarrow & & \downarrow{\scriptstyle \zeta} \\ N & \longrightarrow & G & \longrightarrow & Q . \end{array}$$

Note that ξ and η are epic, hence central with $H_1(\bar{N}) = 0$.

Introduce the commutative diagram

$$(4) \quad \begin{array}{ccccccccc} H_3(\bar{N}) & \longrightarrow & H_3(\tilde{G}) & \longrightarrow & H_3(\tilde{Q}) & \longrightarrow & H_2(\bar{N}) & & & & \\ & & & & \downarrow{\scriptstyle \zeta_*} & & \downarrow{\scriptstyle \eta_*} & & & & \\ & & & & H_3(Q) & \longrightarrow & H_2(N) & \longrightarrow & H_2(G) & \longrightarrow & H_2(Q) & \longrightarrow & 0 \end{array}$$

where ζ_* is epic and η_* is monic by Lemma 3.1. Hence it follows from naturality of action that $\tilde{G}$ operates trivially on $H_2(\bar{N})$. Applying Theorem 1 to the extensions $1 \to \bar{N} \to \tilde{G} \to \tilde{Q} \to 1$ and $1 \to N \to G \to Q \to 1$, we see that the rows of (4) are exact. Since $\xi_*: H_3(\tilde{N}) \to H_3(\bar{N})$ is epic by Lemma 3.1, we may conclude that the sequence stated in Theorem 2 is exact.

4. Proof of Theorem 3. We may take the circle S^1 for B in the fibration $F \xrightarrow{i} E \xrightarrow{p} B$ which realizes (1). We use the Wang sequence for p which is found in Spanier [8; 8.5.5]. There are fibre homotopy equivalences

$$f_-: C_-S^0 \times F \longrightarrow p^{-1}(C_-S^0),\ g_+: p^{-1}(C_+S^0) \longrightarrow C_+S^0 \times F$$

such that $f_-|y_0 \times F$ is homotopic to the map $(y_0, x) \to x$ and $g_+|F$ is homotopic to the map $x \to (y_0, x)$, where y_0 denotes the base point corresponding to $\{0\} \in S^0$ and where C_-S^0 and C_+S^0 are southern and northern hemi-circles. The clutching function $m: S^0 \times F \to F$ is defined by

$$g_+f_-(\{\varepsilon\}, x) = (\{\varepsilon\}, m(\{\varepsilon\}, x)) , \qquad \varepsilon = 0, 1 .$$

Then $m|\{0\} \times F$ is homotopic to the map $(\{0\}, x) \to x$.

Now Spanier has shown that the top row is exact in the following diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow H_{k+1}(E) \longrightarrow & H_{k+1}(C_-S^0 \times F,\ S^0 \times F) & \xrightarrow{m_*\partial} & H_k(F) & \xrightarrow{i_*} & H_k(E) \longrightarrow \cdots \\
 & \uparrow \cong & & \cong \uparrow s & & \\
 & & & H_{k+1}(SF) & & \\
 & & & \uparrow m_* & & \\
 & H_{k+1}(C_{\pi_2}) & \xrightarrow{\quad q_* \quad} & H_{k+1}(S(S^0 \times F)) & & \\
 & \uparrow T_* & & & & \\
 & H_{k+1}(SS^0 \vee S^0 * F) & & & &
\end{array}$$

which is commutative up to sign, where s is the suspension isomorphism, $\pi_2: S^0 \times F \to F$ the projection, q the map pinching F to a point and $T: SS^0 \vee S^0*F \to C_{\pi_2}$ denotes the homotopy equivalence defined in [7; 2.2]; thus, $mqT|SS^0$ is homotopic to the map $(\varepsilon, t) \to (m(\varepsilon, *), t)$ and $mqT|S^0*F$ is homotopic to the map $(1-t) \in \bigoplus tx \to (m(\varepsilon, x), t)$. Hence, using the homeomorphism $h: SF \to S^0*F$ given by

$$h(x, s) = \begin{cases} (1-2s)\{0\} \oplus 2sx & 0 \leqq 2s \leqq 1 \\ (2s-1)\{1\} \oplus (2-2s)x & 1 \leqq 2s \leqq 2 , \end{cases}$$

we see that $mqTh$ induces the homomorphism

$$H_{k+1}(SF) \xrightarrow{(1-S\bar{m})_*} H_{k+1}(SF) ,$$

where $\bar{m}: F \to F$ denotes the map given by $\bar{m}(x) = m(\{1\}, x)$.

Consequently, the proof of Theorems 3 will be completed if the following assertion is proved:

$$(5) \qquad \bar{m}_* = \theta(e)_*$$

Proof of (5). Observe that $+1 \in Z$ is represented by a loop ω in $SS^0 = C_+S^0 \cup C_-S^0$ which emanates at $\{0\}$. By considering g_+f_-

followed by a fiber homotopy inverse f_+ of g_+, we infer easily that ω is lifted to a path $\tilde{\omega}_x$, depending continuously on $x \in F$, with $\tilde{\omega}_x(1) = x$ and such that the map $x \rightarrow \tilde{\omega}_x(0)$ is homotopic to the map $x \rightarrow \bar{m}(x)$. Hence the definition of the action of the fibration and Lemma 1.3 imply the assertion (5).

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