ON A QUESTION OF TARSKI AND A MAXIMAL THEOREM OF KUREPA

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Let *PI* be the statement, "Every Boolean algebra has a prime ideal."; let *SPI* be the statement, "Every infinite set algebra has a nonprincipal prime ideal."; let *K* be the statement, "Every family of sets includes a maximal subfamily of pairwise incomparable (by the inclusion relation) sets.". A model is exhibited of set theory without regularity in which *SPI* and *K* hold, *PI* fails. Furthermore all finitary versions of the axiom of choice fail in this model and the model contains a set which is infinite Dedekind finite in the sense of the model. A proof is given that the local version of *SPI* implies the axiom of choice for families of finite sets in *ZF*.

1. **Introduction and statement of results.** Before 1963, the only method available for constructing interesting models of Zermelo-Fraenkel set theory, (*ZF*), was the Fraenkel-Mostowski-Specker method which produced models in which the axiom of regularity failed. In 1963, Cohen presented his method for producing models in which regularity holds. Questions concerning the axiom of choice, (*AC*), can be considered in both types of models, unlike questions such as the continuum hypothesis which are only interesting in Cohen models. Although an independence result established by the Cohen method is stronger than the same result established by the *FMS* method, this latter method remains of interest for a number of reasons as Lévy points out in [2]. Two such reasons are as follows: (1) There are independence results which hold in set theory without regularity but do not hold in set theory with regularity. (2) The proof of a given independence may be more easily established by means of an *FMS* model. Such a proof may in turn give a clue to the stronger result. In this connection David Pincus has recently indicated a method of converting *FMS* models to Cohen models which gives the stronger result in many cases. In this paper we will give an example of each eventuality mentioned above. Consider the following maximal principles:

PI. Every Boolean algebra has a prime ideal.

SPI. Every infinite set algebra has a nonprincipal prime ideal.

K. (Kurepa's principle). Every family of sets includes a maximal subfamily of pairwise incomparable sets. (Two sets are comparable if one is a subset of the other.)

We shall exhibit an *FMS* model in which *K* holds but C_2 (the

axiom of choice restricted to families whose elements are pairs) does not. On the other hand, Ulrich Felgner, [1], has shown that K, (actually, an equivalent of K called Anti K) together with regularity implies AC. Hence we have an example of (1) above. In pre-Cohen days, some logicians felt that AC might be indeed a consequence of regularity. As we see now, this feeling was incorrect as stated, but not so far off as one might think, since K is a very weak maximal principal by one scale (in the absence of regularity it doesn't imply C_2 or for that matter C_n for any natural number n). Also it does not exclude the existence of infinite Dedekind finite sets.

SPI also holds in the above model. Since $PI \rightarrow C_2$ we obtain the independence result $SPI \nrightarrow PI$ in the absence of regularity, thus answering a question of Tarski, [7]. However using a method of David Pincus we are able to obtain a Cohen model in which $SPI \nrightarrow PI$, thus answering Tarski's question in full. This stronger result will appear in a subsequent paper. Although the original FMS proof of this result did give a clue to the proof of the stronger result, natural ways of converting it to a Cohen model proof did not work. New insight was required which gives rise to anew and conceptually more simple FMS proof. We shall also sketch this new FMS proof.

In §3 we prove, using the axiom of regularity, that what Felgner [1] would call the local version of SPI (actually we state an apparently weaker principle) implies the axiom of choice for families of finite sets. SPI-local and our version are as follows:

DEFINITION. A set X is *surveyable* if there is a choice function on $P_{<\omega}(X)$. ($P_{<\omega}(X)$ is the set of finite subsets of X and a choice function for any set Y is a function which assigns to every nonempty $y \in Y$, a member of y.)

Surveyable is a condition analogous to orderable and wellorderable. Clearly well-orderable $\rightarrow$ orderable $\rightarrow$ surveyable. Well known results of Mostowski and Läuchli establish that the converses are each false.

SPI-local: For any family of infinite sets there is a function which maps each member, Y, of the family to a nonprincipal prime ideal in $P(Y)$. ($P(Y)$ is the power set of Y.)

SPI-ls: (l is for "local" and s is for "surveyable".) The same as SPI-local except that "infinite" is replaced by "infinite and surveyable".

One reason for the interest in SPI is the result of Dana Scott (see [7]) that $SPI^* \leftrightarrow PI$ where SPI^* is "Any proper ideal in a set algebra is included in a prime ideal." Note that SPI is equivalent to: "The ideal of all finite sets in any infinite set algebra is included

in a prime ideal." It is known that PI-local and hence SPI^*-local is equivalent to PI. Our results show that SPI-local is not equivalent to SPI and leave open the question: "Does SPI-local imply PI?"[1]

    An earlier version of this paper contained the claim that SPI-local is true in the model M of §2. We are indebted to Paul Howard for pointing out this error and for other comments which led to the investigations concerning SPI-ls.

    2.   The Fraenkel-Mostowski model,   M.   Let $\mathscr{F}$ denote the set theory whose axioms are those of ZF with regularity and AC excluded.[2] A transitive model of $\mathscr{F}$ or any subtheory of $\mathscr{F}$ is a model whose universe is a transitive set and whose membership relation is the usual membership relation restricted to its universe. We will identify a transitive model with its universe. An element in the universe of the model will be called a set of the model. A supercomplete transitive model is a transitive model satisfying the property that any subset of an element of the model is also an element of the model i.e. $x \in M \rightarrow P(x) \subseteq M$. Existence of a supercomplete model of $\mathscr{F} + AC$ is implied by the existence of a strongly inaccessible cardinal. We start with a supercomplete transitive model M_1 of $\mathscr{F} + AC$ which has an element S such that:
    (1)   Every element of S is reflexive (x is reflexive if $x = \{x\}$).
    (2)   Any nonempty set x of M_1 contains an element y such that $y \cap x = 0$ or $y \cap x \in S$.
    (3) S is denumerably infinite.[3]
Let R be the inductively defined function on the ordinals in M_1, such that

$$R(0) = S$$
$$R(\beta) = \bigcup_{\lambda < \beta} P(R(\lambda)) \ (P \text{ denotes the power set}).$$

From the assumptions on M_1, one easily shows that any set of M_1 is a member of $R(\beta)$ for some ordinal β in M_1. Let G be the group of all automorphisms of M_1, i.e. G consists of all permutations ϕ of M_1 such that $x \in y \leftrightarrow \phi(x) \in \phi(y)$ for any x, y in M_1. Using the above characterization of M_1 one easily verifies that any permutation ϕ of S induces a

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unique automorphism of M_1 coinciding with ϕ on S.

Let the variable e with or without subscripts range over finite subsets of S. Let $G[e] = \{\phi \in G: \phi$ leaves e pointwise fixed$\}$. If $\phi(x) = x$ for all $\phi \in G[e]$, we say that x is *e-symmetric* and e is a *support of x*. x is *finitely symmetric* if it is e-symmetric for some e. Let $M = \{x \in M_1: x$ and every member of the transitive closure of x is finitely symmetric$\}$. M is obviously a transitive set. Furthermore one can use the arguments in [4] to show that M is a model of $\mathscr{F}$; hence M is a transitive model of F. It is easy to verify that S is a Dedekind finite set in M.

The following facts about M will prove useful:

1. If x is finitely symmetric and $x \subseteq M$ then $x \in M$.
2. If x is finite and $x \subseteq M$ then $x \in M$.
3. M is closed under G. G is a group of automorphisms of M. If f is an n-ary operation on M definable in the theory $\mathscr{F}$ then

$$\phi(f(x_1, \cdots, x_n)) = f(\phi x_1, \cdots, \phi x_n) \text{ all } \phi \in G, x_1, \cdots, x_n \in M.$$

4. If $x \in M_1$ and the transitive closure of x has empty intersection with S then $x \in M$. In particular all constructible sets in M_1 are in M.

We will call the definition of a predicate or operation *absolute* if given any transitive model and sequence of elements in the model, the definition is satisfied by the sequence in the model if and only if it is satisfied by the sequence in the universe. A predicate or function will be called absolute if its usual definition is absolute. This notion of absolute is intended to be the semantic analogue of the notion given in [5] Pg.265. The notions probably differ but the difference will not appear in this paper or in any of the existing work on models of set theory. In particular most of the definitions in the development of general set theory are absolute (in both senses). The power set operation is a notable exception. For a list more than sufficient for this paper, see [5] Pgs. 267–269. In addition we note that $P_M x = Px \cap M$ for $x \in M$. (Where P_M denotes the power set operation in the model M.)

Using these remarks it is not difficult to show that the truth of SPI in M is equivalent to:

SPI_M: If $X \in M$ is an infinite set, there is a subset I of PX with the following properties

(0) $I \in M$.

(i) I contains as members all finite subsets of X.

(ii) I is closed under finite union.

(iii) For every $y \in P_M X$, $y \in I \leftrightarrow X - y \notin I$.

The truth of K in M is equivalent to:

K_M: If $F \in M$, there is a maximal set among those sets $y \subseteq F$ with the properties

(0) $M(y)$

(i)   Members of y are pairwise incomparable.

The falsity of C_n in M is equivalent to $(\neg C_n)_M$:   There exists $y \in M$, whose elements are disjoint n-sets and there is no $Z \in M$ whose intersection with each element of Y is a singleten.   We proceed to prove SPI_M, K_M, $(\neg C_n)_M$.

LEMMA 1. $G[e_1 \cap e_2]$ *is the smallest subgroup of* G *including* $G[e_1] \cup G[e_2]$.

Proof.   Clearly $G[e_1]$, $G[e_2]$ are both included in $G[e_1 \cap e_2]$.   It remains to prove that $G[e_1 \cap e_2]$ is included in any subgroup which includes $G[e_1] \cup G[e_2]$.   Let $\phi \in G[e_1 \cap e_2]$.   Let $e_1 - e_2 = \{a_1, \cdots, a_k\}$ and let $c_1, \cdots, c_k$ be distinct elements not in $e_1 \cup \phi^{-1}(e_2) \cup e_2$.   Let $\phi_1 \in G$ such that ϕ_1 permutes a_i with c_i for $i < k$ and is constant elsewhere on S.   Then $\phi_1 \in G[e_2]$.   Let $e_2 - e_1 = \{b_1, \cdots, b_p\}$.   Since $\phi_1 \phi^{-1}(b_i) \notin e_1$ and $b_i \notin e_1$ for $i < p$, there is $\phi_2 \in G$ such that $\phi_2(b_i) = \phi_1 \phi^{-1}(b_i)$ for $i < p$ and $\phi_2 \in G[e_1]$.   Let $\phi_3 = \phi \phi_1^{-1} \phi_2$.   Then $\phi_3(b_i) = \phi \phi_1^{-1} \phi_2(b_i) = \phi \phi_1^{-1} \phi_1 \phi^{-1}(b_i) = b_i$.   Also $\phi_3 \in G[e_1 \cap e_2]$.   Thus $\phi_3 \in G[e_2]$.   But $\phi = \phi_3 \phi_2^{-1} \phi_1$, thus ϕ is any subgroup including $G[e_1] \cup G[e_2]$.

COROLLARY 2.   *The intersection over the set of supports of a set* x *is a support of* x.

Proof.   Since supports are finite, the given intersection is also the intersection over some finite nonempty subset of the set of supports. It follows from Lemma 1 that the intersection of two supports of x is again a support of x since any element in the subgroup generated by $G[e_1] \cup G[e_2]$ can be represented as the composition of a finite number of automorphisms each of which pointwise fixes e_1 or e_2 and hence fixes x.   With a trivial induction one obtains the same result for any finite number of supports.

DEFINITION.   We call the smallest support of x, *the support of* x and denote it by $F(x)$.

LEMMA 3.   *If* $\phi, \psi \in G$ *and* $\phi \mid F(x) = \psi \mid F(x)$ *then* $\phi(x) = \psi(x)$.

Proof.   By hypothesis $\phi^{-1}\psi$ fixes $F(x)$ and hence $\phi^{-1}\psi(x) = x$, i.e. $\psi(x) = \phi(x)$.

LEMMA 4.   *If* $\phi \in G$ *and* $\phi(x) = x$, *then* $\phi(F(x)) = F(x)$.

Proof.   Let ψ be any automorphism pointwise fixing $\phi(F(x))$.   Then $\psi\phi(x) = \phi(x) = x$ by Lemma 3 and the second assumption.   Thus $\phi(F(x))$

is a support of x and hence $F(x) \subseteq \phi(F(x))$. Since $F(x)$ is finite this can only happen if $\phi(F(x)) = F(x)$.

Note. ϕ need not pointwise fix $F(x)$.

LEMMA 5. *If* $\phi \in G$, *then* $\phi(F(x)) = F(\phi(x))$.

Proof. First we note that if ψ pointwise fixes $\phi(F(x))$ then $\psi\phi(x) = \phi(x)$ by Lemma 3. Hence $\phi(F(x)) \supseteq F(\phi(x))$. Since this inclusion holds for all x and ϕ we have $\phi^{-1}F(\phi(x)) \supseteq F(x)$ i.e., $F(\phi(x)) \supseteq \phi(F(x))$ which completes the proof.

In preparation for the proof of SPI_M, let us consider a fixed infinite set X of M. For each $y \in X$, let

$$f(y) = F(y) - F(X)$$
$$T(y) = \{z: \exists\phi[\phi \in G[F(X)] \wedge \phi(y) = z]\} .$$

$T(y)$ is the transitivity class of y under $G[F(X)]$. Using Lemma 5 one easily shows

(1) $$\phi \in G[F(X)] \to \phi(f(y)) = f(\phi(y)) .$$

The following fact is the basic idea for the proof of SPI_M.

(2)   Let $y \in X$ and $|T(y)| > 1$.
        Let $Q(w) \leftrightarrow \exists A$, finite $\forall z[z \in T(y) \cap w \to A \cap f(z) \neq 0]$.
        If $w \in P_M(X)$, then exactly one of $Q(w)$, $Q(X - w)$ hold.

Proof. Suppose neither hold. Then we obtain $z \in T(y) \cap w$, $t \in T(y) \cap (X - w)$ such that $f(z) \cap F(w) = f(t) \cap F(w) = 0$. Since z, $t \in T(y)$, there exists $\phi \in G[F(X)]$ such that $\phi(z) = t$. Since $f(z)$, $f(t)$ are disjoint from $F(X)$, there exists $\psi \in G[F(X)] \cap G[F(w)]$ such that $\psi|f(z) = \phi|f(z)$. By Lemma 3 we have $\psi(z) = \phi(z) = t$. Since ψ pointwise fixes $F(w)$, we conclude $t \in w$ which is a contradiction.

Suppose both $Q(w)$ and $Q(X - w)$ hold. Let A_1, A_2 be the finite sets whose existence is asserted by $Q(w)$ and $Q(X - w)$ respectively. Let $\phi \in G[F(X)]$ map $f(y)$ onto $S - (A_1 \cup A_2)$. Since $\phi(f(y)) = f(\phi(y))$ by (1), we have $f(\phi(y)) \cap (A_1 \cup A_2) = 0$. But $\phi(y) \in T(y) \cap X = (T(y) \cap w) \cup (T(y) \cap (X - w))$ is a contradiction. This completes the proof of (2).

Proof of SPI_M. We consider two cases:

Case 1. There exists $y_0 \in X$ such that $T(y_0)$ is infinite.
Case 2. $T(y)$ is finite for every $y \in X$.

For Case 1 we use (2) to obtain the desired result. No maximal principle is used here. For Case 2 the result follows directly from the fact that SPI is true in M_1.

Proof of Case 1. Let I consist of all $w \in P_M(X)$ such that $Q(w)$. We first show that I is closed under $G[F(X)]$. If $\phi \in G[F(X)]$ and $w \in I$, then $\phi(w) \subseteq X$ since ϕ is an automorphism neutral on X and $\phi(w) \in M$ since M is closed under G. Furthermore $\phi(T(y_0) \cap w) = T(y_0) \cap \phi(w)$ since $T(y_0)$ is closed under $G[F(X)]$ and ϕ is an automorphism. Since $Q(w)$ holds we have an A such that

$$z \in T(y_0) \cap w \rightarrow A \cap f(z) \neq 0 .$$

Thus $\phi(z) \in T(y_0) \cap \phi(w) \rightarrow \phi(A) \cap f(\phi(z)) \neq 0$, using the automorphism property of ϕ and (1). But any element of $T(y) \cap \phi(w)$ is $\phi(z)$ for some $z \in T(y) \cap w$. Thus $\phi(w) \in I$ and I is closed under $G[F(X)]$. Hence I is finitely symmetric and since $I \subseteq M$ we have

$$(0) \qquad\qquad\qquad I \in M .$$

If w is finite and $w \subseteq X$, then $w \in M$. Also $Q(w)$ because $\cup \{f(z) : z \in w\}$ is the required finite A. Thus

(i) I contains as members all finite subsets of X.

If $w, z \in I$, then $w \cup z \in P_M(X)$ and the required A to establish $Q(w \cup z)$ is just the union of the two sets which establish $Q(w)$, $Q(z)$ respectively. Thus,

(ii) I is closed under finite union. We obtain directly from (2) that

(iii) for every $y \in P_M X$, $y \in I \leftrightarrow X - y \notin I$. This completes the proof of case 1.

Case 2. Suppose $T(y)$ is finite for every $y \in X$. Then $F(y) \subseteq F(X)$. For suppose on the contrary that there exists $a \in F(y) - F(X)$. Then for each $b \in S - F(X)$ there exists $\phi \in G[F(X)]$ such that $\phi(a) = b$. From this fact and Lemma 4 one easily concludes the contradiction that $T(y)$ is infinite, Thus $F(y) \subseteq F(X)$. From the definition of $T(y)$ we see that actually $T(y)$ contains only the element y. However the important point is that there is a support which supports every element of X. This is a very strong condition on X. In particular it implies that any subset of $P(X)$ belongs to M. With the aid of SPI, we obtain a nonprincipal prime ideal I, in the algebra of sets PX. By the previous remarks $I \in M$.

Proof of K_M. Let $X \in M$. Then the set $Z = \{Y : Y \subseteq X \wedge Y \in M \wedge$ members of Y are pairwise incomparable $\wedge F(Y) \subseteq F(X)\}$ has a

maximal element: Using the fact that every element of Z is supported by $F(X)$ we establish that the union over a chain in Z is a member of Z. Z is nonempty since $0 \in Z$. With the aid of Zorn's form of AC, we conclude that Z has a maximal element Y_0.

We proceed to show that Y_0 is the desired maximal set. Suppose not. Then there exists $y \in X - Y_0$ such that $Y_0 \cup \{y\}$ consists of incomparable sets. Let $Y_1 = Y_0 \cup \{\phi(y) \colon \phi \in G[F(X)]\}$. Then since $Y_0 \subsetneqq Y_1 \subseteqq X$ and $Y_1 \in M$ and $F(Y_1) \subseteqq F(X)$ and Y_0 is maximal in Z, we conclude that Y_1 includes two comparable elements. We have two cases.

Case 1.   There is $z \in Y_0$ and $\phi \in G[F(X)]$ such that $z \subseteqq \phi(y)$ or $\phi(y) \subseteqq z$.

Case 2.   There are $\phi_1, \phi_2 \in G[F(X)]$ such that $\phi_1(y) \neq \phi_2(y)$ and $\phi_1(y) \subseteqq \phi_2(y)$.

If Case 1 holds we have $\phi^{-1}(z) \subseteqq y$ or $y \subseteqq \phi^{-1}(z)$. Since $\phi^{-1}(z) \in Y_0$, this contradicts $Y_0 \cup \{y\}$ consists of incomparable sets.

If Case 2 holds, we have $y \subseteqq \phi(y)$ and $y \neq \phi(y)$ for $\phi = \phi_1^{-1}\phi_2$. Let

$$A_1 = \{w \in F(y) \colon \exists n[n > 0 \wedge \phi^n(w) = w]\}$$
$$A_2 = \{z \in S \colon \exists w \exists n[w \in A_1, \wedge \phi^n(w) = z]\} \, .$$

A_2 is finite since A_1 is finite and every element of A_1 has only finitely many images under successive iteration of ϕ. Also A_2 is closed under ϕ. Thus $\phi \,|\, A_2$ generates a cyclic group which is a group of permutations of a finite set, and thus has finite order say n_0.

If $z \in F(y) - A_1$, there is a natural number n_z, such that if $m > n_z$, $\phi^m(z) \notin F(y)$. This assertion follows from the finiteness of $F(y)$ and the fact that for each $n > 0$, $\phi^n(z) \neq z$.

Let m = the first multiple of n_0 greater than $\max\{n_z \colon z \in F(y) - A_1\}$. Then $\phi^m(w) = w$ if $w \in A_2$ and, hence, if $w \in A_1$. Also, $\phi^m(z) \notin F(y)$ if $z \in F(y) - A_1$. Thus if $z \in F(y) \cap \{\phi^m(w) \colon w \in F(y)\}$ then $\phi^m(z) = z$. Hence there is a permutation ψ of S such that

   (a) $\psi(z) = \phi^m(z)$ for $z \in F(y)$
   (b) $\psi(\phi^m(z)) = z$ for $z \in F(y)$.
From (a) we obtain
   (c) $\psi(y) = \phi^m(y)$ since $\psi^{-1}\phi^m$ is the identity on $F(y)$. From (b) we obtain
   (d) $\psi\phi^m(y) = y$.
By hypothesis $y \subseteqq \phi(y)$ and since ϕ is an automorphism we have
   (e) $y \subseteqq \phi(y) \cdots \subseteqq \phi^m(y)$.

Thus by (c), $y \subseteqq \psi(y)$. But (e) also yields $\psi(y) \subseteqq \psi\phi^m(y)$. Applying (d), we have $\psi(y) \subseteqq y$. Thus $y = \psi(y)$. From (c) we deduce $y = \phi^m(y)$.

But this together with (e) yields the contradiction $y = \phi(y)$.

Proof of $(\neg C_n)_M$ *for* $n > 0$:

Let $Y = \{\{(X, x): x \in X\}: X \subseteq S \wedge |X| = n\}$. $Y \in M$ since M is a model of $\mathscr{F}$ and the relations and operations used in its definition are absolute and the fact that x finite and $x \subseteq y \in M \Rightarrow x \in M$. Also Y consists of disjoint sets of cardinality n. Suppose Z is a choice set for Y and $Z \in M$. Since S is infinite and $F(Z)$ is finite we can find $X \subseteq S$, $|X| = n$, and $X \cap Z = 0$. Let $\phi \in G[F(Z)]$ such that $\phi(X) = X$ and $\phi(x) \neq x$ all $x \in X$. Then if $(X, x) \in Z$, $(X, \phi(x)) \in Z$ and $x \neq \phi(x)$, contradicting Z is a choice set for Y.

The new proof of $SPI \nrightarrow PI$ to which allusion was made in the introduction comes about as follows:

(1) It is easy to verify that if for every countably infinite Y, $P(Y)$ has a nonprinciple prime ideal then for every Dedekind infinite X, $P(X)$ has a nonprincipal prime ideal. (If X is Dedekind infinite and $Y \subseteq X$ is countably infinite and I is a nonprincipal prime ideal in $P(Y)$ then $\{Z: Z \subseteq X \ \& \ Z \cap Y \in I\}$ is a nonprincipal prime ideal in $P(X)$.)

(2) If M is any *FMS* model and $Y \in M$ and Y is well-orderable within M (i.e. in the M-sense) then $P_M(Y) = P(Y)$ and $P_M P_M(Y) = P(P(Y))$.

Since we assume AC holds in the universe it follows from (1) & (2) that if M is an *FMS* model in which every nonfinite set of the model is Dedekind infinite in the M-sense then SPI holds in the model. In [3] Azriel Levy considers a whole class of such models. It is easy to verify that C_2 fails in these models. Hence any one of these models establishes our independence $SPI \nrightarrow PI$.[4]

Actually this argument establishes the independence (SPI & [Dedekind infinite = infinite] $\nrightarrow C_2$). In light of this argument the following stronger version of the original result is interesting: ($SPI \nrightarrow$ [infinite = Dedekind infinite] or C_2).[4]

3. *SPI-ls* $\rightarrow C_{<\omega}$.

Let $P(X)$ denote the power set of X. Let $P_n(X)$ denote the set of all n element subsets of X, for any natural number n, and let $P_{<\omega}(X)$ denote the set of all finite subsets of X. We prove the following theorem which is apparently stronger than the title of this section since it omits the "nonprinciple".

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[4] The referee has informed us that this proof was discovered independently by Norman Noble.

THEOREM. *If for every set X there is a function which assigns to every infinite surveyable set $Y \subseteq X$ a prime ideal in $P(Y)$, then for every set X there is a choice function for $P_{<\omega}(X)$.*

A function such as the one hypothesized for X in the theorem will be called a *PI* function for X. A disjointed set is one whose elements are pairwise disjoint.

LEMMA 1. *Given f a choice function on $P_{<\omega}(X)$ and g, a PI function for X, one can define a choice function h on the set of finite disjointed subsets of $P(X)$.*

Proof. Let Y be a finite disjointed subset of $P(X)$.

Case 1. $\cup Y$ is finite. Then $\cup Y \in P_{<\omega}(X)$. Since Y is disjointed we can define $h(Y) = $ unique $y \in Y$ such that $f(\cup Y) \in y$.

Case 2. $\cup Y$ is infinite. Then there is exactly one $y \in Y$ such that $y \notin g(\cup Y)$. Define $h(Y)$ to be this unique y.

LEMMA 2. *Given h a choice function on the set of finite disjointed subsets of $P(X)$ one can define a choice function k on $P_{<\omega}(P(X))$.*

Proof. We shall define k on $P_n(P(X))$ by recursion on n. For $u \in P_1(P(x))$ we define $k(u) = h(u)$. Suppose $n > 1$ and $u \in P_n(P(X))$. Let $w = \{z: z$ is an atom in the subalgebra of $P(X)$ generated by $u \wedge z \neq \cap u \wedge z \neq 0 \wedge z \neq \cap \{X - x: x \in u\}\}$. w is a finite disjointed subset of $P(X)$ and since $|u| > 1$, w is nonempty. Let $u_1 = \{x \in u: f(w) \subseteq x\}$, $u_1 \neq 0$ since $f(w) \in w$ and thus is included in some element of u. $|u_1| < n$ since $f(w) \in w$ and thus is included in the complement of some element in u and thus (since $f(w) \neq 0$) $f(w)$ is not included in this element of u. So we can define $k(u) = k(u_1)$.

Proof of the theorem. Let τ be the 1-place term of Zermelo Fraenkel set theory defined by ordinal recursion as follows:

$$\tau(\alpha) = P(\bigcup_{\beta < \alpha} \tau(\beta)) .$$

It follows from the axiom of regularity that $\forall\, x\, \exists\, \alpha[x \subseteq \tau(\alpha)]$. We shall prove the theorem by showing that for each ordinal α there is a function g such that for all $\beta < \alpha$, $g(\beta)$ is a choice function on $P_{<\omega}(\tau(\beta))$. We define g by ordinal recursion up to α. Let f be a *PI* function for $\tau(\alpha)$. Suppose, for each $\gamma < \beta$, $g(\gamma)$ is a choice function on $P_{<\omega}(\tau(\gamma))$ such that $g(\gamma_1) \subseteq g(\gamma_2)$ if $\gamma_1 \leq \gamma_2$. Then $\bigcup_{\gamma < \beta} g(\gamma)$ is a choice function on $P_{<\omega}(\bigcup_{\gamma < \beta} \tau(\gamma))$. Since $\beta < \alpha$, $\bigcup_{\gamma < \beta} \tau(\gamma) \subseteq \tau(\alpha)$. Hence f is a *PI* function for $\bigcup_{\gamma < \beta} \tau(\gamma)$. Applications of Lemmas 1 and 2 allow us to define a

choice function on $P_{<\omega}(\tau(\beta))$.   Call this function g'.   In order to satisfy the recursion condition we define

$$g(\beta) = (g'[P_{<\omega}(\tau(\beta)) - \bigcup_{\gamma < \beta} P_{<\omega}(\tau(\gamma))]) \cup \bigcup_{\gamma < \beta} g(\gamma) \ .$$

REFERENCES

1. Ulrich Felgner, *Die existenz wohlgeordneter konfinaler Tielmengen in Ketten und das Auswahlaxiom*, Math. z., **111** (1969), 221-232.
2. Azriel Lévy, *The Fraenkel-Mostowski method for independence proofs in set theory*, Symposium on the Theory of Models, North-Holland (1965) 221-228.
3. ———, *The interdependence of certain consequences of the axiom of choice*, Fund. Math., **54** (1964), 135-157.
4. A. Mostowski, *Uber die Unabhängigkeit des Wohlordnungssatz vom Ordnungsprinzip*, Fund. Math., **32** (1939), 201-252.
5. J. R. Shoenfield, *Mathematical Logic*, Addison-Wesley Publishing Co. 1967.
6. E. Specker, *Zur Axiomatic der Mengenlehre* (Fundierungs und Auswahlaxiom), Zeitschrift f. Math. Logik und Grundlagen d. Math., **3** (1957), 173-210.
7. A. Tarski, *Prime ideal theorems for set algebras and the axiom of choice*, B.A.M.S. **60** (1954), 391.

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