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ABSOLUTE TOTAL-EFFECTIVE $(N, p_n)(C, 1)$ METHOD

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In the direction of the total-effectiveness of a $(N, p_n)(C, 1)$ method, results concerning the summability of a Lebesgue Fourier series and its conjugate series by such a method are known. Supporting the observation that generally bounded variation is the property associated with absolute summability in the same way in which continuity is associated with ordinary summability, the absolute total-effectiveness of a $(N, p_n)(C, 1)$ method is established in the present paper and the corresponding effectiveness of the (C) method is deduced as a particular case.

Throughout the present paper we use the definitions and notations of [7] without further explanation. The following additional notations for the conditions concerning $\{p_n\}$ are also used.

(1.1) $\{p_n\} \in RS$ means: $p_0 > 0$, $p_n \geqq 0$ $(n \geqq 1)$, $\{R_n\} \in BV$ and $\{S_n\} \in B$;

(1.2) $\{p_n\} \in MS$ means: $p_n > 0$, $p_{n+1}/p_n \leqq p_{n+2}/p_{n+1} \leqq 1$ $(n \geqq 0)$ and $\{S_n\} \in B$;

(1.3) $\{p_n\} \in NS$ means: $p_0 > 0$, $p_n \geqq 0$ $(n \geqq 1)$, $\{R_n\} \in B$, $\{S_n\} \in B$, $\{p_n\}$ and $\{\Delta p_n\}$ monotone .

As we shall see in section 5 of the present paper, $MS \subset NS$, but no interrelation is known between the sets of conditions RS and MS or NS.

Using a result due to Mears [15], Kwee [13] has proved that the following conditions:

$$(1.4) \qquad p_n = o(|P_n|),\ n \to \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \left| \frac{P_n}{P_{n+\nu}} - \frac{P_{n-1}}{P_{n+\nu-1}} \right| < \infty ,$$

for all $\nu \geqq 1$, are necessary and sufficient for the absolute regularity of the (N, p_n) method. It may be observed that Lemma 1 and Lemma 2 of the present paper imply *a fortiori* that the (N, p_n) method is absolutely regular, under each of the conditions: $\{p_n\} \in RS$, $\{p_n\} \in NS$ and $\{p_n\} \in MS$.

Concerning the absolute Fourier-effectiveness, we have the following.

THEOREM A. *If $\{p_n\} \in RS$, then the (N, p_n) method is absolute Fourier-effective.*

THEOREM B. *The (C, δ) method is absolute Fourier-effective for every $\delta > 0$.*

As pointed out by the present author in [4], Theorem A is an apparently lighter but actually equivalent version of results of Pati [17]. Theorem B emerges from Bosanquet [1] and Bosanquet and Hyslop ([2], Th. K, with $\alpha = 0$) and is known to be the best possible in the sense that it breaks down if $\delta = 0$. It may also be mentioned that Theorem B is a special case of Theorem A.

$|F|$-effective part of Theorem B may also be deduced as a special case of the corresponding effectiveness of the (N, p_n) method proved in [6] and [5], under the hypothesis: $\{p_n\} \in MS$ or more generally that $\{p_n\} \in NS$.

The following result emerges from ([10], Ths. 1, 2 and 3), when we observe that its $|F'|$-effective part is deducible from the proof of Theorem 1 in [10], while its absolute Fourier-effective part, follows from the result of Theorem A and the absolute regularity of the $(C, 1)$ method.

THEOREM C. *If $\{p_n\} \in RS$, then the $(C, 1)(N, p_n)$ method is absolute total-effective.*

However, in the direction of the absolute total-effectiveness of the $(N, p_n)(C, 1)$ method[1], we have only succeeded in proving the following (cf. [9]).

THEOREM D. *If $\{p_n\} \in RS$, then the $(N, p_n)(C, 1)$ method is $|F_1|$- and $|F'|$-effective.*

2. **The main results.** That under the hypothesis: $\{p_n\} \in MS$, it is indeed possible to prove a more *powerful* effectiveness of the $(N, p_n)(C, 1)$ method than that obtained in Theorem D is demonstrated by our Theorem 2, where we succeed in establishing absolute total-effectiveness of such a method. The absolute total-effectiveness of the $(C, 1 + \delta)$ method $(\delta > 0)$, which emerges from Bosanquet [1], Bosanquet and Hyslop ([2], Th. 1 for $\alpha = 0$ and Th. 5) and Hyslop [11], reduces to a special case of our Theorem 2.

We first prove the following which corresponds to Theorem D.

THEOREM 1. *If $\{p_n\} \in NS$, then the $(N, p_n)(C, 1)$ method is $|F_1|$- and $|F'|$-effective.*

[1] It is known [18] that the matrix $(C, 1)(N, p_n) \neq (N, p_n)(C, 1)$, unless (N, p_n) is a Cesàro matrix.

Using the result of Theorem 1, we prove:

THEOREM 2. *If* $\{p_n\} \in MS$, *then the* $(N, p_n)(C, 1)$ *method is absolute total-effective.*

In view of the result of Das ([3], Th. 5) that if $p_n > 0$, $p_{n+1}/p_n \leqq p_{n+2}/p_{n+1} \leqq 1$ $(n \geqq 0)$, then

$$|(N, p_n)(C, 1)| \sim |N, P_n| \sim |(C, 1)(N, p_n)| ,$$

the absolute total-effectiveness of the (N, P_n) and the $(C, 1)(N, p_n)$ methods, under the hypotheses: $\{p_n\} \in MS$, follow from the result of Theorem 2.

3. **Some preliminary results.** We need the following lemmas for the proofs of our theorems.

LEMMA 1. *If* $\{p_n\} \in RS$, *then the* (N, p_n) *method is absolutely regular.*

Lemma 1 is included in Lemma 8 of [8].

LEMMA 2. *If* $p_0 > 0$, $p_n \geqq 0$ $(n \geqq 1)$, $\{p_n\}$ *is monotone and* $\{R_n\} \in B$, *then the* (N, p_n) *method is absolutely regular.*

Proof. Since $\{R_n\} \in B$ implies that $p_n = o(P_n)$, $n \to \infty$, in order to prove Lemma 2, it is sufficient to show that for all $\nu \geqq 1$

$$\Sigma^* = \sum_{n=0}^{\infty} \left| \frac{P_n}{P_{n+\nu}} - \frac{P_{n-1}}{P_{n+\nu-1}} \right| = \sum_{n=0}^{\infty} \frac{P_n}{P_{n+\nu-1}} \left| \frac{p_n}{P_n} - \frac{p_{n+\nu}}{P_{n+\nu}} \right| \leqq K . \tag{3.1}$$

The case in which $\{p_n\}$ is monotonic nonincreasing, (3.1) follows directly from Corollary 1 due to Mears [15].

Since $\{P_n\}$ is positive monotonic nondecreasing, we have by suitable changes in orders of summations (cf. [9], proof of Lemma 10)

$$\begin{aligned}
\Sigma^* &\leqq \sum_{n=0}^{\infty} \frac{P_n}{P_{n+\nu-1}} \sum_{k=n}^{n+\nu-1} \left| \Delta\left(\frac{p_k}{P_k}\right) \right| \\
&= \sum_{k=0}^{\nu-1} \left| \Delta\left(\frac{p_k}{P_k}\right) \right| \sum_{n=0}^{k} \frac{P_n}{P_{n+\nu-1}} \\
&\quad + \sum_{k=\nu}^{\infty} \left| \Delta\left(\frac{p_k}{P_k}\right) \right| \sum_{n=k-\nu+1}^{k} \frac{P_n}{P_{n+\nu-1}} \\
&\leqq \frac{1}{P_{\nu-1}} \sum_{k=0}^{\nu-1} (k+1) P_k \left| \frac{\Delta p_k}{P_k} + \frac{(p_{k+1})^2}{P_k P_{k+1}} \right| + \nu \sum_{k=\nu}^{\infty} \left| \frac{\Delta p_k}{P_k} + \frac{(p_{k+1})^2}{P_k P_{k+1}} \right| \\
&= \Sigma_1^* + \Sigma_2^* ,
\end{aligned}$$

say. We now assume, that $\{p_n\}$ is monotonic nondecreasing and therefore

$$\begin{aligned}\Sigma_1^* &\leqq \frac{\nu}{P_{\nu-1}}\sum_{k=0}^{\nu-1}(p_{k+1}-p_k)+\frac{1}{P_{\nu-1}}\sum_{k=0}^{\nu-1}p_{k+1}R_{k+1}\\ &\leqq KR_\nu + K \leqq K\,,\end{aligned}$$

by virture of the hypothesis that $\{R_n\}\in B$, which implies that $P_n/P_{n-1}=O(1)$, $n\to\infty$. To prove that $\Sigma_2^*\leqq K$, we observe that by an application of Abel's transformation,

$$\begin{aligned}&\nu\sum_{k=\nu}^{n}\frac{1}{P_k}(p_{k+1}-p_k)+\nu\sum_{k=\nu}^{n}\frac{(R_{k+1})^2}{(k+1)^2}\\ &\leqq \nu\sum_{k=\nu}^{n-1}\frac{p_{k+1}}{P_kP_{k+1}}\sum_{\mu=\nu}^{k}(p_{\mu+1}-p_\mu)+\nu\frac{1}{P_n}\sum_{\mu=\nu}^{n}(p_{\mu+1}-p_\mu)+K\\ &\leqq K\nu\sum_{k=\nu}^{n-1}\frac{(R_{k+1})^2}{(k+1)^2}+K\nu\frac{R_{n+1}}{n}+K\\ &\leqq K\,,\end{aligned}$$

as $n\to\infty$, by virtue of the conditions: $\{p_n\}$ is monotonic nondecreasing and $\{R_n\}\in B$. Thus, $\Sigma_2^*\leqq K$, and we complete the proof of (3.1) in the case in which $\{p_n\}$ is monotonic nondecreasing. This completes the proof of Lemma 2.

The condition $\{R_n\}\in B$ is automatically satisfied if $\{p_n\}$ is nonnegative and nonincreasing and thus, we observe that the present Lemma 2 extends the result of Corollary 1 of [15].

LEMMA 3. *If* $\{p_n\}\in NS$, *then uniformly in* $0<t\leqq\pi$

$$\sum_{n=1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_k-p_nP_k)\frac{\sin\left(n-k+\frac{1}{2}\right)t}{n-k+\frac{1}{2}}\right|\leqq K\,. \tag{3.2}$$

Proof. The proof of (3.2) is similar to the proof of the following as given in [5].

$$\sum_{n=1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_k-p_nP_k)\frac{\sin(n-k)t}{n-k}\right|\leqq K\,.$$

LEMMA 4. *If* $\theta(t)\in BV(0,\pi)$ *and* $\{p_n\}\in NS$, *then* $\{t_n(u)\}\in BV$, *where* $t_n(u)$ *is the nth* (N,p_n) *mean of* $\{u_n\}$ *defined by*

$$u_n=\int_0^{\pi}\theta(t)\left\{\sin(n+1)t/\sin\left(\frac{1}{2}t\right)\right\}dt\,.$$

Proof. Following the proof of a theorem in Pati ([16], p. 156), we observe that if $\theta(t) \in BV(0, \pi)$, then in order to prove that $\{t_n(u)\} \in BV$, it is sufficient to show that (3.2) holds uniformly in $0 < t \leqq \pi$. Thus Lemma 4, follows from Lemma 3.

LEMMA 5. *If $\theta(t) \in BV(0, \pi)$, then $\{v_n\} \in BV$, where*

$$v_n = \frac{1}{n+1}\int_0^\pi \theta(t)\left\{\sin\left(\frac{1}{2}(n+1)t\right)\Big/\sin\left(\frac{1}{2}t\right)\right\}^2 dt .$$

Proof. Writing

$$v_n = \frac{1}{n+1}\sum_{k=0}^{n}\int_0^\pi \theta(t)\left\{\sin\left(k+\frac{1}{2}\right)t\Big/\sin\left(\frac{1}{2}t\right)\right\}dt ,$$

we observe that under the hypothesis: $\theta(t) \in BV(0, \pi)$, the result of Lemma 5 follows from the proof of $|F|$-effective part of Theorem B, when we appeal to a well known inclusion relation for the absolute (C) method.

LEMMA 6. *Let $t_n^1(s)$ denotes the nth $(N, p_n)(C, 1)$ mean of $\sum_{n=0}^\infty a_n$ and $p_n > 0$, $p_{n+1}/p_n \leqq p_{n+2}/p_{n+1} \leqq 1$, for all $n \geqq 0$. Then $\{t_n^1(s)\} \in BV$, if and only if*

$$\sum_{n=1}^{\infty}\frac{1}{nP_n}\left|\sum_{k=0}^{n} p_{n-k}\frac{1}{k+1}\sum_{r=0}^{k} ra_r\right| \leqq K .$$

Proof. We have by a change of order of summation

$$t_n^1(s) = \frac{1}{P_n}\sum_{k=0}^{n}\sum_{r=k}^{n}\frac{p_{n-r}}{r+1}s_k = \frac{1}{P_n}\sum_{r=0}^{n} p_{n-r}\sigma_r^1(s) , \tag{3.3}$$

where $\sigma_r^1(s)$ is the rth $(C, 1)$ mean of $\sum_{n=0}^\infty a_n$. But, by a well known identity due to Kogbetliantz

$$k\{\sigma_k^1(s) - \sigma_{k-1}^1(s)\} = \frac{1}{k+1}\sum_{r=0}^{k} ra_r . \tag{3.4}$$

In view of (3.3) and (3.4), Lemma 6 follows directly from Theorem 6 of Das [3].

LEMMA 7. *If $\{p_n\}$ is nonnegative and nonincreasing, then for $0 \leqq a \leqq b \leqq \infty$, $0 \leqq t \leqq \pi$ and for any n and a,*

$$\left|\sum_{k=a}^{b} p_k \exp i(n-k)t\right| \leqq KP_\tau .$$

Lemma 7 is contained in [14].

LEMMA 8. *If $p_0 > 0$, $p_n \geqq 0$ $(n \geqq 1)$ and $\{R_n\} \in B$, then the condition: $\{S_n\} \in B$ implies that*

$$P_n \sum_{k=n}^{\infty} \frac{1}{(k+1)P_k} \leqq K, \qquad n = 0, 1, 2, \cdots .$$

The proof of Lemma 8 is contained in [4]. A slight modification in the proof given in [4] shows that the result of Lemma 8 holds, even without the hypothesis: $\{R_n\} \in B$.

4. Proof of Theorem 1. (I). $|F_1|$*-effectiveness*: Denoting by $\sigma_n^1(L(x))$, the nth $(C, 1)$ mean of $L(x)$, we have

$$\begin{aligned}\sigma_n^1(L(x)) &= \frac{1}{\pi(n+1)}\int_0^{\pi}\phi(t)\left\{\sum_{k=0}^{n} \sin\left(k+\frac{1}{2}\right)t/\sin\frac{1}{2}t\right\}dt \\ &= \frac{1}{\pi(n+1)}\int_0^{\pi}\phi(t)\left\{\sin\frac{1}{2}(n+1)t/\sin\frac{1}{2}t\right\}^2 dt .\end{aligned}$$

Writing $\theta(t) = \Phi_1(t)/\sin(1/2)t$, we have on integration by parts

$$\begin{aligned}\sigma_n^1(L(x)) &= \frac{\theta(\pi)}{\pi(n+1)}\left\{\sin\frac{1}{2}(n+1)\pi\right\}^2 - \frac{1}{2\pi}\int_0^{\pi}\theta(t)\left\{\sin(n+1)t/\sin\frac{1}{2}t\right\}dt \\ &\quad + \frac{1}{\pi(n+1)}\int_0^{\pi}\theta(t)\cos\frac{1}{2}t\left\{\sin\frac{1}{2}(n+1)t/\sin\frac{1}{2}t\right\}^2 dt \qquad (4.1) \\ &= w_n + Ku_n + Kv_n ,\end{aligned}$$

say. In view of (3.3), in order to prove the $|F_1|$-effectiveness of the $(N, p_n)(C, 1)$ method, it is enough to show that $\{t_n(u)\} \in BV$, $\{t_n(v)\} \in BV$ and $\{t_n(w)\} \in BV$, where $t_n(u)$, $t_n(v)$ and $t_n(w)$ are the nth (N, p_n) means of $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$, respectively.

Since x is a $|F_1|$-regular point, $\{t^{-1}\Phi_1(t)\} \in BV(0, \pi)$ and that $\{t_n(u)\} \in BV$, follows from Lemma 4. Similarly, $\{t_n(v)\} \in BV$, by virtue of Lemmas 2 and 5. Finally, we write (cf. [9])

$$w_n - w_{n-1} = \alpha_n + \beta_n ,$$

where

$$\alpha_n = (-1)^n A\Big/\left(n+\frac{1}{2}\right) = A\sin\left\{\left(n+\frac{1}{2}\right)\pi\right\}\Big/\left(n+\frac{1}{2}\right) ;$$

$$\beta_n = \begin{cases} -A\Big/\left\{2\left(n+\frac{1}{2}\right)(n+1)\right\} & (n \text{ even}) ; \\ -A/2n\left(n+\frac{1}{2}\right) & (n \text{ odd}) ; \end{cases}$$

and $A = \Phi_1(\pi)/\pi$.

Thus, in order to show that $\{t_n(w)\} \in BV$, it is enough to show that $\{t_n(\alpha)\} \in BV$ and $\{t_n(\beta)\} \in BV$, where $t_n(\alpha)$ and $t_n(\beta)$ are the nth (N, p_n) means of $\sum_{n=0}^{\infty} \alpha_n$ and $\sum_{n=0}^{\infty} \beta_n$, respectively.

Now, we have

$$t_n(\alpha) - t_{n-1}(\alpha) = \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k)\alpha_{n-k}$$

and therefore $\{t_n(\alpha)\} \in BV$ by virtue of (3.2).

That $\{t_n(\beta)\} \in BV$, follows from the absolute convergence of $\sum_{n=0}^{\infty} \beta_n$, when we appeal to the result of Lemma 2.

This completes the proof of $|F_1|$-effective part of Theorem 1.

(II). $|F'|$-*effectiveness*: Denoting by $\sigma_n^1(L'(x))$ the nth $(C, 1)$ mean of $L'(x)$, we have

$$\begin{aligned} \sigma_n^1(L'(x)) &= -\frac{1}{\pi(n+1)}\int_0^{\pi} \psi(t)\frac{d}{dt}\left\{\sin\frac{1}{2}(n+1)t/\sin\frac{1}{2}t\right\}^2 dt \\ &= -\frac{1}{2\pi}\int_0^{\pi}\left\{\psi(t)/\sin\frac{1}{2}t\right\}\left\{\sin(n+1)t/\sin\frac{1}{2}t\right\}dt \\ &\quad + \frac{1}{\pi(n+1)}\int_0^{\pi}\left\{\psi(t)/\tan\frac{1}{2}t\right\}\left\{\sin\frac{1}{2}(n+1)t/\sin\frac{1}{2}t\right\}^2 dt\,. \end{aligned} \tag{4.2}$$

Comparing (4.2) with (4.1), and observing that $\{\psi(t)/t\} \in BV(0, \pi)$, since x is $|F'|$-regular, it follows from Lemmas 2, 4 and 5 that the $(N, p_n)(C, 1)$ method is $|F'|$-effective.

5. **Proof of Theorem 2.(I).** $|F_1|$- *and* $|F'|$-*effectiveness*: We observe that if $p_n > 0$ and $p_{n+1}/p_n \leqq p_{n+2}/p_{n+1} \leqq 1$, for all $n \geqq 0$, then $(n+1)p_n \leqq P_n$, i.e. $\{R_n\} \in B$ and $\{\Delta p_n\}$ is monotonic nonincreasing, for (see [12])

$$\Delta p_n - \Delta p_{n+1} = p_n - 2p_{n+1} + p_{n+2} \geqq (\sqrt{p_n} - \sqrt{p_{n+2}})^2 \geqq 0$$

and therefore $\{p_n\} \in NS$. Thus $|F_1|$ – and $|F'|$-effective parts of Theorem 2 are included in Theorem 1.

(II). $|\tilde{F}_1|$-*effectiveness*: Since

$$nB_n(x) = -\frac{2}{\pi}\int_0^{\pi} \psi(t)\left(\frac{d}{dt}\cos nt\right)dt$$

and $\int_0^{\pi} t^{-1}|\psi(t)|dt \leqq K$, by virtue of the hypothesis that x is a $|\tilde{F}_1|$-regular point, it follows from Lemma 6, that in order to prove the $|\tilde{F}_1|$-effective part of Theorem 2, it is sufficient to show that uniformly in $0 < t \leqq \pi$

$$\Sigma = t\sum_{n=1}^{\infty}\frac{1}{nP_n}\left|\sum_{k=0}^{n}p_{n-k}\frac{1}{k+1}\frac{d}{dt}\left\{\sum_{r=0}^{k}\cos rt\right\}\right| \leqq K. \tag{5.1}$$

Now we write[2]

$$\begin{aligned}\Sigma &\leqq t\sum_{n\leqq\tau}\frac{1}{nP_n}\left|\sum_{k=0}^{n}p_{n-k}\frac{1}{k+1}\sum_{r=1}^{k}r\sin rt\right| \\ &+ t\sum_{n>\tau}\frac{1}{nP_n}\left|\sum_{k=0}^{n}p_{n-k}\frac{1}{k+1}\frac{d}{dt}\left\{\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}\right| \\ &= \Sigma_1 + \Sigma_2,\end{aligned}$$

say. But, we have

$$\Sigma_1 \leqq Kt\sum_{n\leqq\tau}\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k} \leqq K. \tag{5.2}$$

By virtue of Lemmas 7 and 8, we have

$$\begin{aligned}\Sigma_2 &\leqq K\sum_{n>\tau}\frac{1}{nP_n}\left|\sum_{k=0}^{n}p_{n-k}\cos\left(k+\frac{1}{2}\right)t\frac{\left(k+\frac{1}{2}\right)}{k+1}\right| \\ &+ Kt^{-1}\sum_{n>\tau}\frac{1}{nP_n}\left|\sum_{k=0}^{n}p_{n-k}\frac{1}{k+1}\sin\left(k+\frac{1}{2}\right)t\right| \\ &\leqq KP_\tau\sum_{n>\tau}\frac{1}{nP_n} + Kt^{-1}\sum_{n>\tau}\frac{1}{nP_n}\left|\sum_{k=0}^{[n/2]-1}p_{n-k}\frac{1}{k+1}\sin\left(k+\frac{1}{2}\right)t\right| \\ &+ Kt^{-1}\sum_{n>\tau}\frac{1}{nP_n}\left|\sum_{k=[n/2]}^{n}p_{n-k}\frac{1}{k+1}\sin\left(k+\frac{1}{2}\right)t\right| \\ &\leqq K + \Sigma_{21} + \Sigma_{22},\end{aligned} \tag{5.3}$$

say. Since $p_{n+1} \leqq p_n$, for all $n \geqq 0$, we have by Abel's Lemma

$$\begin{aligned}\Sigma_{21} &\leqq Kt^{-1}\sum_{n>\tau}\frac{p_{[n/2]}}{nP_n}\max_{0\leqq\nu<[n/2]}\left|\sum_{k=0}^{\nu}\frac{\sin\left(k+\frac{1}{2}\right)t}{k+1}\right| \\ &\leqq Kt^{-1}\sum_{n>\tau}\frac{1}{n^2}R_{[n/2]} \leqq K,\end{aligned} \tag{5.4}$$

since

$$\left|\sum_{k=0}^{\nu}\left\{\sin\left(k+\frac{1}{2}\right)t\right\}\Big/(k+1)\right| \leqq K$$

and $\{R_n\} \in B$, automatically.

Again by Abel's Lemma and Lemma 7, we have

[2] Throughout $[x]$ denotes the greatest integer not greater than x and $\tau = [2\pi/t]$.

$$
(5.5)\qquad \begin{aligned}
\Sigma_{22} &\leqq Kt^{-1} \sum_{n>\tau} \frac{1}{n^2 P_n} \max_{[n/2]\leqq \nu \leqq n} \left| \sum_{k=[n/2]}^{\nu} p_{n-k} \sin\left(k+\frac{1}{2}\right)t \right| \\
&\leqq Kt^{-1} P_\tau \sum_{n>\tau} \frac{1}{n^2 P_n} \leqq K ,
\end{aligned}
$$

by virtue of Lemma 8.

Combining (5.3)–(5.5), we prove that $\Sigma_2 \leqq K$. This result combined with (5.2), leads to (5.1) and we thus, complete the proof of $|\tilde{F}_1|$-effective part of Theorem 2.

(III). $|F^*|$-*effectiveness*: We have by integration by parts

$$
\begin{aligned}
nB_n(x) &= \frac{2}{\pi}\int_0^\pi \psi(t) n \sin nt \, dt \\
&= \frac{2}{\pi}\psi(\pi)(1-\cos n\pi) - \frac{2}{\pi}\int_0^\pi (1-\cos nt) d\psi(t) .
\end{aligned}
$$

Since x is a $|F^*|$-regular point, we have $\int_0^\pi |d\psi(t)| \leqq K$ and it follows from Lemma 6 that in order to prove $|F^*|$-effectiveness of the $(N, p_n)(C, 1)$ method, it is sufficient to show that uniformly in $0 < t \leqq \pi$

$$
(5.6)\qquad \sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{k=1}^{n} p_{n-k} \frac{1}{k+1} \sum_{r=1}^{k} r\Delta\{1-\cos(r-1)t\} \right| \leqq K .
$$

But, we have

$$
\sum_{r=0}^{k} r\Delta\{\cos(r-1)t\} = \frac{1}{2}(1-\cos kt) + 2\sin\frac{t}{2} \sum_{r=1}^{k} \left(r-\frac{1}{2}\right) \sin\left(r-\frac{1}{2}\right)t .
$$

Thus, to prove (5.6), it is enough to show that uniformly in $0 < t \leqq \pi$

$$
(5.7)\qquad \left| \sin\frac{t}{2} \right| \sum_{n=1}^{\infty} \frac{1}{nP_n} \left| \sum_{k=1}^{n} \frac{p_{n-k}}{k+1} \sum_{r=1}^{k} \left(r-\frac{1}{2}\right) \sin\left(r-\frac{1}{2}\right)t \right| \leqq K
$$

and

$$
(5.8)\qquad \Sigma' = \sum_{n=1}^{\infty} \frac{1}{nP_n} \sum_{k=1}^{n} \frac{p_{n-k}}{k+1} \leqq K .
$$

The proof of (5.7), runs exactly parallel to the proof of (5.1). To prove (5.8), we observe that by a change of order of summations

$$
\Sigma' = \sum_{k=1}^{\infty} \frac{1}{k+1} \sum_{n=k}^{\infty} \frac{p_{n-k}}{nP_n} \leqq K \sum_{k=1}^{\infty} \frac{1}{k+1} \frac{p_0}{P_{k-1}} \leqq K ,
$$

by virtue of Lemma 8 and the condition that $0 < p_{n+1} \leqq p_n$.

We thus complete the proof of $|F^*|$-effective part of Theorem 2.

(IV). *Absolute Fourier-effectiveness*: of the $(N, p_n)(C, 1)$ method follows from the absolute regularity of the (N, p_n) method and the corresponding effectiveness of the $(C, 1)$ method, which is included in the result of Theorem B.

Combing (I)-(IV), we complete the proof of Theorem 2.

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