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CONDITIONAL EXPECTATIONS ASSOCIATED WITH STOCHASTIC PROCESSES

R. A. BROOKS

A stochastic process, $E[x(t, \cdot) \mid \mathscr{F}_t](\omega)$, on a probability space $(\Omega, \mathscr{A}, P)$ and an interval D, where $x \in L_1(D \times \Omega)$ and $\{\mathscr{F}_t, t \in D\}$ is an increasing collection of sigma-fields in $\mathscr{A}$, is considered. Sufficient conditions for the joint measurability of $E[x(t, \cdot) \mid \mathscr{F}_t](\omega)$ in (t, ω) are given, and if $x \in L_2(D \times \Omega)$, it is shown that, under certain fairly general conditions, $E[x(t, \cdot) \mid \mathscr{F}_t](\omega)$ can be identified with the projection of x onto a certain subspace of the Hilbert space associated with $L_2(D \times \Omega)$. The results obtained herein have application in certain classes of stochastic optimization problems.

1. Introduction. Interest in the application of mathematical analysis to problems associated with optimizing control systems has been significant in recent years. [16], [6], [7], [9], [13]. A particular class of stochastic optimization problems [1] which are of interest in certain control system applications can be abstractly formulated in the following manner.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let $(D, \mathscr{B}, m)$ be a measure space, where $D = [a, b] \subset R$, $\mathscr{B}$ is the collection of Borel measurable subsets of D, and m is a finite measure defined on $(D, \mathscr{B})$. Consider the product measure space $(D \times \Omega, \mathscr{B} \times \mathscr{A}, m \times P)$, and denote by $L_2(D \times \Omega)$ the collection of real valued $\mathscr{B} \times \mathscr{A}$-measurable functions defined on $D \times \Omega$ which are square-integrable with respect to $m \times P$. Let $y(t, \omega)$ be a stochastic process on $(\Omega, \mathscr{A}, P)$ and R, and let

$$Y(t, \omega) = [y(\tau, \omega), \tau < t], (t, \omega) \in R \times \Omega .$$

For each $t \in R$, denote by $\mathscr{Y}_t$ the minimal sigma-field of subsets of Ω with respect to which every element of the random vector $Y(t, \cdot)$ is measurable, and observe that $t', t'' \in D$, $t' < t''$ implies $\mathscr{Y}_{t'} \subset \mathscr{Y}_{t''}$.

In the stochastic optimization application under consideration, the random vector $Y(t, \cdot)$ is the observation available to a controller at time $t \in D$, and the control problem is the following: For an arbitrary but fixed element $v \in L_2(D \times \Omega)$ determine a control $u_0 \in L_2(D\times\Omega)$ with the property that $u_0(t, \cdot)$ is $\mathscr{Y}_t$-measurable for each $t \in D$ and

$$\int_{D\times\Omega}[u_0(t, \omega) - v(t, \omega)]^2(m \times P)(d(t, \omega))$$
$$= \inf\left\{\int_{D\times\Omega}[u(t, \omega) - v(t, \omega)]^2(m \times P)(d(t, \omega)); u \in L_2(D\times\Omega)\ ,\right.$$
$$\left. u(t, \cdot) \text{ is } \mathscr{Y}_t\text{-measurable for each } t \in D\right\} .$$

The $\mathscr{Y}_t$-measurability condition is imposed in order that the minimizing control $u_0(t, \cdot)$ be a Baire function of the observation vector $Y(t, \cdot)$ for each $t \in D$.

Now let

$$\mathscr{M} = \{G \in \mathscr{B} \times \mathscr{A}; G_{t\cdot} \in \mathscr{Y}_t \text{ for all } t \in D\}\ ,$$

where $G_{t\cdot} \equiv \{\omega \in \Omega; (t, \omega) \in G\}$, $t \in D$, and observe that $\mathscr{M} \subset \mathscr{B} \times \mathscr{A}$ is a sigma-field. Consider the Hilbert space H of $m \times P$-equivalence classes in $L_2(D \times \Omega)$, and let $S \subset H$ be the collection of $m \times P$-equivalence classes generated by the $\mathscr{M}$-measurable elements in $L_2(D \times \Omega)$. It is readily verified that S is a subspace of H, and hence P_S, the projection operator which maps H onto S, exists. Consequently, for each $v \in H$, there exists a unique $u_0 \in S$ such that

$$\| u_0 - v \| = \inf\{\| u - v \|; u \in S\}\ ,$$

and, by the Projection theorem,

$$u_0 = P_S v\ .$$

Thus, ascertaining a solution to the stochastic optimization problem stated above is tantamount to identifying the projection operator P_S, and this is done in the following theorem, the proof of which will be given later.

THEOREM 1.1. *Let $v \in L_2(D \times \Omega)$. Then there exists an $\mathscr{M}$-measurable real-valued function defined on $D \times \Omega$, denoted by $E[v(t, \cdot) \mid \mathscr{Y}_t](\omega)$, which is a conditional expectation of $v(t, \cdot)$ given $\mathscr{Y}_t$ for m-a.e. $t \in D$, and moreover,*

$$(P_S v)(t, \omega) = E[v(t, \cdot) \mid \mathscr{Y}_t](\omega)$$

for $m \times P$-a.e. $(t, \omega) \in D \times \Omega$.

The crux of the proof of the above theorem is to prove the existence of the $\mathscr{M}$-measurable function $E[v(t, \cdot) \mid \mathscr{Y}_t](\omega)$. The main result obtained in the sequel is that an $\mathscr{M}$-measurable function $E[v(t, \cdot) \mid \mathscr{Y}_t](\omega)$ does indeed exist. Other results which make use of this measurability are also presented.

The utility of the conditional expectation representation of the operator P_S is twofold. First, some interesting and useful facts regarding the solution of the stochastic optimization problem can be deduced using properties of conditional expectations [1]; and second, the problem of calculating conditional expectations of the type encountered above has received much attention in the literature, and efficient computational algorithms have been developed in certain cases. [8], [10], [11], [12], [3], [15], [17], [5], [4].

2. Preliminaries. Let $\{\mathscr{F}_t, t \in D\}$ be a collection of increasing sigma-fields in $\mathscr{A}$, i.e., for each $t \in D$, $\mathscr{F}_t \subset \mathscr{A}$ and $t', t'' \in D$, $t' < t''$ implies $\mathscr{F}_{t'} \subset \mathscr{F}_{t''}$. Let $x \in L_1(D \times \Omega)$. Then $x(t, \cdot) \in L_1(\Omega)$ for m-a.e. $t \in D$, and consequently, [14], a conditional expectation $E[x(t, \cdot) \mid \mathscr{F}_t](\omega)$ exists for m-a.e. $t \in D$. By defining $E[x(t, \cdot) \mid \mathscr{F}_t](\omega)$ to be an arbitrary random variable for each t in the exceptional set, $E[x(t, \cdot) \mid \mathscr{F}_t](\omega)$ can be extended to be a stochastic process on $(\Omega, \mathscr{A}, P)$ and D, and this is the basis of the following definition.

DEFINITION 2.1. Let $x \in L_1(D \times \Omega)$. A conditional expectation of x with respect to $\{\mathscr{F}_t, t \in D\}$, denoted by $E[x(t, \cdot) \mid \mathscr{F}_t](\omega)$, is a stochastic process on $(\Omega, \mathscr{A}, P)$ and D with the property that for m-a.e. $t \in D$,

(i) $E[x(t, \cdot) \mid \mathscr{F}_t]$ is $\mathscr{F}_t$-measurable,

(ii) $E[x(t, \cdot) \mid \mathscr{F}_t] \in L_1(\Omega)$,

(iii) $\int_F E[x(t, \cdot) \mid \mathscr{F}_t](\omega)P(d\omega) = \int_F x(t, \omega)P(d\omega)$ for all $F \in \mathscr{F}_t$.

It has already been mentioned that a conditional expectation of x with respect to $\{\mathscr{F}_t, t \in D\}$ exists, although not necessarily uniquely, for each $x \in L_1(D \times \Omega)$, and the question which is now asked is whether there exists a $\mathscr{B} \times \mathscr{A}$-measurable representation for the conditional expectation of x with respect to $\{\mathscr{F}_t, t \in D\}$. Before addressing this fundamental question, several preliminary results will be given.

LEMMA 2.1. *Let $x \in L_p(D \times \Omega)$, $1 \leq p < \infty$. If $E[x(t, \cdot) \mid \mathscr{F}_t](\omega)$ is $\mathscr{B} \times \mathscr{A}$-measurable, then $E[x(t, \cdot) \mid \mathscr{F}_t](\omega) \in L_p(D \times \Omega)$. Moreover, if $E[x(t, \cdot) \mid \mathscr{F}_t]'(\omega)$ is any other $\mathscr{B} \times \mathscr{A}$-measurable conditional expectation of x with respect to $\{\mathscr{F}_t, t \in D\}$, then*

$$E[x(t, \cdot) \mid \mathscr{F}_t]'(\omega) = E[x(t, \cdot) \mid \mathscr{F}_t](\omega) \ m \times P\text{-a.e.}$$

Proof. The uniqueness of $\mathscr{B} \times \mathscr{A}$-measurable representations for the conditional expectation of x with respect to $\{\mathscr{F}_t, t \in D\}$ up to $m \times P$-equivalence on $D \times \Omega$ is immediate, and an application of Jensen's inequality for conditional expectation, together with Tonelli's theorem, proves $E[x(t, \cdot) \mid \mathscr{F}_t](\omega) \in L_p(D \times \Omega)$.

LEMMA 2.2. *Let $\mathscr{F} \subset \mathscr{A}$ be a sigma-field and let $x \in L_1(D \times \Omega)$. Then there exists a $\mathscr{B} \times \mathscr{F}$-measurable stochastic process on $(\Omega, \mathscr{A}, P)$ and D, denoted by $E[x(t, \cdot) \mid \mathscr{F}](\omega)$, which is a conditional expectation of $x(t, \cdot)$ given $\mathscr{F}$ for each $t \in D$ such that $x(t, \cdot) \in L_1(\Omega)$.*

Proof. It is sufficient to prove that the lemma holds on $L_1'(D \times \Omega)$, where $L_1'(D \times \Omega) = \{x \in L_1(D \times \Omega);\ x(t, \cdot) \in L_1(\Omega) \text{ for all } t \in D\}$, and to this end let S be the collection of elements in $L_1'(D \times \Omega)$ for which the lemma holds.

(1) To show that S contains the characteristic functions of measurable rectangles, observe that if $F = B \times A$, where $B \in \mathscr{B}$, $A \in \mathscr{A}$, then $\chi_F(t, \omega) = \chi_B(t)\chi_A(\omega)$, $(t, \omega) \in D \times \Omega$, and, for each $t \in D$,

$$(1) \qquad E[\chi_F(t, \cdot) \mid \mathscr{F}](\omega) = \chi_B(t) E[\chi_A \mid \mathscr{F}](\omega)$$

for P-a.e. $\omega \in \Omega$, where $E[\chi_A \mid \mathscr{F}](\omega)$ is a conditional expectation of χ_A given $\mathscr{F}$. Thus the right-hand side of (1) is a $\mathscr{B} \times \mathscr{F}$-measurable stochastic process on $(\Omega, \mathscr{A}, P)$ and D which is a conditional expectation of $\chi_F(t, \cdot)$ given $\mathscr{F}$ for each $t \in D$.

(2) Observe that S is closed with respect to linear operations in the sense that if $x_1(t, \omega), \cdots, x_n(t, \omega) \in S$, $c_1, \cdots, c_n \in R$ and $x(t, \omega) = \sum_{i=1}^n c_i x_i(t, \omega)$, $(t, \omega) \in D \times \Omega$, then $x(t, \omega) \in S$.

(3) S is also closed with respect to dominated convergence in a certain sense. To show this let $\{x_n(t, \omega), n = 1, 2, \cdots\} \subset S$, $x(t, \omega)$, $w(t, \omega) \in L_1'(D \times \Omega)$, and suppose that $x(t, \omega) = \lim_{n \to \infty} x_n(t, \omega)$ for all $(t, \omega) \in D \times \Omega$, and also that $|x_n(t, \omega)| \leqq w(t, \omega)$ for all $(t, \omega) \in D \times \Omega$, $n = 1, 2, \cdots$. It is now shown that $x \in S$. For each $n = 1, 2, \cdots$, let $E[x_n(t, \cdot) \mid \mathscr{F}](\omega)$ be a $\mathscr{B} \times \mathscr{F}$-measurable stochastic process on $(\Omega, \mathscr{A}, P)$ and D which is also a conditional expectation of $x_n(t, \cdot)$ given $\mathscr{F}$ for each $t \in D$. Now for each $t \in D$,

$$E[x(t, \cdot) \mid \mathscr{F}](\omega) = \lim_{n \to \infty} E[x_n(t, \cdot) \mid \mathscr{F}](\omega)$$

for P-a.e. $\omega \in \Omega$ [2, p. 23]. Let $z(t, \omega)$ be defined on $D \times \Omega$ by

$$z(t, \omega) = \begin{cases} \lim_{n \to \infty} E[x_n(t, \cdot) \mid \mathscr{F}](\omega) \ , & \text{when this limit exists} \ , \\ 0 & , \text{otherwise} \ . \end{cases}$$

Then $z(t, \omega)$ is a $\mathscr{B} \times \mathscr{F}$-measurable stochastic process on $(\Omega, \mathscr{A}, P)$ and D, and, for each $t \in D$, $z(t, \omega)$ is a conditional expectation of $x(t, \cdot)$ given $\mathscr{F}$. Thus $x \in S$.

(4) Let $\mathscr{C} = \{F \in \mathscr{B} \times \mathscr{A};\ \chi_F \in S\}$. From (1), it follows that $\mathscr{C}$ contains the measurable rectangles, in $\mathscr{B} \times \mathscr{A}$, and (2) then implies

that $\mathscr{C}$ contains the field generated by the measurable rectangles in $\mathscr{B} \times \mathscr{A}$. From (3) it follows that $\mathscr{C}$ is a monotone class, and consequently, by the monotone class lemma, $\mathscr{C} = \mathscr{B} \times \mathscr{A}$.

(5) Let $x \in L_1'(D \times \Omega)$. Then there exists a sequence of simple functions $\{x_n, n = 1, 2, \cdots\} \subset L_1'(D \times \Omega)$ such that $x_n^+(t, \omega) \uparrow x^+(t, \omega)$ $x_n^-(t, \omega) \uparrow x^-(t, \omega)$, as $n \to \infty$, for all $(t, \omega) \in D \times \Omega$. From (2), (4) it follows that $\{x_n, n = 1, 2, \cdots\} \subset S$, and (3) then implies $x \in S$, which proves the lemma.

The next result is a Fubini-type theorem for conditional expectation.

LEMMA 2.3. *Let $\mathscr{F} \subset \mathscr{A}$ be a sigma-field, let $x \in L_1(D \times \Omega)$, and let $E[x(t, \cdot) \mid \mathscr{F}](\omega)$ be a $\mathscr{B} \times \mathscr{F}$-measurable conditional expectation of $x(t, \cdot)$ given $\mathscr{F}$ for m-a.e. $t \in D$. Then*

$$E\left[\int_D x(t, \cdot)m(dt) \mid \mathscr{F}\right](\omega) = \int_D E[x(t, \cdot) \mid \mathscr{F}](\omega)m(dt)$$

for P-a.e. $\omega \in \Omega$.

Proof. Immediate consequence of Fubini's theorem.

LEMMA 2.4. *Let $D_0 \subset D$ be countable, and suppose that $\mathscr{F}_t = \sigma\{\bigcup_{t'<t} \mathscr{F}_{t'}\}$ for each $t \in D \backslash D_0$. Then for each $A \in \mathscr{A}$, there exists a $\mathscr{B} \times \mathscr{A}$-measurable conditional expectation of χ_A with respect to $\{\mathscr{F}_t, t \in D\}$.*

Proof. Let $S' \subset D$ be countable and dense in D and let $S = S' \cup D_0 \cup \{a\}$. Then $S \subset D$ is also countable and dense in D. Let $S = \{s_1, s_2, \cdots\}$ be an arbitrary denumeration of S for which $s_1 = a$, and for each $t \in D$ put $s(t, n) = \max\{s_i \in S; s_i \leqq t, i = 1, 2, \cdots n\}$, $n = 1, 2, \cdots$.

Let $A \in \mathscr{A}$ and for each $t \in D$ let $u(t, \omega)$ be a conditional expectation of χ_A given $\mathscr{F}_t$. Then for each $t \in D$ it is clear that $|u(t, \omega)| < 1$ for P-a.e. $\omega \in \Omega$.

Let $u_n(t, \omega) = u(s(t, n), \omega)$, $(t, \omega) \in D \times \Omega$, $n = 1, 2, \cdots$, and observe that $\{u_n, n = 1, 2, \cdots\}$ is a sequence of $\mathscr{B} \times \mathscr{A}$-measurable functions on $D \times \Omega$. Furthermore, for each $n = 1, 2, \cdots$, $t \in D$, $u_n(t, \cdot)$ is $\mathscr{F}_t$-measurable since $s(t, n) \leqq t$.

Now if $t \in S$, then $s(t, n) = t$ for n sufficiently large, and hence $\lim_{n\to\infty} u_n(t, \omega) = u(t, \omega)$ for all $\omega \in \Omega$. On the other hand, if $t \in D \backslash S$, then $s(t, n) \uparrow t$ as $n \to \infty$, and thus

$$\bigcup_{n=1}^{\infty} \mathscr{F}_{s(t,n)} = \bigcup_{t'<t} \mathscr{F}_{t'} .$$

Therefore, by hypothesis,

$$\mathscr{F}_t = \sigma\{\bigcup_{n=1}^{\infty} \mathscr{F}_{s(t,n)}\} ,$$

and consequently, [2, Theorem 4.3, p. 331],

$$u(t, \omega) = \lim_{n\to\infty} u_n(t, \omega)$$

for P-a.e. $\omega \in \Omega$. Hence for each $t \in D$

$$u(t, \omega) = \lim_{n\to\infty} u_n(t, \omega) \tag{2}$$

for P-a.e. $\omega \in \Omega$.

By the Bounded Convergence theorem, it follows that for each $t \in D$

$$\lim_{n\to\infty} \int_\Omega | u(t, \omega) - u_n(t, \omega) | P(d\omega) = 0 ,$$

and hence

$$\lim_{k,n\to\infty} \int_\Omega | u_k(t, \omega) - u_n(t, \omega) | P(d\omega) = 0 .$$

For each $t \in D$, $n = 1, 2, \cdots$, let

$$v_n(t) = \sup_{k \geqq n} \int_\Omega | u_k(t, \omega) - u_n(t, \omega) | P(d\omega) .$$

Then $\{v_n, n = 1, 2, \cdots\}$ is a sequence of $\mathscr{B}$-measurable functions on D, and $| v_n(t) | \leqq 2$, $t \in D$, $n = 1, 2, \cdots$ Thus, applying the Bounded Convergence theorem once more, it follows that

$$\lim_{n\to\infty} \int_D v_n(t) m(dt) = 0 ,$$

and consequently,

$$\lim_{k,n\to\infty} \int_{D\times\Omega} | u_k(t, \omega) - u_n(t, \omega) | (m \times P)(d(t, \omega)) = 0 . \tag{3}$$

Now let $\mathscr{S} = \{F \in \mathscr{B} \times \mathscr{A};\ F_{t\cdot} \in \mathscr{F}_t,\ t \in D\}$, where $F_{t\cdot} \equiv \{\omega \in \Omega; (t, \omega) \in F\}$, $t \in D$. It is readily verified that $\mathscr{S}$ is a sigma-field, and it is also clear that $\{u_n, n = 1, 2, \cdots\}$ is a sequence of $\mathscr{S}$-measurable functions on $D \times \Omega$. Thus, by the Riesz-Fischer theorem, it follows from (3) that there exists a $\mathscr{S}$-measurable function $w \in L_1(D \times \Omega)$ such that

$$\lim_{n\to\infty} \int_{D\times\Omega} | w(t, \omega) - u_n(t, \omega) | (m \times P)(d(t, \omega)) = 0 ,$$

and hence there exists a subsequence $\{n_k\} \subset \{n\}$ such that

$$w(t, \omega) = \lim_{k\to\infty} u_{n_k}(t, \omega)$$

for $m \times P$-a.e. $(t, \omega) \in D \times \Omega$. Therefore, for m-a.e. $t \in D$,

$$w(t, \omega) = \lim_{k\to\infty} u_{n_k}(t, \omega)$$

for P-a.e. $\omega \in \Omega$, and from (2) it follows that, for m-a.e. $t \in D$, $w(t, \omega) = u(t, \omega)$ for P-a.e. $\omega \in \Omega$. The $\mathscr{S}$-measurability of w implies $w(t, \cdot)$ is $\mathscr{F}_t$-measurable for each $t \in D$, and consequently w is a $\mathscr{B} \times \mathscr{A}$-measurable conditional expectation of χ_A with respect to $\{\mathscr{F}_t, t \in D\}$.

3. Main results. Let $y(t, \omega)$ be a stochastic process on $(\Omega, \mathscr{A}, P)$ and R, let

$$Y(t, \omega) = [y(\tau, \omega), \tau < t],\ (t, \omega) \in R \times \Omega ,$$

and for each $t \in R$, let

$$\mathscr{Y}_t = \sigma\{Y(t, \cdot)\} .$$

LEMMA 3.1. $\mathscr{Y}_t = \sigma\{\bigcup_{t'<t} \mathscr{Y}_{t'}\},\ t \in R$.

Proof. Since $\mathscr{Y}_t$ is a sigma-field and $(\bigcup_{t'<t} \mathscr{Y}_{t'}) \subset \mathscr{Y}_t$, it follows that $\sigma\{\bigcup_{t'<t} \mathscr{Y}_{t'}\} \subset \mathscr{Y}_t$ for each $t \in R$.

Let $t \in R$ be fixed. Then for each $\tau < t$, the random variable $y(\tau, \cdot)$ is measurable with respect to $\mathscr{Y}_{t'}$ for each $t' > \tau$, and hence $y(\tau, \cdot)$ is measurable with respect to $\sigma\{\bigcup_{t'<t} \mathscr{Y}_{t'}\}$. Thus

$$\mathscr{Y}_t \equiv \sigma\{[y(\tau, \cdot), \tau < t]\} \subset \sigma\left\{\bigcup_{t'<t} \mathscr{Y}_{t'}\right\} ,$$

and this proves the lemma.

THEOREM 3.2. *Let $x \in L_1(D \times \Omega)$. Then there exists a $\mathscr{B} \times \mathscr{A}$-measurable conditional expectation of x with respect to $\{\mathscr{Y}_t, t \in D\}$.*

Proof. From Lemmas 2.4 and 3.1, it follows that the theorem holds for $x = \chi_A$, $A \in \mathscr{A}$. By an argument which parallels the proof of Lemma 2.2, it can be shown that the theorem holds for all $x \in L_1(D \times \Omega)$.

Now let $\mathscr{M} \subset \mathscr{B} \times \mathscr{A}$ be the sigma-field defined by

$$\mathscr{M} = \{G \in \mathscr{B} \times \mathscr{A};\, G_{t\cdot} \in \mathscr{Y}_t \text{ for all } t \in D\}\ ,$$

and observe that Theorem 3.2 implies that for each $x \in L_1(D \times \Omega)$ there exists an $\mathscr{M}$-measurable conditional expectation of x with respect to $\{\mathscr{Y}_t, t \in D\}$. Let H be the Hilbert space of $m \times P$-equivalence classes in $L_2(D \times \Omega)$, with innerproduct denoted by $\langle \cdot, \cdot \rangle$, let $S \subset H$ be the collection of $m \times P$-equivalence classes generated by the $\mathscr{M}$-measurable elements in $L_2(D \times \Omega)$, and let P_S be the projection of H onto S.

THEOREM 3.3. *Let $x \in L_2(D \times \Omega)$ and let $E[x(t, \cdot) \mid \mathscr{Y}_t](\omega)$ be an $\mathscr{M}$-measurable conditional expectation of x with respect to $\{\mathscr{Y}_t, t \in D\}$. Then*

$$(P_S x)(t, \omega) = E[x(t, \cdot) \mid \mathscr{Y}_t](\omega)$$

for $m \times P$-a.e. $(t, \omega) \in D \times \Omega$.

Proof. The existence of an $\mathscr{M}$-measurable conditional expectation of x with respect to $\{\mathscr{Y}_t, t \in D\}$ follows from Theorem 3.2, since $L_2(D \times \Omega) \subset L_1(D \times \Omega)$.

Let $M \in \mathscr{M}$ and observe that

$$\begin{aligned}
&\int_M [(P_S x)(t, \omega) - E[x(t, \cdot) \mid \mathscr{Y}_t](\omega)](m \times P)(d(t, \omega)) \\
&= \langle P_S x, \chi_M \rangle - \int_D \int_{M_{t\cdot}} E[x(t, \cdot) \mid \mathscr{Y}_t](\omega) P(d\omega) m(dt) \\
&= \langle x, \chi_M \rangle - \int_D \int_{M_{t\cdot}} x(t, \omega) P(d\omega) m(dt) \\
&= \int_M x(t, \omega)(m \times P)(d(t, \omega)) - \int_M x(t, \omega)(m \times P)(d(t, \omega)) \\
&= 0\ .
\end{aligned}$$

From the arbitrariness of $M \in \mathscr{M}$, it follows that

$$(P_S x)(t, \omega) = E[x(t, \cdot) \mid \mathscr{Y}_t](\omega)$$

for $m \times P$-a.e. $(t, \omega) \in D \times \Omega$.

4. Extensions. The theorem which follows extends some of the results obtained in §§ 2, 3, and it has application in the study of the class of stochastic optimization problems defined in § 1.

THEOREM 4.1. *Let $(D', \mathscr{B}')$ be a copy of $(D, \mathscr{B})$, let m' be an arbitrary finite measure on $(D', \mathscr{B}')$, and let $x \in L_1(D' \times D \times \Omega)$.*

Then there exists a $\mathscr{B}' \times \mathscr{B} \times \mathscr{A}$*-measurable real-valued function defined on* $D' \times D \times \Omega$*, denoted by* $E[x(s, t, \cdot) \mid \mathscr{Y}_t](\omega)$*, which for* m-a.e. $t \in D$*, is a conditional expectation of* $x(s, t, \cdot)$ *given* $\mathscr{Y}_t$ *for every* $s \in D'$ *such that* $x(s, t, \cdot) \in L_1(\Omega)$*. Furthermore, for every* $t \in D$*,* $E[x(s, t, \cdot) \mid \mathscr{Y}_t](\omega)$ *is* $\mathscr{B}' \times \mathscr{Y}_t$*-measurable on* $D' \times \Omega$*, and, for* m-a.e. $t \in D$*,*

$$E\left[\int_{D'} x(s, t, \cdot)m'(ds) \mid \mathscr{Y}_t\right](\omega) = \int_{D'} E[x(s, t, \cdot) \mid \mathscr{Y}_t](\omega)m'(ds)$$

for P-a.e. $\omega \in \Omega$.

Proof. The existence of $E[x(s, t, \cdot) \mid \mathscr{Y}_t](\omega)$ with the stated conditional expectation and measurability properties is established by an extension of the proof of Lemma 2.2, and the validity of interchanging conditional expectation and integration is guaranteed by Lemma 2.3.

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UNIVERSITY OF CALIFORNIA, SANTA BARBARA