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WEAK ORTHOGONALITY

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Two Hilbert space-valued functions, f and g, are orthogonal iff $\int_\Omega < f(\omega), g(\omega) > \mu(d\omega) = 0$, where $<, >$ denotes the inner product of the Hilbert space. This paper concerns an analogous condition for Banach space-valued functions where, in general, no inner product structure is given.

DEFINITION 1.1. Let (Ω, Σ, μ) be a measure space and let f be a function from Ω into a Banach space X. Let X^* denote the dual of X. f is said to be *almost separably-valued* iff there exists a separable subspace $X_0 \subset X$ such that $\mu(f^{-1}(X/X_0)) = 0$. f is said to be *strongly measurable* iff f is almost separably-valued and, for each Borel set $B \subset X$, $f^{-1}(B)$ is measurable. f is said to be an *X-valued function* iff the range of f is contained in X and f is strongly measurable.

DEFINITION 1.2. Two X-valued functions, f and g, are said to be *weakly orthogonal* iff, for each $x^* \in X^*$,

$$\int_\Omega x^* f(\omega) \cdot x^* g(\omega) \mu(d\omega) = 0 .$$

A sequence of X-valued functions is said to be a *weakly orthogonal sequence* iff each pair of its functions is weakly orthogonal.

It is easy to see that if f and g are independent, X-valued functions with $\int_\Omega f = \int_\Omega g = 0$, then f and g are weakly orthogonal. In the event that X is also a Hilbert space where f and g satisfy the above conditions, then f and g will be orthogonal in the classical sense. Thus both orthogonality and weak orthogonality generalize the concept of independence and as such have a role in probability theory. In a subsequent paper we will consider some results connecting weak orthogonality and the strong law of large numbers for X-valued functions. Here, however, we will restrict our attention to some of the structural aspects of weak orthogonality.

2. Hilbert space.

Orthogonality and weak orthogonality can only be compared in inner-product spaces. In this context, namely in a Hilbert space, weak orthogonality is a more restrictive condition than orthogonality. Let H denote a Hilbert space with an inner-product $<, >$ and let $L_2(\Omega, H)$ be the space of L_2-integrable H-valued functions.

THEOREM 2.1. *If f and g are weakly orthogonal H-valued func-*

tions belonging to $L_2(\Omega, H)$, *then* f *and* g *are orthogonal.*

We need two technical lemmas.

LEMMA 2.2. *Let* $\{u_\alpha : \alpha \in A\}$ *be any orthonormal basis for* H. *If* f *is an* H*-valued function, then there exists a countable subcollection of basis elements* $u_{\alpha(1)}, u_{\alpha(2)}, \cdots$, *such that*

$$f(\omega) = \sum_{k=1}^{\infty} < f(\omega), \mu_{\alpha(k)} > \mu_{\alpha(k)} \quad a.e.$$

Proof. Since f is strongly measurable, the range of f is almost separably-valued. This implies that there exists a closed, separable subspace H_0 of H containing the essential range of f. Since H_0 is separable, it has a countable orthonormal basis, say $v_1, v_2, \cdots$. Furthermore, for fixed i, it is a well known fact that $< v_i, u_\alpha > \neq 0$ for at most countably many values of α. Denote these values by $\alpha(i, j)$, $j = 1, 2, \cdots$. Thus, for $i = 1, 2, \cdots$,

$$v_i = \sum_{j=1}^{\infty} <v_i, u_{\alpha(i,j)}> u_{\alpha(i,j)} \ .$$

Let

$B = \{u_{\alpha(i,j)}: i, j = 1, 2, \cdots\}$. Clearly B is a countable subset of $\{u_\alpha : \alpha \in A\}$ and, for $f(\omega) \in H_0$, it follows that

$$f(\omega) = \sum_{k=1}^{\infty} <f(\omega), u_{\alpha(k)}> u_{\alpha(k)}$$

where

$$u_{\alpha(k)} \in B \ .$$

LEMMA 2.3. *Let* f *and* g *be* H*-valued functions with both* f *and* g *in* $L_2(\Omega, H)$. *Let* $\{u_\alpha : \alpha \in A\}$ *be an orthonormal basis for* H *and suppose*

$$f(\omega) = \sum_{k=1}^{\infty} < f(\omega), u_{\alpha(k)} > u_{\alpha(k)} \quad \text{a.e.}$$

$$g(\omega) = \sum_{k=1}^{\infty} < g(\omega), u_{\alpha(k)} > u_{\alpha(k)} \quad \text{a.e.}$$

are the representations given in Lemma 2.2. *Then*

$$\int_\Omega \sum_{k=1}^{\infty} <f(\omega), u_{\alpha(k)}><g(\omega), u_{\alpha(k)}> = \sum_{k=1}^{\infty} \int_\Omega <f(\omega), u_{\alpha(k)}><g(\omega), u_{\alpha(k)}> \ .$$

Proof. Clearly $< f(\omega), g(\omega)> = \sum_{k=1}^{\infty} < f(\omega), u_{\alpha(k)} >< g(\omega), u_{\alpha(k)} >$ a.e.. For these ω, define

$$h_n(\omega) = \sum_{k=1}^{n} < f(\omega), u_{\alpha(k)} > < g(\omega), u_{\alpha(k)} > .$$

It is easy to show, using the inequalities of Cauchy and Bessel, that

$$|h_n(\omega)| \leqq ||f(\omega)|| \cdot ||g(\omega)|| .$$

As a consequence of Holder's inequality, the expression on the right is integrable. The conclusion now follows from the Lebesgue limit theorem.[1]

Proof of Theorem 2.1. Let $\{u_\alpha : \alpha \in A\}$ be an orthonormal basis for H. The Riesz-Fisher theorem implies that each u_α defines a unique element $x_\alpha^* \in H^*$ such that

$$x_\alpha^*(h) = < h, u_\alpha > \quad \text{for each } h \in H .$$

Lemmas 2.2 and 2.3 and the fact that f and g are weakly orthogonal imply that

$$\begin{aligned} \int_\Omega < f(\omega), g(\omega) > &= \int_\Omega \sum_{k=1}^{\infty} < f(\omega), u_{\alpha(k)} > < g(\omega), u_{\alpha(k)} > \\ &= \sum_{k=1}^{\infty} \int_\Omega < f(\omega), u_{\alpha(k)} > < g(\omega), u_{\alpha(k)} > \\ &= \sum_{k=1}^{\infty} \int_\Omega x_{\alpha(k)}^*(f(\omega)) x_{\alpha(k)}^*(g(\omega)) \\ &= 0 . \end{aligned}$$

The next example shows that orthogonal H-valued functions are not necessarily weakly orthogonal.

EXAMPLE 2.4. Let $f \equiv (1, 1, 0, 0, \cdots)$ and $g \equiv (1, -1, 0, 0, \cdots)$, so that f and g are l_2-valued functions. Clearly f and g are orthogonal, but if x^* is the vector $(1, 0, 0, 0, \cdots)$ regarded as an element of l_2^*, then we have

$$\int x^* f(\omega) \cdot x^* g(\omega) = 1$$

so that f and g are not weakly orthogonal.

3. Characterization and examples. The next result provides some insight into the structure of weakly orthogonal X-valued functions for a wide class of sequence spaces.

[1] Cf. E. Hille and R. S. Phillips [1], p. 83.

THEOREM 3.1. *Let X be either an l_p-space, with $1 \leqq p < \infty$, or the space c_0 (the space of sequences which converge to zero). Let f and g be X-valued functions and suppose that*

$$f(\omega) = (\beta_1(\omega), \beta_2(\omega), \cdots, \beta_i(\omega), \cdots)$$

$$g(\omega) = (\gamma_1(\omega), \gamma_2(\omega), \cdots, \gamma_i(\omega), \cdots)$$

are representations for f and g in X. Suppose, further, that $\int_\Omega ||f(\omega)||_X \cdot ||g(\omega)||_X \mu(d\omega)$ exists. Then f and g are weakly orthogonal if and only if both

(i) $\int_\Omega \beta_i(\omega)\gamma_i(\omega)\mu(d\omega) = 0, \forall i$

(ii) $\int_\Omega [\beta_i(\omega)\gamma_j(\omega) + \beta_j(\omega)\gamma_i(\omega)]\mu(d\omega) = 0, \forall i \neq j.$

Proof. We remark that for $1 \leqq p < \infty$, $l_p^* = l_q$ where $1 < q \leqq \infty$ and $c_0^* = l_1$. We define

$$x_k^* = (\delta_{1k}, \delta_{2k}, \cdots)$$

and, for $i \neq j$,

$$x_{i,j}^* = x_i^* + x_j^* .$$

x_k^* has a 1 in the kth coordinate and 0's elsewhere whereas $x_{i,j}^*$ has 1's in the ith and jth coordinates with 0's elsewhere. It is clear that for any positive integers i, j, k, both x_k^* and $x_{i,j}^*$ belong to X^*.

Now suppose that $\int_\Omega x^*(f(\omega))x^*(g(\omega)) = 0, \forall x^* \in X^*$. Then (i) and (ii) follow from evaluations of this integral using the functionals x_i^* and $x_{i,j}^*$.

Now suppose that (i) and (ii) are satisfied. Choose an arbitrary $x^* \in X^*$ and let it be represented as

$$x^* = (\alpha_1, \alpha_2, \cdots) .$$

Define, for each n,

$$x_n^* = (\alpha_1, \alpha_2, \cdots, \alpha_n, 0, 0, \cdots) .$$

Clearly $x_n^* \in X^*$ and $||x_n^*|| \leqq ||x^*||$ for $n = 1, 2, \cdots$. Also $x_n^* \rightarrow x^*$ as $n \rightarrow \infty$. Moreover, for each n,

$$\begin{aligned}&\int_\Omega x_n^*(f(\omega))x_n^*(g(\omega)) \\ &= \int_\Omega \Big(\sum_{k=1}^n \alpha_k\beta_k(\omega)\Big)\cdot\Big(\sum_{k=1}^n \alpha_k\gamma_k(\omega)\Big) \\ &= \int_\Omega \sum_{k=1}^n \alpha_k^2\beta_k(\omega)\gamma_k(\omega)) + \sum_{i\neq j} \alpha_i\alpha_j(\beta_i(\omega)\gamma_j(\omega) + \beta_j(\omega)\gamma_i(\omega))\end{aligned}$$

$$= \sum_{k=1} \alpha^2 \int_\Omega \beta_k(\omega)\gamma_k(\omega) + \sum_{i \neq j}^{n} \alpha_i \alpha_j \int (\beta_i(\omega)\gamma_j(\omega) + \beta_j(\omega)\gamma_i(\omega))$$
$$= 0$$

as a consequence of (i) and (ii). Choose an arbitrary $\varepsilon > 0$ and let n be so large that $\|x_n^* - x^*\| < \varepsilon$ if $n \geqq n_0$. Then, if $n \geqq n_0$,

$$\left| \int_\Omega x^*(f(\omega))x^*(g(\omega)) - \int_\Omega x_n^*(f(\omega))x_n^*(g(\omega)) \right|$$
$$\leqq \int_\Omega |x^*(g(\omega))| \cdot |(x^* - x_n^*)(f(\omega))| + \int_\Omega |x_n^*(f(\omega))| \cdot |(x^* - x_n^*)(g(\omega))|$$
$$\leqq 2\varepsilon \|x^*\| \int_\Omega \|f(\omega)\| \cdot \|g(\omega)\| \ .$$

Since ε is arbitrary, this implies that

$$\int_\Omega x^*(f(\omega))x^*(g(\omega)) = 0$$

which completes the proof.

It is not difficult to find functions which satisfy both conditions (i) and (ii) of Theorem 3.1 in a rather trivial way.

EXAMPLE 3.2. Let α_k, $k = 1, 2, \cdots$ be a sequence of mutually orthogonal real-valued functions with $|\alpha_k(\omega)| \leqq 1$ for all k. The Radamacher functions would do here. Let $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ be sequences of scalars which both belong to one of the spaces l_p, $1 \leqq p < \infty$ or c_0. Let

$$f(\omega) = (a_1\alpha_2(\omega), a_2\alpha_4(\omega), \cdots a_i\alpha_{2i}(\omega), \cdots)$$
$$g(\omega) = (b_1\alpha_1(\omega), b_2\alpha_3(\omega), \cdots b_i\alpha_{2i-1}(\omega), \cdots) \ .$$

Then, clearly, f and g are weakly orthogonal by Theorem 3.1.

Note that condition (ii) of Theorem 3.1 is trivially satisfied because, in terms of the coordinate functions of f and g, everything in sight is mutually orthogonal. Loosely speaking, condition (ii) suggests a sort of "cross-product" orthogonality. For the sake of curiosity, it is worth inquiring what sorts of functions satisfy this condition in a non-trivial manner. The following is an example of a weakly orthogonal sequence of c_0-valued functions which are, in the sense of these remarks, as nontrivial as they possibly can be. That is, the coordinate functions are orthogonal only when they have to be as necessitated by condition (i).

EXAMPLE 3.3. We shall construct a sequence of c_0-valued functions f_i, $i = 1, 2, 3, \cdots$. Each f_i will be described in terms of its coordinate functions

$$f_i(\omega) = (\gamma_{i,1}(\omega), \gamma_{i,2}(\omega), \cdots, \gamma_{i,j}(\omega), \cdots) .$$

The objective of the construction is to have

(i) $\int_\Omega \gamma_{a,j}(\omega)\cdot\gamma_{b,j}(\omega)\mu(d\omega) = 0$ for $a \neq b$ and $\forall_j$.

(ii) $\int_\Omega [\gamma_{a,c}(\omega)\cdot\gamma_{b,d}(\omega) + \gamma_{a,d}(\omega)\cdot\gamma_{b,c}(\omega)]\mu(d\omega) = 0$ for $a \neq b$.

(iii) $\int_\Omega \gamma_{a,c}(\omega)\cdot\gamma_{b,d}(\omega)\mu(d\omega) \neq 0$ for $a \neq b$ and $c \neq d$.

The order of business is to first construct the $\gamma_{i,j}$ and then show that they satisfy the above.

Let S be the set of all 4-tuples (i, j, m, n) taken from the positive integers. S is countable and hence may be put into one-to-one correspondence with the set of positive integers I. For $k \in I$, let $(i, j, m, n)_k$ be the 4-tuple in correspondence with k. We will associate with each such 4-tuple a 2-dimensional array of scalar-valued functions (called the kth array). The functions of the kth array are denoted by

$$\{r^k_{u,v}(\omega)\}^\infty_{u,v=1}$$

and will be chosen to meet the requirement that

$$\begin{cases} r^k_{i,j} \equiv -r^k_{i+m,j+n} \\ r^k_{i+m\ j} \equiv r^k_{i,j+n} \end{cases}$$

and otherwise when $(u, v) \neq (p, q)$ or $k \neq h$ it will always be the case that $|r^k_{u,v}| \not\equiv |r^h_{p,q}|$. The collection of functions $\{r^k_{u,v}\colon u, v, k=1, 2, 3, \cdots\}$ is countable and so can be replaced by the Radamacher functions defined on $\Omega = [0, 1]$. Assume that this has already been done to conform with the above requirement.

By this construction, distinct arrays do not have any elements in common and furthermore, when $k \neq h$, we have $\int_\Omega r^k_{a,b}(\omega)\cdot r^h_{c,d}(\omega) = 0$. Also in the kth array there are four special elements which distinguish, so to speak, the four corners of a rectangle. These elements are in relation to one another as

$$\begin{matrix} \alpha & \cdots & \beta \\ \vdots & & \vdots \\ \beta & \cdots & -\alpha \end{matrix} .$$

Notice that for an arbitrary 2-tuple of positive integers (i_0, j_0), all possible rectangles of the above sort with corners (upper left, lower left, upper right, lower right) at (i_0, j_0)th position occurs. Each is in its own array corresponding to some value of k.

Let α_j, $j = 1, 2, \cdots$, be any sequence of positive real numbers which converges to zero. Define

$$\gamma_{i,j} = \alpha_j \sum_{k=1}^{\infty} 2^{-k} r_{i,j}^k .$$

It is almost immediate that for each i, f_i is c_0-valued. Indeed, for all $\omega \in \Omega$,

$$|\gamma_{i,j}(\omega)| = |\alpha_j| \cdot \left| \sum_{k=1}^{\infty} 2^{-k} r_{i,j}^k(\omega) \right| \leqq |\alpha_j| \sum_{k=1}^{\infty} 2^{-k} = \alpha_j ,$$

so that the jth coordinate function of f_i is uniformly bounded by the jth term of a convergent sequence.

From Dominated Convergence, it follows that, if $a \neq b$, then

$$\begin{aligned} \int_\Omega \gamma_{a,j}(\omega) \cdot \gamma_{b,j}(\omega) \mu(d\omega) &= \alpha_j^2 \int_\Omega \left(\sum_{k=1}^{\infty} 2^{-k} r_{a,j}^k(\omega) \right) \left(\sum_{l=1}^{\infty} 2^{-l} r_{b,j}^l(\omega) \right) \mu(d\omega) \\ &= \alpha_j^2 \sum_{k,l=1}^{\infty} 2^{-(k+l)} \int_\Omega r_{a,j}^k(\omega) \cdot r_{b,j}^l(\omega) \mu(d\omega) \\ &= 0. \end{aligned}$$

So that condition (i) is satisfied. On the other hand, if we consider the coordinate functions of two different coordinates, say $\gamma_{a,c}$ and $\gamma_{b,d}$ where $a \neq b$, $c \neq d$, it follows that there is exactly one array in our construction where (a, c) and (b, d) determine two diagonally opposite corners of a rectangle of the form given above. Suppose this occurs for $k = \pi$. Thus it must be that either $r_{a,c}^\pi \equiv -r_{b,d}^\pi$ or $r_{a,c}^\pi \equiv r_{b,d}^\pi$. Now, as before, it follows that

$$\begin{aligned} \int_\Omega \gamma_{a,c}(\omega) \cdot \gamma_{b,d}(\omega) \mu(d\omega) &= \alpha_c \alpha_d \sum_{k,l=1}^{\infty} 2^{-(k+l)} \int_\Omega r_{a,c}^k(\omega) \cdot r_{b,d}^l(\omega) \mu(d\omega) \\ &= \alpha_c \alpha_d \sum_{k=l} 4^{-k} \int r_{a,c}^k(\omega) \cdot r_{b,d}^l(\omega) \mu(d\omega) \\ &= \pm 4^{-\pi} \alpha_c \alpha_d . \end{aligned}$$

Furthermore, (a, d) and (b, c) determine the "other" two corners of the rectangle in this πth array. Thus, by the same considerations,

$$\int_\Omega \gamma_{a,d}(\omega) \cdot \gamma_{b,c}(\omega) \mu(d\omega) = - \int_\Omega \gamma_{a,c}(\omega) \cdot \gamma_{b,d}(\omega) \mu(d\omega) .$$

This verifies conditions (ii) and (iii).

4. General properties. In this section we explore some of the structure of weakly orthogonal sequences of Banach space-valued functions which parallels the structure of orthogonal sequences of Hilbert space-valued functions.

Weak orthogonality of X-valued functions is an invariant with respect to all linear transformations on X. This may be compared with the fact that orthogonality of Hilbert space-valued functions is an invariant only with respect to all unitary operators (an operator is called unitary if and only if composition with its adjoint produces the identity operator). In the following $\mathscr{B}(X, Y)$ denotes the space of linear transformations from the space X into the space Y.

THEOREM 4.1. *Let X and Y be arbitrary Banach spaces over the same field of scalars Φ. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of X-valued functions. Then $\{f_n\}_{n=1}^{\infty}$ is a weakly orthogonal sequence of X-valued functions if and only if, for each $T \in \mathscr{B}(X, Y)$, the sequence of Y-valued functions $\{Tf_n\}_{n=1}^{\infty}$ is also weakly orthogonal.*

Proof. Suppose that f_n, $n = 1, 2, \cdots$, is a weakly orthogonal sequence of X-valued functions. Fix T, where $T \in \mathscr{B}(X, Y)$. Fix an arbitrary element $\psi \in Y^*$. Define x_ψ^* mapping X to Φ by $x_\psi^*(x) = \psi T(x)$ for $x \in X$. Clearly x_ψ^* is a continuous linear functional and hence belongs to X^*. Thus, for each $\psi \in Y^*$,

$$\int \psi(Tf_n(\omega)) \cdot \psi(Tf_m(\omega)) = \int x_\psi^*(f_n(\omega)) \cdot x_\psi^*(f_m(\omega)) = 0$$

when $n \neq m$. This means that Tf_n is a weakly orthogonal sequence of Y-valued functions and this is true for each $T \in \mathscr{B}(X, Y)$.

On the other hand, suppose that Tf_n, $n = 1, 2, \cdots$, is a weakly orthogonal sequence for each $T \in \mathscr{B}(X, Y)$. Choose any vector $\bar{u} \in Y$ such that $\|\bar{u}\| = 1$. Let $V = \{y \in Y\colon y = \alpha\bar{u}$ where $\alpha \in \Phi\}$. Note that V is a 1-dimensional subspace of Y. Define $\mathscr{T} = \{T \in \mathscr{B}(X, Y)\colon T(X) \subseteqq V\}$. That is, $T \in \mathscr{T}$ means that T has 1-dimensional range. For $T \in \mathscr{T}$, let g_T, mapping X to Φ, be defined by

$$g_T(x)\bar{u} = T(x) \qquad \text{for} \quad x \in X .$$

Since $|g_T(x)| \cdot \|\bar{u}\| = \|T(x)\| \leqq \|T\| \cdot \|x\|$, it follows that g_T is continuous and hence $g_T \in X^*$.

Now, for each $T \in \mathscr{T}$, define Y_T^*, mapping $T(X)$ to Φ, by

$$Y_T^*(Tx) = g_T(x) \qquad \text{for} \quad x \in X .$$

It is clear that Y_T^* is linear since g_T is linear. Also Y_T^* is continuous on $T(X)$ since $|Y_T^*(Tx)| = \|Tx\|$. Since $T(X)$ is a linear subspace of Y, the Hahn-Banach theorem implies that we can extend Y_T^* to all of Y. Denote this extension by ψ_T. Note that $\psi_T \in Y^*$. For any $T \in \mathscr{T}$ and $n \neq m$, we have by assumption that

$$\int g_T(f_n(\omega) \cdot g_T(f_m(\omega)) = \int \psi_T(Tf_n(\omega)) \cdot \psi_T(Tf_m(\omega)) = 0 .$$

Finally pick any $\rho \in X^*$ and define T_ρ mapping X to V by

$$T_\rho(x) = \rho(x)\bar{u} .$$

Certainly $T_\rho \in \mathscr{T}$ and $g_{T_\rho}(x) = \rho(x)$ for all $x \in X$ so that $g_{T_\rho} = \rho$. Thus, for any $\rho \in X^*$ and $n \neq m$, it follows that

$$\int \rho(f_n(\omega)) \cdot \rho(f_m(\omega)) = 0 .$$

As a direct corollary, the preceding theorem provides an alternative and more general characterization of weak orthogonality than that given in I.3.1.

COROLLARY 4.2. *Let X be a Banach space and H be a Hilbert space having the same field of scalars Φ. Then f_n, $n = 1, 2, \cdots$, is a weakly orthogonal sequence of X-valued functions if and only if, for every $T \in \mathscr{B}(X, H)$, Tf_n, $n = 1, 2, \cdots$, is an orthogonal sequence of H-valued functions.*

Proof. Suppose f_k, $k = 1, 2, \cdots$, is a weakly orthogonal sequence of X-valued functions. Then, by Theorem 4.1, Tf_k, $k = 1, 2, \cdots$, is a weakly orthogonal sequence of H-valued functions. But, by Theorem 2.1, this implies that Tf_k, $k = 1, 2, \cdots$, is orthogonal in the usual sense.

Now assume that Tf_k, $k = 1, 2, \cdots$, is an orthogonal sequence for each $T \in \mathscr{B}(X, H)$. Choose any vector $\bar{u} \in H$ such that $\|\bar{u}\| = 1$. Let

$$V = \{h \in H: h = \alpha\bar{u} \quad \text{where} \quad \alpha \in \Phi\}.$$

Define

$$\mathscr{T} = \{T \in \mathscr{B}(X, H): T(X) \subseteqq V\} .$$

For $T \in \mathscr{T}$, define g_T, mapping X to Φ, by $g_T(x)\bar{u} = T(x)$ for $x \in X$. Exactly as in the previous proof, we can verify that $g_T \in X^*$. Furthermore,

$$\int < Tf_n(\omega), Tf_m(\omega) > = \|\bar{u}\|_H^2 \int g_T(f_n(\omega)) g_T(f_m(\omega)) .$$

Thus, by our assumption, for $m \neq n$, we have that

$$\int g_T(f_n(\omega)) g_T(f_m(\omega)) = 0 .$$

Pick any $\rho \in X^*$. Define T_ρ, mapping X to V, by

$$T_\rho(x) = x^*(x)\bar{u} .$$

Hence $T_\rho \in \mathscr{T}$ and $g_{T_\rho} = \rho$. This implies, for each $\rho \in X^*$ and $n \neq m$, that we have

$$\int \rho(f_n(\omega)) \cdot \rho(f_m(\omega)) = 0 \; .$$

By now it should be obvious that the term weak orthogonality is somewhat of a misnomer since this condition is really stronger than orthogonality in a Hilbert space. Indeed, many of the nice properties of the orthogonal sequences in Hilbert space are not present in the analogous weakly orthogonal sequences of a Banach space. For example, there is no appropriate analog of the fact that, in a Hilbert space, a maximal (i.e., complete) class of mutually orthogonal functions is a basis for the space. By a maximal orthogonal (or weakly orthogonal) class M, we mean a collection of mutually orthogonal (weakly orthogonal) functions such that if f is orthogonal (weakly orthogonal) to each function in M, then either f belongs to M or $f = 0$ a.e. Maximal weakly orthogonal classes are not necessarily "rich" enough to provide the space with a basis. This is shown in example 4.4 and is a consequence of the following theorem. Here $L_2(\Omega, X)$ denotes the space of L_2-integrable, X-valued functions.

THEOREM 4.3. *Let X be the space l_p, with $1 \leqq p < \infty$, over the scalar field Φ. Let $\{f_n\}_{n=1}^\infty$ be a sequence of X-valued functions. Furthermore, let $f_i = g_i \bar{u}_i$, where $\bar{u}_i$ is the* ith *unit vector in l_p and $g_i \in L_2(\Omega, \Phi)$. If $\{g_n\}_{n=1}^\infty$ is a maximal* (i.e., *complete*) *orthogonal set in the Hilbert space $L_2(\Omega, \Phi)$, then $\{f_n\}_{n=1}^\infty$ is a maximal weakly orthogonal set in $L_2(\Omega, X)$.*

Proof. Suppose that the theorem is false. That is, there exists a function $\phi \in L_2(\Omega, X)$ of the form

$$\phi(\omega) = (\alpha_1(\omega), \alpha_2(\omega), \cdots)$$

such that for each $x^* \in X^*$

$$\int x^*(\phi(\omega))x^*(f_k(\omega))d\omega = 0 \; , \qquad k = 1, 2, \cdots$$

We will show that this will imply that each α_j must be zero a.e.

Consider first a functional $x^* \in X^* = l_p^*$ of the form $x_i^* = (\delta_{1,i}, \delta_{2,i}, \cdots)$. This functional has a 1 in the ith coordinate and zeros elsewhere. Then, for every i,

$$0 = \int x_i^*(\phi(\omega))x_i^*(f_i(\omega)) = < \alpha_i, g_i > \; .$$

Next fix an arbitrary integer j, where $j \neq i$, and consider the functional in l_p^* of the form $x_{i,j}^* = x_i^* + x_j^*$. This functional has a 1 in both the ith and jth coordinates and zeros elsewhere. Then, for every

$i \neq j$,

$$0 \equiv x^*_{i,j}(\phi(\omega))x^*_{i,j}(f_i(\omega)) = < \alpha_i, g_i > + < \alpha_j, g_i > .$$

Therefore $< \alpha_j, g_i > \equiv 0$ for $i = 1, 2, \cdots$, which implies that $\alpha_j = 0$ a.e. since g_i, $i = 1, 2, \cdots$ constitutes a complete orthonormal set. Since j was arbitrary this completes the proof.

EXAMPLE 4.4. Let $p = 2$ in Theorem 4.3 in which case $L_2(\Omega, l_2)$ is a Hilbert space. Suppose furthermore that $\{g_n\}_{n=1}^\infty$ is a complete orthonormal set in $L_2(\Omega, l_2)$ so that according to the theorem, the class of functions $\Gamma = \{f_i = g_i\bar{u}_i : i = 1, 2, \cdots\}$ is a maximal weakly orthogonal class. However, this class does not contain a basis for the space $L_2(\Omega, l_2)$. If it did, then since $L_2(\Omega, l_2)$ is a Hilbert space, it would be necessary that Γ also be a maximal orthogonal class. To see that this is not the case, consider the function given by $f = g_k\bar{u}_h$, $k \neq h$. It is clear that $f \notin \Gamma$ and yet

$$\begin{aligned} \langle f, f_i \rangle_{L_2(\Omega, l_2)} &= \int_\Omega \langle f(\omega), f_i(\omega) \rangle_{l_2} \\ &= \int_\Omega g_k(\omega) \cdot g_h(\omega) \\ &= 0 \quad \text{for all } i , \end{aligned}$$

since the g_n's are orthonormal.

REFERENCE

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