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ON THE ZEROS OF A POLYNOMIAL AND ITS DERIVATIVE

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Let all the zeros of a polynomial $p(z)$ of degree n lie in $|z| \leqq 1$. Given a complex number a what is the radius of the smallest disk centred at a containing at least one zero of the polynomial $((z-a)p(z))'$? According to Theorem 1 the answer is $(|a|+1)/(n+1)$ if $|a| > (n+2)/n$. Theorem 2 which states that if both the zeros of the quadratic polynomial $p(z)$ lie in $|z| \leqq 1$ and $|a| \leqq 2$ then $((z-a)p(z))'$ has at least one zero in

$$|z-a| \leqq \{3|a| + (12-3|a|^2)^{1/2}\}/6$$

completely settles the case $n = 2$.

For $|a| \leqq 1$ the question is equivalent to a problem in [1, (see problem 4.5)] which reads as follows: Is it true that if all the zeros $z_1, z_2, \cdots, z_n$ of the polynomial $p(z) = c\prod_{\nu=1}^{n}(z-z_\nu)$ lie in the disk $|z| \leqq 1$ then $p'(z)$ has at least one zero in each of the disks $|z-z_\nu| \leqq 1$, $\nu = 1, 2, \cdots, n$? It has been shown by Rubinstein [2] that if all the zeros of the polynomial $p(z)$ lie in $|z| \leqq 1$ and $p(1) = 0$ then at least one zero of $p'(z)$ lies in the disk $|z-1| \leqq 1$. On the other hand, the example $z^n - 1$ shows that a disk of radius less than 1 may not contain a zero of $p'(z)$. Thus when $|a| = 1$ the answer to our question is 1.

If a is arbitrary the problem is trivial for $n = 1$ and the answer to the question is $(|a|+1)/2 = (|a|+1)/(n+1)$.

For polynomials of arbitrary degree n we prove

THEOREM 1. *If all the zeros of a polynomial $p(z)$ of degree n lie in the closed unit disk then $((z-a)p(z))'$ has one and only one zero in $|z-a| \leqq (|a|+1)/(n+1)$ provided $|a| > (n+2)/n$. The remaining $n-1$ zeros of $((z-a)p(z))'$ lie in $|z| \leqq 1$. The example $p(z) = (z+e^{i\alpha})^n$ where $\alpha = \arg a$ shows that the result is best possible.*

The disk $|z-a| \leqq (|a|+1)/(n+1)$ may contain more than one zero of $((z-a)p(z))'$ if $|a| = (n+2)/n$. That it contains at least one follows from the fact that the zeros of $((z-a)p(z))'$ are continuous functions of a.

The next theorem gives a solution of the problem when

$$| a | \leqq (n+2)/n \quad \text{and} \quad n = 2 \, .$$

THEOREM 2. *If both the zeros of the quadratic polynomial* $p(z)$ *lie in* $| z | \leqq 1$ *and* $| a | \leqq 2$ *then* $((z-a)p(z))'$ *has at least one zero in*

$$| z-a | \leqq \{3 | a | + (12-3 | a |^2)^{1/2}\}/6 \, .$$

The example

$$p(z) = z^2 - 2 \, [\{3-a(12-3a^2)^{1/2}\}/\{3a-(12-3a^2)^{1/2}\}] \, z + 1, \; 0 \leqq a \leqq 2$$

shows that the result is best possible.

For the proof of Theorem 2 we shall need the following lemma [3, p. 36].

LEMMA. *If both the zeros of the polynomial*

$$A(z) = a_0 + \binom{2}{1} a_1 z + a_2 z^2$$

lie in $| z | \geqq r$ *and those of*

$$B(z) = b_0 + \binom{2}{1} b_1 z + b_2 z^2$$

lie in $| z | > s$ *then both the zeros of the polynomial*

$$C(z) = a_0 b_0 + \binom{2}{1} a_1 b_1 z + a_2 b_2 z^2$$

lie in $| z | > rs$.

Proof of Theorem 1. Let

$$p(z) = c \prod_{\nu=1}^{n} (z-z_\nu)$$

where by hypothesis $| z_\nu | \leqq 1$, $\nu = 1, 2, \cdots, n$. For a given z_0 with $| z_0 | > 1$ the transformation $1/(z_0-z)$ maps the closed unit disk onto some disk $D(z_0)$ in the finite plane. Thus all the numbers $1/(z_0-z_1)$, $1/(z_0-z_2), \cdots, 1/(z_0-z_n)$ belong to $D(z_0)$ and hence so does their arithmetic mean $\mu(z_0)$. But there exists a unique point $\phi(z_0)$ in the disk $| z | \leqq 1$ such that $\mu(z_0) = 1/(z_0-\phi(z_0))$. Consequently

$$p'(z_0)/p(z) = n/(z_0-\phi(z_0)) \, .$$

Since z_0 in an arbitrary point outside the unit disk we get the representation

$$p'(z)/p(z) = n/(z-\phi(z))$$

where $\phi(z) = z-n\{p(z)/p'(z)\}$ is holomorphic and of absolute value at most 1 in $|z| > 1$.

If $|a| > 1$ then

$$p'(z)/p(z) = n\psi(z)/\{(z-a)\psi(z)-1\}$$

where $\psi(z) = 1/(\phi(z)-a)$ is holomorphic in $|z| > 1$ and

$$(1) \qquad 1/(|a|+1) \leqq |\psi(z)| \leqq 1/(|a|-1) .$$

Since

$$\{(z-a)p'(z)+p(z)\}/p(z) = \{(n+1)(z-a)\psi(z)-1\}/\{(z-a)\psi(z)-1\}$$

the zeros of $((z-a)p(z))'$ are the same as the zeros of $(n+1)(z-a)\psi(z)-1$. Now if $|a| > (n+2)/n$ and $(|a|+1)/(n+1) < |z-a| < |a| - 1$ then from (1)

$$|(n+1)(z-a)\psi(z)| > 1 .$$

Hence by Rouché's theorem $(n+1)(z-a)\psi(z)-1$, $(n+1)(z-a)\psi(z)$ have the same number of zeros in $|z-a| \leqq (|a|+1)/(n+1)$, namely 1. Given $\xi \in \{z: |z| \leqq 1\} \cup \{z: |z-a| \leqq (|a|+1)/(n+1)\}$ we can draw a contour C such that $\{z: |z-a| \leqq (|a|+1)/(n+1)\}$ and the point ξ lie in C_i (the bounded domain determined by C) whereas $\{z: |z| \leqq 1\}$ lies in C_e (the unbounded domain determined by C). According to the above reasoning $((z-a)p(z))'$ has one and only one zero in C_i. Since we know that the zero lies in $|z-a| \leqq (a|+1)/(n+1)$ the point ξ cannot be a zero of $((z-a)p(z))'$. Hence the remaining $n-1$ zeros of $((z-a)p(z))'$ lie in $|z| \leqq 1$.

REMARK. Theorem 1 may be refined by observing that $(n+1)(z-a)\psi(z) - 1 \equiv (n+1)(z-a)(\phi(z)-a)^{-1} - 1$ can vanish only if $z - na/(n+1) = \phi(z)/(n+1)$. Hence in fact $((z-a)p(z))'$ has one and only one zero in $D = \{z: |z - na/(n+1)| \leqq 1/(n+1)\}$. By considering $p(z) = (z-z_0)^n$ with an appropriate z_0 in the closed unit disk we see that any given point of D can be a zero of $((z-a)p(z))'$.

Proof of Theorem 2. Without loss of generality we may suppose $0 \leqq a \leqq 2$. Let

$$p(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2$$

and put

$$f(z) = ((z-a)p(z))' = (\alpha_0 - a\alpha_1) + 2(\alpha_1 - a\alpha_2)z + 3\alpha_2 z^2 ,$$
$$s = \{3a + (12-3a^2)^{1/2}\}/6 .$$

We wish to prove that $f(z)$ must vanish is $|z-a| \leqq s$. If not, both the zeros of

$$B(z) = f(z+a) = \alpha_0 + a\alpha_1 + a^2\alpha_2 + \binom{2}{1}(\alpha_1+2a\alpha_2)z + 3\alpha_2 z^2$$

lie in $|z| > s$. Since both the zeros of

$$A(z) = 1 + \binom{2}{1}(1/2)z + (1/3)z^2$$

lie on $|z| = \sqrt{3}$ the lemma implies that both the zeros of the polynomial

$$C(z) = \alpha_0 + a\alpha_1 + a^2\alpha_2 + (\alpha_1+2a\alpha_2)z + \alpha_2 z^2 \equiv p(z+a)$$

lie in $|z| > \sqrt{3}s$, i.e., the polynomial $p(z)$ does not vanish in $|z-a| \leqq \sqrt{3}s$. We can therefore find a positive number ε such that the disk $|z-(a-2s)| \leqq s - \varepsilon$ contains both the zeros of $p(z)$. Now it can be easily deduced from Theorem 1 that $((z-a)p(z))'$ has one and only one zero in $|z-a| \leqq s - \varepsilon/3$. This completes the proof of Theorem 2.

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