

ISOMORPHIC POWER SERIES RINGS

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Let A and B be commutative rings with identity, let X be an indeterminate over A and B , and let $A[[X]]$ and $B[[X]]$ be the formal power series rings over A and B , respectively.

The motivation for this paper was to consider the analogue of a question raised by Coleman and Enochs for the polynomial ring. Specifically, the following question is considered:

(*) If $A[[X]] \cong B[[X]]$, must $A \cong B$?

(The author knows no counterexample.)

The paper is divided into three parts. The first part, § 2, deals with the problem of extending a given homomorphism ρ of A into $B[[X]]$ to a homomorphism ϕ of $A[[X]]$ into $B[[X]]$ such that ϕ maps X onto an arbitrary element β of $B[[X]]$. Necessary and sufficient conditions are given in order that there exists an extension of ρ to an isomorphism ϕ of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta$. In § 3, it is shown that if ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$ such that either $\phi(A) \subseteq B$ or $B \subseteq \phi(A)$, then $\phi(A) = B$, so that ϕ itself is an isomorphism of A onto B . In § 4, sufficient conditions are imposed on A to guarantee that the condition $A[[X]] \cong B[[X]]$ implies that $A \cong B$. In particular, if A is semisimple (the Jacobson radical of $A = (0)$), then the condition $A[[X]] \cong B[[X]]$ implies that $A \cong B$.

All rings considered in this paper are assumed to be commutative and contain an identity element. The symbols ω and ω_0 are used throughout the paper to denote the sets of positive and non-negative integers, respectively. If $g = \sum_{i=0}^{\infty} c_i X^i$ is a nonzero element of $B[[X]]$ such that the first nonzero coefficient of g is c_k , then we say that g has order k and we write $O(g) = k$.

Throughout the paper, our notation will be the same as that of [4]. In particular, if $\beta \in B[[X]]$ and if T is a subring of $B[[X]]$ containing β and the identity of B , then $(\beta^n T)$ will denote the ideal of T generated by β^n , and $(T, (\beta T))$ the topological ring T in the (βT) -adic topology. It is well known that $(T, (\beta T))$ is a Hausdorff space if and only if $\bigcap_{n \in \omega} (\beta^n T) = (0)$. [6; p. 252]. We say that $(T, (\beta T))$ is complete if each Cauchy sequence of T converges to a point of T . If U is a subring of $B[[X]]$, then $U[\beta]$ will be the subring of $B[[X]]$ consisting of all finite sums of the form: $\sum_{i=0}^n u_i \beta^i$, $u_i \in U$.

We denote the Jacobson radical of a ring A by $J(A)$ and we make frequent use of the two facts:

(i) $J(A) = \{a \in A \mid 1 - ra \text{ is a unit of } A \text{ for all } r \in A\}$. [5; p. 206].

(ii) $J(A[[X]]) = J(A)A[[X]] + (X)$. [3; p. 62].

2. Homomorphisms of power series rings. Let $\beta \in B[[X]]$ and suppose that ρ is a homomorphism of A into $B[[X]]$. In this section we consider the problem of extending ρ to a homomorphism ϕ of $A[[X]]$ into $B[[X]]$ such that $\phi(X) = \beta$. O'Malley [3] and O'Malley and Wood [4] have considered this problem for the case when $A = B$ and ρ is the identity map. Several, but not all, of the results of [4], together with their proofs, extend to the general case with only minor modifications. We summarize the results of interest here and reference the corresponding result in [4]. Throughout this section, we suppose that ρ maps the identity of A onto the identity of B , and since there will be no ambiguity, we write 1 to denote either identity. Note that if an extension ϕ of ρ exists for which $B \subseteq \phi(A[[X]])$, then the condition $\rho(1) = 1$ is necessarily satisfied. Throughout this section, we denote $\rho(A)$ by U .

We first observe that if T is a subring of $B[[X]]$ containing $U[\beta]$ and such that $(T, (\beta T))$ is a complete Hausdorff space, then we may define a mapping ϕ from $A[[X]]$ into $B[[X]]$ by $\phi(f) = \lim_n \sum_{i=0}^n \rho(f_i) \beta^i$, for each $f = \sum_{i=0}^{\infty} f_i X^i \in A[[X]]$, where the limit is taken in $(T, (\beta T))$. Following the proof for Theorem (2.2) given in [4], it can easily be shown that ϕ , as defined above, is the unique homomorphism of $A[[X]]$ into $B[[X]]$ that extends ρ , maps X onto β , and has range contained in T .

We note that if $\bigcap_{n \in \omega} (\beta^n) = (0)$ (in $B[[X]]$), and if ϕ is a homomorphism of $A[[X]]$ into $B[[X]]$ such that $\phi/A = \rho$ and $\phi(X) = \beta$, then it can be shown that ϕ is the unique homomorphism of $A[[X]]$ into $B[[X]]$ with these two properties. In particular, if ϕ is an isomorphism (onto $B[[X]]$), then $\bigcap_{n \in \omega} (\beta^n) = \bigcap_{n \in \omega} (\phi(X))^n = \phi(\bigcap_{n \in \omega} (X^n)) = \phi((0)) = (0)$, and hence, ϕ is the unique homomorphism of $A[[X]]$ into $B[[X]]$ that extends ρ and maps X onto β .

We now observe that if we impose the added condition on T that each element of T be the limit of a Cauchy sequence of $(U[\beta], (\beta U[\beta]))$, then we have the following theorem.

THEOREM 2.1. *Let T be a subring of $B[[X]]$ containing $U[\beta]$ and satisfying each of the following:*

(i) *$(T, (\beta T))$ is a complete Hausdorff space;*

(ii) *Each element of T is the limit of a Cauchy sequence of $(U[\beta], (\beta U[\beta]))$.*

Then there exists a unique homomorphism ϕ of $A[[X]]$ into $B[[X]]$

that extends ρ , maps X onto β , and has range equal to T .

Conversely, let ϕ be a homomorphism of $A[[X]]$ into $B[[X]]$ such that $\phi/A = \rho$ and $\phi(X) = \beta$. If $T = \phi(A[[X]])$, and if $\bigcap_{n \in \omega} (\beta^n T) = (0)$, then T is a subring of $B[[X]]$ containing $U[\beta]$ and satisfying conditions (i) and (ii).

Proof. See [4; Theorem (2.3)].

For the one-to-one case, we have the following characterization. We note that this is the first result that does not carry over intact from [4]. Part (iii) of Example (2.6) will provide an example of an isomorphism ϕ of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta = \sum_{i=0}^{\infty} b_i X^i$ and such that $\beta - b_0 = 0$. In particular, $\beta - b_0$ is not regular in $U[\beta]$, and hence the change here in condition (i) of Theorem (2.2) from that of condition (i) of Theorem (3.2) in [4] is necessary.

THEOREM 2.2. *Let $\beta = \sum_{i=0}^{\infty} b_i X^i \in B[[X]]$ and suppose that ρ is a homomorphism of A into $B[[X]]$. There exists a one-to-one homomorphism ϕ of $A[[X]]$ into $B[[X]]$ such that $\phi(X) = \beta$ and $\phi/A = \rho$ if and only if the following conditions are satisfied:*

- (i) β is transcendental over U .
- (ii) ρ is one-to-one.
- (iii) There exists a subring T of $B[[X]]$ such that $(T, (\beta T))$ is a completion of $(U[\beta], \beta U[\beta])$.

Proof. [4; Lemma (3.1) and Theorem (3.2)].

Observe that by Theorem 2.1, if $\bigcap_{n \in \omega} (\beta^n) = (0)$ (in $B[[X]]$), then there exists a homomorphism ϕ of $A[[X]]$ onto $B[[X]]$ such that $\rho(X) = \beta$ and $\phi/A = \rho$ if and only if $(B[[X]], (\beta))$ is complete and $(U[\beta], (\beta U[\beta]))$ is dense in $(B[[X]], (\beta))$. Thus with Theorem 2.2 and its proof, we have the following.

THEOREM 2.3. *Under the hypothesis of Theorem 2.2, there exists an isomorphism ϕ of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta$ and $\phi/A = \rho$ if and only if the following conditions are satisfied:*

- (i) β is transcendental over U .
- (ii) ρ is one-to-one.
- (iii) $(B[[X]], (\beta))$ is a completion of $(U[\beta], (\beta U[\beta]))$. Moreover, when such an isomorphism ϕ exists, ϕ is the unique isomorphism of $A[[X]]$ onto $B[[X]]$ that extends ρ and maps X onto β .

REMARK 2.4. It is clear from the preceding results of this section that the conditions obtained here to guarantee the existence

of an extension of ρ to a homomorphism (isomorphism) ϕ of $A[[X]]$ into (onto) $B[[X]]$ such that $\phi(X) = \beta$ are very similar to the conditions obtained in [4] to guarantee the existence of a B -endomorphism (automorphism) ψ of $B[[X]]$ such that $\psi(X) = \beta$. In particular, if ϕ is an extension of ρ to an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta$, then $(B[[X]], (\beta))$ is a complete Hausdorff space (Theorem 2.3), and hence, by [4; Theorem 2.2], there exists a B -endomorphism ψ of $B[[X]]$ such that $\psi(X) = \beta$. A natural question then is: must ψ be an automorphism of $B[[X]]$? The answer is no in general, but we are able to obtain necessary and sufficient conditions in order that the answer be yes. We provide these conditions (under slightly more general hypothesis) in Theorem 2.5, and in part (iii) of Example 2.6 we provide an example of rings A and B and an isomorphism ϕ of $A[[X]]$ onto $B[[X]]$ for which the corresponding B -endomorphism of $B[[X]]$ is neither one-to-one nor onto.

However, before doing this, we first observe that if $A[[X]] \cong B[[X]]$ under ϕ , and if the corresponding B -endomorphism ψ of $B[[X]]$ such that $\psi(X) = \phi(X)$ is, in fact, an automorphism, then we have an affirmative answer to our question (*). For, in this case, if $\tau = \psi^{-1} \circ \phi$, then τ is an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $\tau(X) = X$. Hence,

$$A \cong A[[X]]/(X) \cong B[[X]]/(\tau(X)) = B[[X]]/(X) \cong B.$$

This observation is a useful tool in seeking to obtain sufficient conditions on A to guarantee an affirmative answer to (*). It was used extensively in [1] in considering the analogous question for the polynomial ring, and we use it in § 4 of this paper.

THEOREM 2.5. *Let $\beta \in B[[X]]$ be such that $\bigcap_{n \in \omega} (\beta^n) = (0)$ and let ϕ be a homomorphism of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta$. Then the unique B -endomorphism ψ of $B[[X]]$ such that $\psi(X) = \beta$ is an automorphism if and only if $U = \phi(A) \subseteq \psi(B[[X]])$.*

Proof. First note that since ϕ is onto and since $\bigcap_{n \in \omega} (\beta^n) = (0)$, then by the comment preceding Theorem 2.3 $(B[[X]], (\beta))$ is a complete Hausdorff space and $(U[\beta], (\beta U[\beta]))$ is dense in $(B[[X]], (\beta))$. In particular, since $(B[[X]], (\beta))$ is a complete Hausdorff space, there exists a unique B -endomorphism ψ of $B[[X]]$ such that $\psi(X) = \beta$.

If ψ is an automorphism, then clearly $\phi(A) \subseteq \psi(B[[X]]) = B[[X]]$. Conversely, if $U = \phi(A) \subseteq \psi(B[[X]])$, then, since $\beta \in \psi(B[[X]])$, we have that $U[\beta] \subseteq \psi(B[[X]])$. Thus, since $(U[\beta], (\beta U[\beta]))$ is dense in $(B[[X]], (\beta))$, it follows that $(\psi(B[[X]]), (\beta \psi(B[[X]])))$ is dense in $(B[[X]], (\beta))$. But $(\psi(B[[X]]), (\beta \psi(B[[X]])))$ is a complete Hausdorff

space ([4; Theorem 2.3]), and thus it follows that $\psi(B[[X]]) = B[[X]]$. Therefore, ψ is onto and hence is an automorphism of $B[[X]]$. [4; Corollary 4.6].

One of the main results of [4; Theorem 4.7] shows that if ψ is a B -endomorphism of $B[[X]]$ such that $\psi(X) = \sum_{i=0}^{\infty} b_i X^i = \beta$, then the following conditions are equivalent.

- (a) ψ is onto.
- (b) ψ is an automorphism.
- (c) b_1 is a unit of B .

The only implication true for the general case is the obvious one (b) \rightarrow (a). We provide counterexamples to show that none of the other implications are true. It suffices to show that (c) \rightarrow (a), (a) \rightarrow (b), and (b) \rightarrow (c). Note that for each of our examples $A = B$.

EXAMPLE 2.6. Let R be a ring, let $\{Y_i\}_{i \in \omega}$ be an infinite collection of indeterminates over R , and let $A = B = R[\{Y_i\}_{i \in \omega}]$ be the polynomial ring in an infinite number of indeterminates over R . Let X be an indeterminate over B ; we use $B[[X]]$ and $\beta = X$ for the first two examples.

(i) (c) \rightarrow (a). Define $\rho: B \rightarrow B[[X]]$ by $\rho(r) = r$, for each $r \in R$, and $\rho(Y_i) = Y_{i+1}$ for each $i \in \omega$. Then ρ determines a unique one-to-one homomorphism of B into $B[[X]]$, and, since $(B[[X]], (X))$ is a complete Hausdorff space, we may extend ρ to an endomorphism ϕ of $B[[X]]$ such that $\phi(X) = X = \beta$. Since $Y_1 \notin \text{range of } \phi$, ϕ is not onto, but $b_1 = 1$ is a unit of B .

(ii) (a) \rightarrow (b). Define $\rho: B \rightarrow B[[X]]$ by $\rho(r) = r$, for each $r \in R$, $\rho(Y_{i+1}) = Y_i$ for each $i \in \omega$, and $\rho(Y_1) = X$. Then the unique one-to-one homomorphism of B into $B[[X]]$ determined by ρ has range = $B[X]$. Since $B[X][X] = B[X]$ and since $(B[[X]], (X))$ is the completion of $(B[X], (XB[X]))$, there exists an extension of ρ to an endomorphism ϕ of $B[[X]]$ onto $B[[X]]$ such that $\phi(X) = X$. Clearly, ϕ is not one-to-one.

(iii) (b) \rightarrow (c). Let R be a ring, X, Y indeterminates over R , and $R[[X, Y]] = R[[Y]][[X]]$ the formal power series ring in X and Y . If $A = B = R[[Y]]$, then the mapping ϕ from $B[[X]]$ into itself that is the identity on R and interchanges X and Y is clearly an automorphism of $B[[X]]$ and, in particular, $\phi(X) = \sum_{i=0}^{\infty} b_i X^i$, where $b_0 = Y$ and $b_i = 0$, for each $i \geq 1$.

3. Isomorphisms of $A[[X]]$ onto $B[[X]]$ that are isomorphisms of A onto B . In [1], it is shown that if $\sigma: A[X] \rightarrow B[X]$ is an isomorphism such that $\sigma(A) \subseteq B$, then $\sigma(A) = B$. As an easy consequence of this result is the fact that the same conclusion holds, if the reverse containment is assumed. The main purpose of this

section will be to prove the corresponding results for the power series ring.

THEOREM 3.1. *Let $\beta = \sum_{i=0}^{\infty} b_i X^i \in B[[X]]$ and suppose that ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta$. If $\phi(A) \subseteq B$, then $\phi(A) = B$.*

We require a few preliminary observations before proving Theorem 3.1. We make strong use of the results of [4]. We begin with a lemma.

LEMMA 3.2. *Let $\sigma: A[[X]] \rightarrow B[[X]]$ be a homomorphism such that $\sigma(X) = \gamma = \sum_{i=0}^{\infty} c_i X^i$. If $B \subseteq \sigma(A[[X]])$, then $c_0 \in J(B)$. In particular, if σ is onto, then $c_0 \in J(B)$.*

Proof. For each $g \in A[[X]]$, $1 + gX$ is a unit of $A[[X]]$, and hence, $\sigma(1 + gX) = 1 + \sigma(g)\gamma$ is a unit of $\sigma(A[[X]])$. Thus, if $B \subseteq \sigma(A[[X]])$, then, for each $b \in B$, there exists a $g \in A[[X]]$ such that $\sigma(g) = b$, and hence $\sigma(1 + gX) = 1 + b\gamma = (1 + bc_0) + \sum_{i=1}^{\infty} bc_i X^i$ is a unit of $B[[X]]$. Therefore, by [6; p. 131], $1 + bc_0$ is a unit of B for each $b \in B$, and thus, $c_0 \in J(B)$. [5; p. 206].

Suppose that ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$ satisfying the hypothesis of Theorem 3.1. If ψ denotes the unique B -endomorphism of $B[[X]]$ that maps X onto β , then $\phi(A) \subseteq B \subseteq \psi(B[[X]])$, and hence, by Theorem 2.5, ψ is an automorphism. Therefore, b_1 is a unit of B . [4; Theorem 4.5].

Of prime importance to us is the following.

THEOREM 3.3. *Under the hypothesis of Theorem 3.1, there exists an A -automorphism τ of $A[[X]]$ such that $\tau(X) = \phi^{-1}(-b_0) + X$.*

Proof. Let $\phi(h) = -b_0$; $h = \sum_{i=0}^{\infty} h_i X^i \in A[[X]]$. Then $\phi(h + X) = \beta - b_0$ and $\phi^{-1}(\beta - b_0) = h + X$. Thus, since $\phi^{-1}: (B[[X]], (\beta - b_0)) \rightarrow (A[[X]], (h + X))$ is continuous, and since $(B[[X]], (\beta - b_0))$ is a complete Hausdorff space, it follows that $(A[[X]], (h + X))$ is a complete Hausdorff space. Therefore, by [4; Theorem 2.2], there exists a unique A -endomorphism τ of $A[[X]]$ such that $\tau(X) = h + X = \phi^{-1}(\beta - b_0) = \phi^{-1}(-b_0) + X$. We show that τ is an automorphism.

Consider $\phi(h) = \phi(\sum_{i=0}^{\infty} h_i X^i) = \lim_n \sum_{i=0}^n \phi(h_i) \beta^i = -b_0$, where the limit is taken in the (β) -adic topology on $B[[X]]$. It follows that, for $k \geq 2$, there exists $N_k \in \omega$ such that $\sum_{i=0}^n \phi(h_i) \beta^i + b_0 \in (\beta)^k$ for $n \geq N_k$. Thus, if $\sum_{i=0}^n \phi(h_i) \beta^i + b_0 = v\beta^k$, then, expanding both sides of the equality in powers of X , we have that the coefficient of X on the left-hand side has the form $\phi(h_1)b_1 + \sum_{i=1}^{n-1} \phi(h_{i+1})c_i b_0^i$, where $c_i \in B$

for each i , while the coefficient of X on the right-hand side belongs to (b_0) . Therefore, since $b_0 \in J(B)$ (Lemma 3.2), and since b_1 is a unit of B , it follows that $\phi(h_1) \in J(B)$, and hence $h_1 \in J(A)$. Thus, $1 + h_1$, the coefficient of X in $\tau(X)$, is a unit of A , so that τ is an A -automorphism of $A[[X]]$. [4; Theorem 4.7].

Proof of Theorem 3.1. Consider the isomorphism $\mu = \phi \circ \tau$ of $A[[X]]$ onto $B[[X]]$. Since $\phi \circ \tau(X) = \phi(\phi^{-1}(-b_0) + X) = \beta - b_0$, and since $\phi \circ \tau(a) = \phi(a)$ for each $a \in A$, it follows from the definition and comments preceding Theorem 2.1 that, for each $g = \sum_{i=0}^{\infty} g_i X^i \in A[[X]]$, $\mu(g) = \lim_n \sum_{i=0}^n \phi(g_i)(\beta - b_0)^i = \sum_{i=0}^{\infty} \phi(g_i)(\beta - b_0)^i$. Thus, since μ is onto, for each $b \in B$, there exists $g = \sum_{i=0}^{\infty} g_i X^i \in A[[X]]$ such that $\mu(g) = \sum_{i=0}^{\infty} \phi(g_i)(\beta - b_0)^i = b$. But since $O(\beta - b_0) = 1$ and since $\phi(A) \subseteq B$, it follows that $\phi(g_0) = b$. Therefore, $B \subseteq \phi(A)$, and thus we have equality.

COROLLARY 3.4. *Let $\beta = \sum_{i=0}^{\infty} b_i X^i \in B[[X]]$ and suppose that ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta$. If $B \subseteq \phi(A)$, then $\phi(A) = B$.*

Proof. Since $B \subseteq \phi(A)$, $\phi^{-1}(B) \subseteq A$. Thus, ϕ^{-1} is an isomorphism of $B[[X]]$ onto $A[[X]]$ such that $\phi^{-1}(B) \subseteq A$. Therefore, by Theorem 3.1, $\phi^{-1}(B) = A$, and hence, $B = \phi(\phi^{-1}(B)) = \phi(A)$.

In connection with Theorem 3.1 and Corollary 3.4 it should be remarked here that if $\rho: A \rightarrow B$ is an isomorphism and if $\beta \in B[[X]]$, then the existence of an extension of ρ to a homomorphism ϕ of $A[[X]]$ into $B[[X]]$ such that $\phi(X) = \beta$ is equivalent to the existence of a B -endomorphism ψ of $B[[X]]$ such that $\psi(X) = \beta$. It can be shown that this correspondence establishes a one-to-one relationship between the set of homomorphisms of $A[[X]]$ into $B[[X]]$ which restrict to ρ on A and the set of B -endomorphisms of $B[[X]]$.

In particular, we have the following theorem. We note that it is the analogue of a result of Gilmer for the polynomial ring. [2; Theorem 4].

THEOREM 3.5. *Let ρ be an isomorphism of A onto B and let $\beta \in B[[X]]$. There exists an extension of ρ to a homomorphism ϕ of $A[[X]]$ into $B[[X]]$ such that $\phi(X) = \beta$ if and only if there exists a B -endomorphism ψ of $B[[X]]$ such that $\psi(X) = \beta$. Moreover, if such an extension ϕ exists, then:*

(i) ϕ is the unique extension of ρ to a homomorphism of $A[[X]]$ into $B[[X]]$ that maps X onto β if and only if ψ is the unique B -endomorphism of $B[[X]]$ that maps X onto β .

(ii) ϕ is one-to-one if and only if ψ is one-to-one.

(iii) ϕ is onto if and only if ψ is onto. Hence, by (ii) and (iii), ϕ is an isomorphism if and only if ψ is a B -automorphism of $B[[X]]$.

Proof. Observe that if such an extension ϕ exists, then $\psi: B[[X]] \rightarrow B[[X]]$ defined by $\psi(\sum_{i=0}^{\infty} f_i X^i) = \phi(\sum_{i=0}^{\infty} \rho^{-1}(f_i) X^i)$ is a B -endomorphism of $B[[X]]$ such that $\psi(X) = \beta$. Conversely, if such a B -endomorphism ψ exists, then $\phi: A[[X]] \rightarrow B[[X]]$ defined by $\phi(\sum_{i=0}^{\infty} a_i X^i) = \psi(\sum_{i=0}^{\infty} \rho(a_i) X^i)$ is an extension of ρ to a homomorphism of $A[[X]]$ into $B[[X]]$ such that $\phi(X) = \beta$.

The verification of these statements, together with (i), (ii), and (iii), is straightforward, and we omit it.

4. Strongly power invariant rings. In this section, in contrast to § 3, we attempt to impose conditions on A (rather than ϕ) to guarantee that the condition $A[[X]] \cong B[[X]]$ implies that $A \cong B$.

Extending the terminology used in [1], we say that a ring A is *power invariant* if whenever B is a ring such that $A[[X]]$ is isomorphic to $B[[X]]$, then A and B are isomorphic. A ring A is said to be *strongly power invariant* if whenever B is a ring and ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$, then there exists a B -automorphism ψ of $B[[X]]$ such that $\psi(X) = \phi(X)$. We have already implicitly observed in Remark 2.4 that a strongly power invariant ring is power invariant. However, part (iii) of Example 2.6 shows that $R[[X]]$ is not strongly power invariant for any ring R .

In this section we shall be concerned only with the property: strongly power invariant. Our main result is that any semisimple ring A ($J(A) = (0)$) is strongly power invariant.

Observe that if R and S are rings and if ϕ is a homomorphism of R onto S , then $\phi(J(R)) \subseteq J(S)$, and if, in addition, the kernel of ϕ is contained in $J(R)$, equality holds.

We have a simple, but key lemma.

LEMMA 4.1. *Let $\beta = \sum_{i=0}^{\infty} b_i X^i \in B[[X]]$ and suppose that ϕ is a homomorphism of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta$. If the kernel of ϕ is contained in $J(A[[X]]) = J(A) + (X)$, and if $\phi(J(A)) \subseteq B$, then b_1 is a unit of B .*

Proof. By the previous remark, $\phi(J(A) + (X)) = \phi(J(A)) + (\beta) = J(B) + (X)$. Thus, there exists $g \in \phi(J(A))$, $h = \sum_{i=0}^{\infty} h_i X^i \in B[[X]]$ such that $X = g + h\beta$. Since $\phi(J(A)) \subseteq B$, $g \in B$, and therefore, equating coefficients of X , we have that $1 = h_0 b_1 + h_1 b_0$. By Lemma

3.2, $b_0 \in J(B)$, and thus $h_0 b_1 = 1 - h_1 b_0$ is a unit of B , so that b_1 is a unit of B .

LEMMA 4.2. *Under the hypothesis of Lemma 4.1, if ϕ is an isomorphism, then there exists a B -automorphism ψ of $B[[X]]$ such that $\psi(X) = \beta$.*

Proof. We have already observed (Remark 2.4) that if ϕ is an isomorphism then there exists a B -endomorphism ψ of $B[[X]]$ such that $\psi(X) = \beta$. Hence, by Lemma 4.1, b_1 is a unit of B , and thus ψ is an automorphism. [4; Theorem 4.7].

RESULT 4.3. *For any ring A , $A/J(A)$ is strongly power invariant. In particular, if A is a semisimple ring, then A is strongly power invariant.*

Proof. Let $C = A/J(A)$ and let B be a ring having the property that there exists an isomorphism ϕ of $C[[X]]$ onto $B[[X]]$. If $\phi(X) = \beta = \sum_{i=0}^{\infty} b_i X^i$ and if ψ is the corresponding B -endomorphism of $B[[X]]$ mapping X onto β , then, since $J(C) = (0)$, Lemma (4.2) implies that b_1 is a unit of B , and therefore, ψ is an automorphism.

Observing that the units of $B[[X]]$ coincide with the units of B if and only if B has no nonzero nilpotent element, we have the following.

COROLLARY 4.4. *Let $\beta = \sum_{i=0}^{\infty} b_i X^i \in B[[X]]$ and suppose that ϕ is an isomorphism of $A[[X]]$ onto $B[[X]]$ such that $\phi(X) = \beta$. If A has no nonzero nilpotent element and if $\phi(A) \subseteq B[[X]]$, then $A \cong B$.*

Proof. Since A has no nonzero nilpotent element, it follows that $A[[X]]$ has no nonzero nilpotent element. Thus, since ϕ is an isomorphism, we have that $B[[X]]$, and in particular, B , has no nonzero nilpotent element. Thus, if $t \in J(A)$, then $1 - t$ is a unit of A , and hence, $\phi(1 - t) = 1 - \phi(t)$ is a unit of $\phi(A) \subseteq B[[X]]$. By the preceding observation, it follows that $1 - \phi(t) \in B$, so that $\phi(t) \in B$. Thus, $\phi(J(A)) \subseteq B$, and hence, it follows from Lemma 4.2 that there exists a B -automorphism ψ of $B[[X]]$ such that $\psi(X) = \beta$. Therefore, as observed in Remark 2.4, $A \cong B$.

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