# OPEN MAPPINGS ON 2-MANIFOLDS

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This paper will be concerned with the local structure of open, continuous functions $f: M^2 \to N^1$ from a 2-manifold into the real line or the circle. The two main results are:

THEOREM 1. Let $f: M^2 \to N^1$ be an open mapping, and suppose that $f$ has isolated branch points. Then for each point $p$ in $M^2$ there are a neighborhood $U$ of $p$ and a positive integer $d$ (depending on $p$) such that $f|_U$ is topologically equivalent to the real analytic mapping $z \to Re(z^d)$.

THEOREM 2. If $f: M^2 \to N^1$ is an open, real analytic mapping, then $f$ has isolated branch points, and (hence) the conclusion of Theorem 1 holds.

2. **Definitions and notation.** Let $f: X \to Y$ be any continuous function. Then $f$ is an *open mapping* if $f(U)$ is an open subset of $Y$ whenever $U$ is an open subset of $X$. If $f': X' \to Y'$ is another mapping then $f$ and $f'$ are *topologically equivalent* if there exist homeomorphisms $\alpha: X \to X'$, $\beta: Y \to Y'$ such that $f'\alpha = \beta f$.

If $A$ is any subset of $X$, then $f|_A$ will denote the restriction of $f$ to $A$.

Let $f: M^n \to N^p$ be a mapping from an $n$-manifold into a $p$-manifold, $n \geq p$. The *branch set*, $B_f$, of $f$ is the subset of $M^n$ defined as follows: a point $x$ is in $M^n$-$B_f$ if there exists a neighborhood $U$ of $x$ in $M^n$ such that $f|_U$ is topologically equivalent to the canonical projection of $n$-space onto $p$-space.

If $M^n$ and $N^p$ are differentiable manifolds, and $f: M^n \to N^p$ is differentiable, then $R_q(f)$ will denote the set of points in $M^n$ at which the rank of the Jacobian of $f$ is at most $q$, $q = 0, 1, \cdots$.

The boundary of a set $A$ will be denoted by $\partial A$ and its closure by $\bar{A}$.

3. *Proof of Theorem 1.* We first need some lemmas.

LEMMA 3.1. Let $f: [0, 1] \times [0, 1] \to [0, 1]$ with $\{i\} \times [0, 1] = f^{-1}(i)$, $i = 0, 1$. Suppose there exists a homeomorphism $\alpha: [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ which fixes the four vertices of $[0, 1] \times [0, 1]$ such that $\Pi\alpha = f$, where $\Pi$ is the projection $(x, y) \to x$. Let $h: \{0\} \times [0, 1] \to \{0\} \times [0, 1]$ be any homeomorphism with $h(0, 0) = (0, 0)$. Then there exists a homeomorphism $\beta: [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ such that $\Pi\beta = f$ and $\beta|_{\{0\} \times [0,1]} = h$.

*Proof.* The desired homeomorphism is given by $\beta(x, y) = (f(x, y),$

$h^2\alpha^{-1}(0, \alpha^2(x, y)))$, where $h^2$ and $\alpha^2$ are the second coordinate functions of $h$ and $\alpha$, respectively.

LEMMA 3.2. *Let* $f\colon [0, 1] \times [0, 1] \to [0, 1]$ *be an open mapping. Suppose that the following conditions are satisfied:*

( i )  $f^{-1}(0) = \{0\} \times [0, 1]$, $f^{-1}(1) = \{1\} \times [0, 1]$,

( ii )  *if* $g$ *is the restriction of* $f$ *to* $(0, 1) \times (0, 1)$, *then* $B_g = \varnothing$,

(iii)  *for each point* $y$ *in the open interval* $(0, 1)$, $y \neq 1/2$ *there is a neighborhood of the point* $(1, y)$ *in* $[0, 1] \times [0, 1]$ *on which* $f$ *is topologically equivalent to the projection of* $(0, 1] \times (0, 1)$ *onto* $(0, 1]$, *and for each point* $y$ *in the open interval* $(0, 1)$ *there is a neighborhood of* $(0, y)$ *in* $[0, 1] \times [0, 1]$ *on which* $f$ *is topologically equivalent to the projection of* $[0, 1) \times (0, 1)$ *onto* $[0, 1)$,

(iv)  *for each point* $x$ *in the open interval* $(0, 1)$, *there is a neighborhood of the point* $(x, 0)$ *in* $[0, 1] \times [0, 1]$ *and a neighborhood of the point* $(x, 1)$ *in* $[0, 1] \times [0, 1]$ *on which* $f$ *is topologically equivalent to the projection of* $(0, 1) \times [0, 1)$ *onto* $(0, 1)$, *and*

( v )  *if* $x = 0$ *or* $1$, *and* $y = 0$ *or* $1$, *then there exists a neighborhood of the point* $(x, y)$ *in* $[0, 1] \times [0, 1]$ *on which* $f$ *is topologically equivalent to the first coordinate projection of* $[0, 1) \times [0, 1)$ *onto* $[0, 1)$.

*Then* $f$ *is topologically equivalent to the first coordinate projection of* $[0, 1] \times [0, 1]$ *onto* $[0, 1]$.

*Proof.*  Let $\Pi\colon [0, 1] \times [0, 1] \to [0, 1]$ denote the projection $(x, y) \to x$. Let $a_0 = [0, 1] \times \{1\}$, and let $b_0 = [0, 1] \times \{0\}$. Let $\{y_n^+\}$ be a monotone decreasing sequence in the open interval $(1/2, 1)$ converging to $1/2$, and let $\{y_n^-\}$ be a monotone increasing sequence in the open interval $(0, 1/2)$ converging to $1/2$.

In [7, p. 16 (3.1)], Timourian showed that a proper mapping with empty branch set between manifolds of codimension one is a locally trivial fiber map; hence the restriction of $f$ to $[0, 1) \times [0, 1]$ is topologically equivalent to the projection of $[0, 1) \times [0, 1]$ onto $[0, 1)$. For each positive integer $n$, let $a_n$ be an arc in $[0, 1] \times [0, 1]$ whose endpoints are $(0, y_n^+)$ and $(1, y_n^+)$, and let $b_n$ be an arc in $[0, 1] \times [0, 1]$ with endpoints $(0, y_n^-)$ and $(1, y_n^-)$ satisfying the following conditions (these arcs exist by the above reference and condition (iii)):
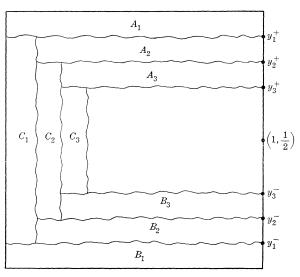
( i )  the collection $\{a_1, b_1, a_2, b_2, \cdots\}$ is pairwise disjoint, and

( ii )  each $a_i$ and each $b_i$ is mapped by $f$ homeomorphically onto $[0, 1]$.

For each $n$, let $A_n'$ denote the closed 2-cell bounded by $a_n \cup a_{n-1} \cup (\{0, 1\} \times [y_n^+, y_{n-1}^+])$, and let $B_n'$ be the closed 2-cell bounded by $b_n \cup b_{n-1} \cup (\{0, 1\} \times [y_{n-1}^-, y_n^-])$, where $y_0^+ = 1$, and $y_0^- = 0$.

Let $\{x_n\}$ be a monotone increasing sequence in the open interval $(0, 1)$ converging to $1$, and let $A_n = A_n' \cap f^{-1}([x_{n-1}, 1])$ and $B_n = B_n' \cap$

$f^{-1}([x_{n-1}, 1])$, where $x_0 = 0$. Finally, let $C_n$ denote the closure of

$$f^{-1}([x_{n-1}, x_n]) - \left( \bigcup_{i=1}^{n} (A_i \cup B_i) \right).$$      (See figure.)



FIGURE 1.

By the aforementioned reference to [7], there exist homeomorphisms

$$h_n^A : A_n \longrightarrow [x_{n-1}, 1] \times [y_n^+, y_{n-1}^+] ,$$
$$h_n^B : B_n \longrightarrow [x_{n-1}, 1] \times [y_{n-1}^-, y_n^-] , \quad \text{and}$$
$$h_n^C : C_n \longrightarrow [x_{n-1}, x_n] \times [y_n^-, y_n^+]$$

such that $\Pi h_n^A = f|_{A_n}$, $\Pi h_n^B = f|_{B_n}$ and $\Pi h_n^C = f|_{C_n}$. We assume here that $h_n^A((1, y_i^+)) = (1, y_i^+)$ and $h_n^B((1, y_i^-)) = (1, y_i^-)$ for $i = n - 1, n$.

Clearly, $h_n^A$ agrees with $h_n^C$ on $A_n \cap C_n$, and $h_n^B$ agrees with $h_n^C$ on $B_n \cap C_n$, so there exists a homeomorphism

$$h_n : A_n \cup B_n \cup C_n \longrightarrow ([x_{n-1}, 1] \times [y_n^+, y_{n-1}^+]) \cup ([x_{n-1}, 1] \times [y_{n-1}^-, y_n^-])$$

$$\cup ([x_{n-1}, x_n] \times [y_n^-, y_n^+])$$

such that $\Pi h_n = f|_{A_n \cup B_n \cup C_n}$, $n = 1, 2, \cdots$. Clearly, $h_{n+1}$ agrees with $h_n$ on $A_{n+1} \cap A_n$ and on $B_{n+1} \cap B_n$; moreover, by 3.1, we may assume that $h_{n+1}$ agrees with $h_n$ on $C_n \cap (A_{n+1} \cup B_{n+1} \cup C_{n+1})$.

Thus, we get a continuous bijection $h; [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ by

$$h(p) = \begin{cases} h_n(p) , & \text{if } p \in A_n \cup B_n \cup C_n \\ \left(1, \dfrac{1}{2}\right) , & \text{if } p = \left(1, \dfrac{1}{2}\right) , \end{cases}$$

satisfying the condition $\Pi h = f$.

LEMMA 3.3. *Let $f\colon M^n \to N^1$ be open, let $U \subset M^n$ be open, and suppose that $f(U)$ is contained in a closed arc on $N^1$. Let $q \in U$, and let $Q$ be a closed subset of $f^{-1}(f(q)) \cap U$. Then no component of $U\text{-}Q$ is separated from $\partial U$. (Two sets are* separated *if their closures are disjoint.)*

*Proof.* If $V$ is a component of $U\text{-}Q$, then $V$ is an open connected subset of $M^n$, so $f(V)$ is an open interval (whose closure is a closed interval). If $V$ is separated from $\partial U$, then $\partial V \subset Q$. Thus $f(\bar{V}) = f(V) \cup f(\partial V) = \{f(q)\} \cup f(V)$ is not compact.

*Proof of Theorem 1.* Let $U'$ be an open disk about $p$ whose closure is a closed disk such that $B_f \cap \bar{U}' \subset \{p\}$ and $f(\bar{U}')$ is a closed interval on $N^1$. By 3.3, for each point $x$ in $U'$, $f^{-1}(f(x)) \cap U'$ contains no simple closed curve. Thus, the closure of each component of $(f^{-1}(f(p)) \cap \bar{U}') - \{p\}$ is a closed arc with a point of $\partial U'$ as one endpoint and either $p$ or a point of $\partial U'$ as its other endpoint, and for each point $q$ in $\bar{U}' - f^{-1}(f(p))$, each component of $f^{-1}(f(q)) \cap \bar{U}'$ is a (possibly degenerate) closed arc each of whose endpoints is on $\partial U'$.

Let $A_1, A_2, \cdots, A_k$ be the closures of the component arcs of $(f^{-1}(f(p)) \cap \bar{U}') - \{p\}$ which have $p$ as an endpoint, and let $a_i$ be the endpoint of $A_i$ on $\partial U'$. (There are clearly finitely many such $A_i$.) Orient $\partial U'$, and index the $a_i$ in order of increasing argument.

Let $R'_1, \cdots, R'_k$ be the components of $U' - \bigcup_{i=1}^{k} A_i$, indexed so that $a_j \in \bar{R}'_i$ if, and only if, $j = i$ or $j \equiv (i+1)(\mathrm{mod}\, k)$.

For each $i = 1, 2, \cdots, k$, there exist a neighborhood $U_i$ of $a_i$ and a homeomorphism $\alpha'_i \colon U_i \to E^2$ such that $\Pi\alpha'_i = f\vert_{U_i}$, where $\Pi\colon E^2 \to E^1$ is projection onto the first coordinate. We may as well assume that $U_i \cap U_j = \varnothing$ when $i \neq j$, $\alpha'_i(U_i \cap \partial U')$ is the $x$-axis, and $\Pi\alpha'_i(q) > 0$ for each $q \in R'_i \cap U_i$.

Let $\{Y_\alpha\}$ be the collection of all components of $(f^{-1}(f(p)) \cap \bar{U}') - \{p\}$ which do not have $p$ as a limit point, and let $Y = \cup Y_\alpha$. Clearly, $p$ is not a limit point of $Y$, so there exists a neighborhood $U''$ of $p$ such that

$$(*) \qquad \left( \bigcup_{i=1}^{k} U_i \right) \cap \bar{U}' = \left( \bigcup_{i=1}^{k} U_i \right) \cap \bar{U}'' \quad \text{and} \quad \bar{U}'' \cap f^{-1}(f(p)) = \bigcup_{i=1}^{k} A_i \,.$$

Let $R_i = R'_i \cap U''$, and fix $i$. We claim that there exists a point $t_i \in (\bar{R}_i \cap \partial U'' \cap U_i) - \{a_i\}$ such that if $x$ is any point in $(\bar{R}_i \cap \partial U'' \cap U_i)$ with $0 < \Pi\alpha'_i(x) \leq \Pi\alpha'_i(t_i)$ and $X$ is the component arc of $f^{-1}(f(x)) \cap \bar{U}''$ containing $x$, then $x' \in U_j$, where $j \equiv (i+1)(\mathrm{mod}\, k)$, and $x'$ is the other endpoint of $X$. For suppose not. Then there exists a sequence $\{x_n\}$ in $(\bar{R}_i \cap \partial U'' \cap U_i) - \{a_i\}$ converging to $a_i$ such that $x'_n \notin U_j$, for each $n$. There is a convergent subsequence of $\{x'_n\}$ whose limit is in

$f^{-1}(f(p)) \cap \partial U''$ but is not in $\bigcup_{i=1}^{k} A_i$. This contradicts $(*)$, above.

Write $f(\bar{U}'')$ as the closed interval $[a', b']$, where $a' < f(p) < b'$. We may assume that $f(p) < f(t_i) < b'$ if $i$ is even, and $a' < f(t_i) < f(p)$ if $i$ is odd. Note that $k$ is even. Let $b = \min\{f(t_2), f(t_4), \cdots, f(t_k)\}$, and let $a = \max\{f(t_1), f(t_3), \cdots, f(t_{k-1})\}$. Finally, let $U = \bar{U}'' \cap f^{-1}([a, b])$, let $d = k/2$, and assume $a = 0$ and $b = 1$.

If $S_i = \bar{R}_i \cap U$, then $f|_{S_i}$ satisfies the hypotheses of 3.2. Note that $f(S_i) = [f(p), b]$ if $i$ is even, and $f(S_i) = [a, f(p)]$ if $i$ is odd, so by 3.2 there exists a homeomorphism $\alpha_i: S_i \to [-1, 1] \times [0, 1]$ such that $f|_{S_i} = \Pi'\alpha_i$, where $\Pi': [-1, 1] \times [0, 1] \to [0, 1]$ is the natural projection. We may assume that $\alpha_i(p) = (0, 1)$, and that if $j \equiv (i + 1)(\mathrm{mod}\, k)$, then $\alpha_i(A_j) = \alpha_j(A_j)$, $i = 1, 2, \cdots, k$.

Let $X_1, X_2, \cdots, X_k$ be disjoint copies of $[-1, 1] \times [0, 1]$ indexed so that $\alpha_i(S_i) = X_i$. Form the disjoint union $X = X_1 + X_2 + \cdots + X_k$. Now, make the following identifications on $X$: if $x_j \in \alpha_j(A_j)$ and $j \equiv (i + 1)(\mathrm{mod}\, k)$, then identify $x_j$ with $\alpha_i \alpha_j^{-1}(x_j)$.

Let $X/\alpha$ denote the resulting identification space, and let $P_\alpha: X \to X/\alpha$ be the quotient map. Let $\alpha: U \to X/\alpha$ by $\alpha(u) = P_\alpha \alpha_i(u)$, if $u \in S_i$. Then $\alpha$ is a homeomorphism.

Let $\Pi_i: X_i \to [0, 1]$ denote the projection mapping, $i = 1, 2, \cdots, k$, and let $\sum_{i=1}^{k} \Pi_i: X \to [0, 1]$ by $(\sum_{i=1}^{k} \Pi_i)(x_j) = \Pi_j(x_j)$, where $x_j \in X_j$. Then $\sum_{i=1}^{k} \Pi_i$ can be factored through $P_\alpha$, obtaining a map $G: X/\alpha \to [0, 1]$ such that $GP_\alpha = \sum_{i=1}^{k} \Pi_i$. But then $f|_U = G\alpha$; i.e., $f|_U$ is topologically equivalent to $G$. Theorem 1 follows.

In [8, Theorem 1], Tôki gives a topological characterization of mappings on a 2-manifold which are pseudo-harmonic. His results imply that if $f: M^2 \to N^1$ is an open map with $B_f$ discrete, then $f$ is topologically equivalent to a harmonic function. It is well known (see, for example, [5, pp. 8–9]) that harmonic functions have the local structure of (our) Theorem 1. The advantage of the present approach must lie in its directness and brevity.

4. **The real analytic case.** In this section we prove Theorem 2. A few facts from the theory of analytic varieties will be needed, and we include them here for completeness. For the reader who is unfamiliar with the terminology, we recommend [6].

LEMMA 4.1. *Every real analytic variety is locally connected.*

LEMMA 4.2. *If $V$ is a $k$-dimensional variety in the real analytic manifold $M$, then for every point $p \in V$, there is a neighborhood $U$ of $p$ in $M$ such that $V \cap U$ is the disjoint union $V' + V''$, where $V'$ is a $k$-dimensional real analytic submanifold of $U$, and $V''$ is a variety in $U$ of dimension at most $k - 1$.*

The proof of 4.2 is in [1].

If $f: M \to N$ is a real analytic mapping, then clearly the sets $f^{-1}(y)$, $y \in N$, are varieties, as are the sets $R_q(f)$, defined in §2. Moreover, it was proved by Church [2, p. 88 (1.3)] that $\dim (f(R_q(f))) \leqq q$, $q = 0, 1, \cdots$.

LEMMA 4.3. *Let* $f: M^n \to N^1$ *be a proper real analytic mapping. Then* $f(R_0(f))$ *is discrete.*

*Proof.* Let $y$ be a limit point of $f(R_0(f))$. Let $\{y_n\}$ be a sequence in $f(R_0(f))$, converging to $y$, and let $x_n$ be a point in $f^{-1}(y_n) \cap R_0(f)$, $n = 1, 2, \cdots$. By the properness of $f$, the sequence $\{x_n\}$ has an accumulation point; call it $x$. Since $R_0(f)$ is closed, $x \in R_0(f)$. Then, if $U$ is any neighborhood of $x$, $U \cap R_0(f) - \{x\}) \neq \varnothing$. Then, by the 0-dimensionality of $f(R_0(f))$ (see remark, above), $U \cap R_0(f)$ is not connected; i.e., $R_0(f)$ is not locally connected at $x$, contradicting 4.1.

*Proof of Theorem 2.* Let $D$ be an open disk in $M^2$ with compact closure, and let $g = f|_D$. Then $g(R_0(g))$ is discrete by 4.3. Let $q \in g(R_0(g))$. It suffices to show that $g^{-1}(q) \cap B_g$ is discrete.

Since $\dim (g^{-1}(q)) = 1$, we can remove from $g^{-1}(q)$, by 4.2, a discrete set $A'$ such that $g^{-1}(q) - A'$ is a 1-dimensional real analytic submanifold of $D$. Let $p$ be a point in $g^{-1}(q) - A'$, and let $U \subset D$ be a coodinate chart about $p$ with real analytic coordinates $(x, y)$ such that $g^{-1}(q)$ is the $y$-axis and $g(R_0(g) \cap U) = \{q\}$. Let $h = f|_U$. By Fox's Spoke Theorem [4, p. 347 (5.1)], it suffices to show that $h^{-1}(q) \cap B_h$ is discrete.

Suppose $W$ is an open disk about a point on $h^{-1}(q) \cap U$ such that each horizontal line in $W$ given by $y = $ constant is mapped homeomorphically by $h$ onto $h(W)$. Then $W \cap B_h = \varnothing$. For define $F: W \to h(W) \times R$ by $F(x, y) = (h(x, y), y)$; this will give the desired equivalence.

For such fixed $\bar{y}$, consider the set $X(\bar{y})$ of points $(x, \bar{y})$ at which $h|_{y=\bar{y}}$ is not interior; i.e., the set of points at which $(\partial/\partial_x)(h|_{y=\bar{y}})$ changes sign. Now, $X = \bigcup_{\bar{y}} X(\bar{y})$ is contained in the set $A = \{(x, y): (\partial h/\partial x)(x, y) = 0\}$. $A$ is clearly a variety, and the openness of $f$ implies that $\dim (A) \leqq 1$. If $A$ is discrete, then the comment in the last paragraph applies and there is nothing to prove. Hence, we may as well assume that $\dim (A) = 1$. Let $E$ be the set of points at which $A$ is not a 1-submanifold.

Consider the intersection of $A$ with the $y$-axis. It is a variety of dimension at most one. If it has dimension zero (i.e., if it is discrete), then there is nothing to prove. So suppose it is 1-dimensional,

in which case it is the entire $y$-axis. Now, the openness of $h$ implies that all points of $U$ with positive abscissa map on one side of $q$, and the points with negative abscissa map on the other side of $q$. Hence, $\partial h/\partial x$ does not change sign at any point of the $y$-axis. Thus, every point on the $y$-axis which is not in $E$ has a neighborhood disjoint from $X$, and $E$ is discrete.

For more extensive results on the topological properties of real analytic mappings, see [3].

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